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A note on zero-one laws in metrical Diophantine approximation

by

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Dedicated to Wolfgang Schmidt on the occasion of his 75th birthday

1. Introduction. Given $\psi : \mathbb{N} \to [0, +\infty)$, let $\mathcal{A}(\psi)$ denote the set of $x \in [0, 1]$ such that

$$|qx + p| < \psi(q) \tag{1}$$

holds for infinitely many $(p, q) \in \mathbb{Z} \times \mathbb{Z} \setminus \{0\}$. In 1924, Khinchin [14] established a beautiful and strikingly simple criterion for the "size" of $\mathcal{A}(\psi)$ expressed in terms of Lebesgue measure. Under the condition that ψ is monotonic, Khinchin's theorem states that the measure of $\mathcal{A}(\psi)$ is one (respectively, zero) if the sum $\sum_q \psi(q)$ diverges (respectively, converges). The monotonicity condition is only required in the divergence case and moreover it is absolutely crucial. Duffin and Schaeffer [10] constructed a non-monotonic function ψ for which $\sum_q \psi(q)$ diverges but $\mathcal{A}(\psi)$ is of zero measure. In other words, without the monotonicity assumption, Khinchin's theorem is false and the famous Duffin–Schaeffer conjecture provides the appropriate statement. The key difference is that in (1), we require coprimality of the integers p and q. Let $\mathcal{A}'(\psi)$ denote the resulting subset of $\mathcal{A}(\psi)$. The Duffin–Schaeffer conjecture states that the measure of $\mathcal{A}'(\psi)$ is one (respectively, zero) if the sum $\sum_q \varphi(q)\psi(q)q^{-1}$ diverges (respectively, converges). Although various partial results have been obtained, the full conjecture represents a key unsolved problem in metric number theory—see [4, 13] for details. Returning to the raw set $\mathcal{A}(\psi)$, without monotonicity and coprimality the appropriate analogue of Khinchin's theorem has been formulated by Catlin [9]. The Catlin conjecture also remains open.

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The upshot of the above discussion is that currently we are unable to prove analogues of Khinchin's theorem for either of the fundamental sets $\mathcal{A}(\psi)$ and $\mathcal{A}'(\psi)$. However, it is known that the Lebesgue measure of $\mathcal{A}(\psi)$ and $\mathcal{A}'(\psi)$ is either 0 or 1. In the case of $\mathcal{A}(\psi)$ this zero-one law is due to Cassels [8] and in the case of $\mathcal{A}'(\psi)$ it is due to Gallagher [11]. The goal of this note is to establish the higher dimensional analogues of these classical zero-one laws. For a discussion concerning the higher dimensional analogues of the conjectures of Duffin-Schaeffer and Catlin see [4, 12, 15, 19, 20].

Throughout, $m \geq 1$ and $n \geq 1$ are integers. Given $\Psi : \mathbb{Z}^m \to [0, +\infty)$, let $\mathcal{A}_{n,m}(\Psi)$ be the set of $\mathbf{X} \in [0,1]^{nm}$ such that

$$|\mathbf{q}\mathbf{X} + \mathbf{p}| < \Psi(\mathbf{q}) \tag{2}$$

holds for infinitely many $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{\mathbf{0}\}$. Here $|\cdot|$ denotes the supremum norm in $\mathbb{R}^m$, $\mathbf{X}$ is regarded as an $n \times m$ matrix and $\mathbf{q}$ is regarded as a row vector. Thus, $\mathbf{q}\mathbf{X} \in \mathbb{R}^m$ represents a system of m real linear forms in n variables. In higher dimensions the set $\mathcal{A}'(\psi)$ has two natural generalizations:

$$\mathcal{A}'_{n,m}(\Psi) := \{\mathbf{X} : (2) \text{ holds for i.m. } (\mathbf{p}, \mathbf{q}) \text{ with } \gcd(\mathbf{p}, \mathbf{q}) = 1\},$$

$$\mathcal{A}''_{n,m}(\Psi) := \{\mathbf{X} : (2) \text{ holds for i.m. } (\mathbf{p}, \mathbf{q}) \text{ with } \gcd(p_i, \mathbf{q}) = 1,\ i = \overline{1, m}\},$$

where $\mathbf{X} \in [0,1]^{nm}$, "i.m." stands for "infinitely many" and $\gcd(\mathbf{p}, \mathbf{q})$ denotes the greatest common divisor of all the components of $\mathbf{p}$ and $\mathbf{q}$.

Before we state our main result, let us agree on the following notation: $\mathcal{A}^{\circ}_{n,m}(\Psi)$ will denote any of the fundamental sets $\mathcal{A}_{n,m}(\Psi)$, $\mathcal{A}'_{n,m}(\Psi)$ or $\mathcal{A}''_{n,m}(\Psi)$. Thus, a statement for $\mathcal{A}^{\circ}_{n,m}(\Psi)$ is valid for $\mathcal{A}_{n,m}(\Psi)$, $\mathcal{A}'_{n,m}(\Psi)$ and $\mathcal{A}''_{n,m}(\Psi)$. Also, $|X|$ will denote the k-dimensional Lebesgue measure of the set $X \subset \mathbb{R}^k$.

THEOREM 1. *For any n, m and Ψ we have $|\mathcal{A}^{\circ}_{n,m}(\Psi)| \in \{0, 1\}$.*

2. Auxiliary results. In this section we group together various self contained statements that we appeal to during the course of establishing Theorem 1. Most are higher dimensional analogues of well known one-dimensional statements. Indeed, the one-dimensional version of our first result can be found in [8].

LEMMA 1. *Let $\{B_i\}$ be a sequence of balls in $\mathbb{R}^k$ with $|B_i| \to 0$ as $i \to \infty$. Let $\{U_i\}$ be a sequence of Lebesgue measurable sets such that $U_i \subset B_i$ for all i. Assume that for some $c > 0$,*

$$|U_i| \geq c|B_i| \quad \text{for all } i. \tag{3}$$

Then the sets

$$\mathcal{U} = \limsup_{i\to\infty} U_i := \bigcap_{j=1}^{\infty} \bigcup_{i \geq j} U_i \quad \text{and} \quad \mathcal{B} = \limsup_{i\to\infty} B_i := \bigcap_{j=1}^{\infty} \bigcup_{i \geq j} B_i$$

have the same Lebesgue measure.

Proof. Let $\mathcal{U}_j := \bigcup_{i\ge j} U_i$ and $\mathcal{D}_j := \mathcal{B}\setminus\mathcal{U}_j$. Then $\mathcal{D} := \mathcal{B}\setminus\mathcal{U} = \bigcup_j \mathcal{D}_j$ and Lemma 1 states that $\mathcal{D}$ has measure zero, or equivalently, that every $\mathcal{D}_j$ has measure zero. Assume the contrary. Then there is an $l\in\mathbb{N}$ such that $|\mathcal{D}_l|>0$ and therefore there is a density point $\mathbf{x}_0$ of $\mathcal{D}_l$. Since $\mathbf{x}_0\in\mathcal{B}$, we know that $x_0\in B_{j_i}$ for a sequence j_i. Since $|B_{j_i}|\to 0$, we see that $|\mathcal{D}_l\cap B_{j_i}|\sim|B_{j_i}|$ as $i\to\infty$. Since $\mathcal{D}_j\supset\mathcal{D}_l$ for all $j\ge l$, it follows that

$$|\mathcal{D}_{j_i}\cap B_{j_i}|\sim|B_{j_i}|\quad\text{as } i\to\infty. \tag{4}$$

On the other hand, by construction $\mathcal{D}_{j_i}\cap U_{j_i}=\emptyset$. Thus, in view of (3) we have

$$|B_{j_i}|\ge|U_{j_i}|+|\mathcal{D}_{j_i}\cap B_{j_i}|\ge c|B_{j_i}|+|\mathcal{D}_{j_i}\cap B_{j_i}|,$$

i.e. $|\mathcal{D}_{j_i}\cap B_{j_i}|\le(1-c)|B_{j_i}|$ for all sufficiently large i. This contradicts (4). ∎

The following lemma is the higher dimensional analogue of the well known one-dimensional “ergodic” property of rational transformations—see for example [11, Lemma 3], [13, Lemma 2.2] or [22, Lemma 7].

LEMMA 2. *For any integer $l\ge 2$ and $\mathbf{s}\in\mathbb{Z}^k$ consider the transformation of the unit cube $[0,1]^k$ into itself given by*

$$T:\mathbf{x}\mapsto l\mathbf{x}+\frac{1}{l}\mathbf{s}\ (\mathrm{mod}\ 1).$$

Let A be a subset of $[0,1]^k$ such that $T(A)\subseteq A$. Then A is of Lebesgue measure 0 or 1.

Proof. Let A be as in the statement. Then $T^\nu(A)\subseteq A$, where $T^\nu:\mathbf{x}\mapsto l^\nu\mathbf{x}+\mathbf{s}/l\ (\mathrm{mod}\ 1)$ is the νth iterate of T. Let χ_A be the characteristic function of A. It follows that

$$\chi_A(\mathbf{x})\le\chi_A\biggl(l^\nu\mathbf{x}+\frac{\mathbf{s}}{l}\biggr). \tag{5}$$

Suppose that $|A|>0$. Then there is a density point $\mathbf{x}_0$ of A. Let C_ν be the cube in $[0,1]^k$ centred at $\mathbf{x}_0$ of sidelength $l^{-\nu}$. Then $|A\cap C_\nu|$ equals

$$\int\limits_{C_\nu}\chi_A(\mathbf{x})\,d\mathbf{x}\overset{(5)}{\le}\int\limits_{C_\nu}\chi_A\biggl(l^\nu\mathbf{x}+\frac{\mathbf{s}}{l}\biggr)\,d\mathbf{x}=\frac{1}{l^{\nu k}}\int\limits_{[0,1]^k}\chi_A(\mathbf{x})\,d\mathbf{x}=|C_\nu|\cdot|A|.$$

Since $\mathbf{x}_0$ is a density point of A and $\operatorname{diam}C_\nu\to 0$ as $\nu\to\infty$, the left hand side of the above equality is asymptotically $|C_\nu|$. Therefore, $|A|=1$. ∎

Given a ball $B=B(x,r)$ and a real number $c>0$, we denote by cB the “scaled” ball $B(x,cr)$. The next lemma is a basic covering result from geometric measure theory usually referred to as the $5r$-lemma. For further details and proof the reader is referred to [18].

LEMMA 3. *Every collection $\mathcal{C}$ of balls of uniformly bounded diameter in a metric space contains a disjoint subcollection $\mathcal{G}$ such that*

$$\bigcup_{B\in\mathcal{C}} B \subset \bigcup_{B\in\mathcal{G}} 5B.$$

We immediately make use of the covering lemma to show that the Lebesgue measure of a reasonably general lim sup set is unchanged with respect to "scaling" by a constant factor.

LEMMA 4. *Let $\{S_i\}_{i\in\mathbb{N}}$ be a sequence of subsets in $[0,1]^k$, $\{\delta_i\}_{i\in\mathbb{N}}$ be a sequence of positive numbers such that $\delta_i \to 0$ as $i \to \infty$ and let*

$$\Delta(S_i,\delta_i) := \{x \in [0,1]^k : \operatorname{dist}(S_i,x) < \delta_i\}.$$

Then for any real number $C > 1$, the sets

$$A := \limsup_{i\to\infty} \Delta(S_i,\delta_i) \quad \textit{and} \quad B := \limsup_{i\to\infty} \Delta(S_i,C\delta_i)$$

have the same Lebesgue measure.

Proof. First of all notice that the sets $\Delta(S_i,\delta_i)$ are open and therefore Lebesgue measurable. Since $C > 1$ we have $A \subset B$. For each $i \in \mathbb{N}$, let $\mathcal{B}_i$ denote the collection of balls $\{B(x,\delta_i) : x \in S_i\}$. Thus, $\Delta(S_i,\delta_i) = \bigcup_{B\in\mathcal{B}_i} B$. By Lemma 3, there is a disjoint subcollection $\mathcal{G}_i$ of $\mathcal{B}_i$ such that

$$\overset{\circ}{\bigcup_{B\in\mathcal{G}_i}} B \subset \Delta(S_i,\delta_i) = \bigcup_{B\in\mathcal{B}_i} B \subset \bigcup_{B\in\mathcal{G}_i} 5B. \tag{6}$$

Since $S_i \subset [0,1]^k$, every ball $B \in \mathcal{G}_i$ is contained in the cube $[-\delta_i, 1+\delta_i]^k$. It follows that $\mathcal{G}_i$ is a finite disjoint collection of balls.

If $z \in \Delta(S_i,C\delta_i)$, then there is a $y \in S_i$ such that $|z-y| < C\delta_i$. Furthermore, by (6) there exists a ball $B = B(x,\delta_i) \in \mathcal{G}_i$ such that $y \in 5B$. Therefore, $|z-x| \le |z-y| + |y-x| < (5+C)\delta_i$. Thus we have shown that

$$\Delta(S_i,C\delta_i) \subset \bigcup_{B\in\mathcal{G}_i} (5+C)B. \tag{7}$$

Now given a constant $\lambda > 0$, let $\mathcal{C}(\lambda) := \limsup_{i\to\infty} \bigcup_{B\in\mathcal{G}_i} \lambda B$. This is the set of x such that $x \in \lambda B$ for some $B \in \mathcal{G}_i$ for infinitely many i. Then (6) and (7) imply that

$$\mathcal{C}(1) \subset A \subset B \subset \mathcal{C}(5+C). \tag{8}$$

By Lemma 1, the sets $\mathcal{C}(\lambda)$ with $\lambda > 0$ have the same Lebesgue measure irrespective of λ. Therefore, in view of (8) the sets A and B must have the same Lebesgue measure. ∎

3. Proof of Theorem 1. Following the arguments of [11], it is easily verified that $\mathcal{A}^{\circ}_{n,m}(\Psi) = [0,1]^{nm}$ if the condition

$$\Psi(\mathbf{q})/|\mathbf{q}| \to 0 \quad \text{as } |\mathbf{q}| \to \infty \tag{9}$$

is violated. Therefore, without loss of generality we assume that (9) is satisfied.

When considering $\mathcal{A}^{\circ}_{n,m}(\Psi)$, the error of approximation is rigidly determined by the function Ψ. In proving Theorem 1, it is extremely useful to introduce a certain degree of flexibility within the error of approximation. Given $\mathcal{A}^{\circ}_{n,m}(\Psi)$, let

$$\mathcal{F}^{\circ}_{n,m}(\Psi) = \bigcup_{k=1}^{\infty} \mathcal{A}^{\circ}_{n,m}(k\Psi).$$

Clearly, $\mathcal{F}^{\circ}_{n,m}(\Psi) \supset \mathcal{A}^{\circ}_{n,m}(\Psi)$. However, as a consequence of Lemma 4 we have

$$|\mathcal{F}^{\circ}_{n,m}(\Psi)| = |\mathcal{A}^{\circ}_{n,m}(\Psi)|. \tag{10}$$

Clearly, Theorem 1 follows on establishing the analogous statement for $\mathcal{F}^{\circ}_{n,m}(\Psi)$.

THEOREM 2. *For any n, m and Ψ we have $|\mathcal{F}^{\circ}_{n,m}(\Psi)| \in \{0,1\}$.*

Proof. We establish the theorem by considering the sets $\mathcal{F}_{n,m}(\Psi)$, $\mathcal{F}'_{n,m}(\Psi)$ and $\mathcal{F}''_{n,m}(\Psi)$ separately.

The set $\mathcal{F}_{n,m}(\Psi)$: Clearly, $\mathcal{F}_{n,m}(\Psi)$ is invariant under the translation $T : \mathbf{X} \mapsto 2\mathbf{X} \pmod 1$. Thus, the desired statement is a trivial consequence of Lemma 2.

The set $\mathcal{F}'_{n,m}(\Psi)$: By definition, $\mathcal{F}'_{n,m}(\Psi)$ consists of points $\mathbf{X} \in [0,1]^{nm}$ for which there exists a constant $C = C(\mathbf{X}) > 0$ such that

$$|\mathbf{q}\mathbf{X} + \mathbf{p}| < C\Psi(\mathbf{q}) \quad \text{and} \quad \gcd(\mathbf{p},\mathbf{q}) = 1 \tag{11}$$

for infinitely many $(\mathbf{p},\mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{0\}$. Now, for each prime l consider the following subsets of $\mathcal{F}'_{n,m}(\Psi)$:

$\mathcal{S}_0(l) = \{\mathbf{X} \in [0,1]^{nm} : \exists C > 0$ so that (11) holds for i.m. $(\mathbf{p},\mathbf{q})$ with $l \nmid d = \gcd(\mathbf{q})\}$,

$\mathcal{S}_1(l) = \{\mathbf{X} \in [0,1]^{nm} : \exists C > 0$ so that (11) holds for i.m. $(\mathbf{p},\mathbf{q})$ with $l \,\|\, d = \gcd(\mathbf{q})\}$,

$\mathcal{S}_2(l) = \{\mathbf{X} \in [0,1]^{nm} : \exists C > 0$ so that (11) holds for i.m. $(\mathbf{p},\mathbf{q})$ with $l^2 \mid d = \gcd(\mathbf{q})\}$.

Here $l \,\|\, d$ means that l divides d but l^2 does not. Note that

$$\mathcal{F}'_{n,m}(\Psi) = \mathcal{S}_0(l) \cup \mathcal{S}_1(l) \cup \mathcal{S}_2(l). \tag{12}$$

Suppose $\mathbf{X} \in \mathcal{S}_0(l)$. Then (11) is satisfied for infinitely many $(\mathbf{p}, \mathbf{q})$ with $l \nmid d = \gcd(\mathbf{q})$. On setting $\mathbf{q}' := \mathbf{q}$ and $\mathbf{p}' := l\mathbf{p}$, we find that

$$|\mathbf{q}'(l\mathbf{X}) + \mathbf{p}'| < lC\Psi(\mathbf{q}')$$

for infinitely many $(\mathbf{p}', \mathbf{q}') \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{0\}$ with

$$\gcd(\mathbf{p}', \mathbf{q}') = 1. \tag{13}$$

The coprimality condition is readily verified by making use of the fact that $l \nmid \gcd(\mathbf{q})$. Thus, if $\mathbf{X} \in \mathcal{S}_0(l)$ then $l\mathbf{X} \in \mathcal{S}_0(l)$. Therefore the set $\mathcal{S}_0(l)$ is invariant under the transformation $T : \mathbf{X} \mapsto l\mathbf{X} \pmod 1$ and Lemma 2 implies that $|\mathcal{S}_0(l)|$ is 0 or 1.

For $j \in \{1, \ldots, n\}$, let $\mathcal{S}_{1,j}(l)$ denote the set of $\mathbf{X} \in [0,1]^{nm}$ such that (11) is satisfied for infinitely many $(\mathbf{p}, \mathbf{q})$ with $l \parallel q_j$. Recall that q_j is the jth coordinate of $\mathbf{q} = (q_1, \ldots, q_n)$. Clearly,

$$\mathcal{S}_1(l) = \bigcup_{j=1}^{n} \mathcal{S}_{1,j}(l)\,.$$

Suppose $\mathbf{X} \in \mathcal{S}_{1,j}(l)$ for some $j \in \{1, \ldots, n\}$. Let $\mathbf{S}_j \in \mathbb{Z}^{nm}$ denote the integer matrix with zero entries everywhere except in the jth row where every entry is 1. Then $\mathbf{q}\mathbf{S}_j = (q_j, \ldots, q_j) \in \mathbb{Z}^m$. By definition, (11) is satisfied for infinitely many $(\mathbf{p}, \mathbf{q})$ with $l \parallel q_j$. On setting $\mathbf{q}' := \mathbf{q}$ and $\mathbf{p}' := l\mathbf{p} - \frac{1}{l}\mathbf{q}\mathbf{S}_j$, we see that

$$\left|\mathbf{q}'\left(l\mathbf{X} + \frac{1}{l}\,\mathbf{S}_j\right) + \mathbf{p}'\right| < lC\Psi(\mathbf{q}') \tag{14}$$

for infinitely many $(\mathbf{p}', \mathbf{q}') \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{0\}$ satisfying (13). Thus, if $\mathbf{X} \in \mathcal{S}_{1,j}(l)$ then $l\mathbf{X} + (1/l)\mathbf{S}_j \in \mathcal{S}_{1,j}(l)$. Therefore the set $\mathcal{S}_{1,j}(l)$ is invariant under the transformation

$$T : \mathbf{X} \mapsto l\mathbf{X} + \frac{1}{l}\,\mathbf{S}_j \pmod 1$$

and Lemma 2 implies that $|\mathcal{S}_{1,j}(l)|$ is 0 or 1. Thus, $\mathcal{S}_1(l)$ is a finite union of sets with measure 0 or 1 and so $|\mathcal{S}_1(l)|$ is also 0 or 1.

In view of (12), the upshot of the above results for $|\mathcal{S}_0(l)|$ and $|\mathcal{S}_1(l)|$ is that if there exists a prime l such that $\mathcal{S}_0(l)$ or $\mathcal{S}_1(l)$ is of positive measure then $|\mathcal{F}'_{n,m}(\Psi)| = 1$. Thus, without loss of generality we can assume that such a prime does not exist and so by (12) we have

$$|\mathcal{S}_2(l)| = |\mathcal{F}'_{n,m}(\Psi)| \quad \text{for every prime } l. \tag{15}$$

Suppose $\mathbf{X} \in \mathcal{S}_2(l)$ and fix any $\mathbf{S} \in \mathbb{Z}^{nm}$. Then (11) is satisfied for infinitely many $(\mathbf{p}, \mathbf{q})$ with $l^2 \mid d = \gcd(\mathbf{q})$. On setting $\mathbf{q}' := \mathbf{q}$ and $\mathbf{p}' := \mathbf{p} - (1/l)\mathbf{q}\mathbf{S}$, we see that

$$\left|\mathbf{q}'\left(\mathbf{X} + \frac{1}{l}\,\mathbf{S}\right) + \mathbf{p}'\right| < \ C\Psi(\mathbf{q}) \tag{16}$$

for infinitely many $(\mathbf{p}', \mathbf{q}') \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{0\}$ satisfying (13). Thus, if $\mathbf{X} \in \mathcal{S}_2(l)$ then $\mathbf{X} + (1/l)\mathbf{S} \in \mathcal{S}_2(l)$ for any $\mathbf{S} \in \mathbb{Z}^{nm}$. Therefore the set $\mathcal{S}_2(l)$ is invariant under any transformation $\mathbf{X} \mapsto \mathbf{X} + (1/l)\mathbf{S} \pmod 1$ with $\mathbf{S} \in \mathbb{Z}^{nm}$. In other words, $\mathcal{S}_2(l)$ is $1/l$-periodic in every coordinate. Thus, for any cube $\mathcal{C}_l$ in $[0,1]^{nm}$ of sidelength $1/l$ we have

$$|\mathcal{S}_2(l) \cap \mathcal{C}_l| = |\mathcal{S}_2(l)| \cdot |\mathcal{C}_l|.$$

In view of (15), it follows that

$$|\mathcal{F}'_{n,m}(\Psi) \cap \mathcal{C}_l| = |\mathcal{F}'_{n,m}(\Psi)| \cdot |\mathcal{C}_l| \tag{17}$$

for any prime l. Now suppose that $|\mathcal{F}'_{n,m}(\Psi)| > 0$. Then there is a density point $\mathbf{X}_0$ of $\mathcal{F}'_{n,m}(\Psi)$. For each prime l, let $\mathcal{C}_l$ denote the cube in $[0,1]^{nm}$ centred at $\mathbf{X}_0$ of sidelength $1/l$. Then

$$|\mathcal{F}'_{n,m}(\Psi) \cap \mathcal{C}_l| \sim |\mathcal{C}_l| \quad \text{as } l \to \infty.$$

This together with (17) implies that $|\mathcal{F}'_{n,m}(\Psi)| = 1$ and thereby completes the proof of Theorem 1 for the set $\mathcal{F}'_{n,m}(\Psi)$.

The set $\mathcal{F}''_{n,m}(\Psi)$: To establish the desired zero-one statement for the set $\mathcal{F}''_{n,m}(\Psi)$, we modify in the obvious manner the argument for $\mathcal{F}'_{n,m}(\Psi)$. Naturally, "$\gcd(p_j, \mathbf{q}) = 1$ for all $j = 1, \ldots, m$" will replace "$\gcd(\mathbf{p}, \mathbf{q}) = 1$" appearing in (11). Similarly, the condition that "$\gcd(p'_j, \mathbf{q}') = 1$ for all $j = 1, \ldots, m$" will replace the coprimality condition (13). The rest remains pretty much unchanged. ∎

4. Further results and questions

Ψ-well approximable points. Various sets of *Ψ-well approximable points* are defined by requiring that the constant $C > 0$ appearing in (11) can be made arbitrarily small. More precisely, set

$$\mathcal{W}^\circ_{n,m}(\Psi) := \bigcap_{k=1}^{\infty} \mathcal{A}^\circ_{n,m}(k^{-1}\Psi).$$

Lemma 4 readily implies the following statement.

THEOREM 3. *For any n, m and Ψ we have* $|\mathcal{W}^\circ_{n,m}(\Psi)| = |\mathcal{A}^\circ_{n,m}(\Psi)|$.

Theorem 3 combined with (10) and Theorem 2 trivially implies the zero-one law for Ψ-well approximable sets.

COROLLARY 1. *For any n, m and Ψ we have* $|\mathcal{W}^\circ_{n,m}(\Psi)| \in \{0, 1\}$.

Ψ-badly approximable points. Naturally, various sets of *Ψ-badly approximable* points can be thought of as being complementary to Ψ-well approximable sets. More precisely,

$$\mathcal{B}^\circ_{n,m}(\Psi) := \mathcal{A}^\circ_{n,m}(\Psi) \setminus \mathcal{W}^\circ_{n,m}(\Psi).$$

An immediate consequence of Theorem 3 is the following result.

Corollary 2. *For any n, m and Ψ we have $|\mathcal{B}^\circ_{n,m}(\Psi)| = 0$.*

The classical set of badly approximable real numbers $\mathbf{Bad} := \mathcal{B}_{1,1}(q \mapsto q^{-1})$ is known to have full Hausdorff dimension, i.e. $\dim \mathbf{Bad} = 1$. For a general function $\psi : \mathbb{N} \to [0, +\infty)$ with various mild growth conditions, Bugeaud [7], answering a question posed in [5], has shown that $\mathcal{B}_{1,1}(\psi)$ has full Hausdorff dimension, i.e. $\dim \mathcal{B}_{1,1}(\psi) = \dim \mathcal{A}_{1,1}(\psi)$. In view of this and Corollary 2 it is reasonable to ask the following question.

Question 1. *Does $\mathcal{B}_{n,m}(\Psi)$ have full Hausdorff dimension, i.e.*

$$\dim \mathcal{B}_{n,m}(\Psi) = \dim \mathcal{A}_{n,m}(\Psi)\ ?$$

A weaker form of this question, in which $\mathcal{B}_{n,m}(\Psi)$ is replaced by $\mathcal{A}_{n,m}(\Psi) \setminus \mathcal{A}_{n,m}(\Psi')$ with $\Psi'(\mathbf{q}) = o(\Psi(\mathbf{q}))$ as $|\mathbf{q}| \to \infty$, can be found in [5]. Note that if the answer to the above question is yes, then automatically $\dim \mathcal{B}^\circ_{n,m}(\Psi) = \dim \mathcal{A}^\circ_{n,m}(\Psi)$.

Multi-error approximation. Observe that the inequality (2) can be rewritten as a system of m inequalities

$$|\mathbf{q}\mathbf{X}^{(j)} + p_j| < \Psi(\mathbf{q}), \quad j = 1, \ldots, m,$$

where $\mathbf{X}^{(j)}$ is the jth column of $\mathbf{X}$. Thus, the error of approximation associated with each linear form is determined by Ψ and is the same. More generally, we consider the system

$$|\mathbf{q}\mathbf{X}^{(j)} + p_j| < \Psi_j(\mathbf{q}), \qquad j = 1, \ldots, m, \tag{18}$$

with $\Psi_j : \mathbb{Z}^n \to [0, +\infty)$ and so the error of approximation is allowed to differ from one linear form to the next. Let $\mathcal{A}^\circ_{n,m}(\Psi_1, \ldots, \Psi_m)$ denote the "multi-error" analogue of $\mathcal{A}^\circ_{n,m}(\Psi)$, obtained by replacing (2) with (18) in the definition of $\mathcal{A}^\circ_{n,m}(\Psi)$. Naturally, this enables us to define the multi-error analogues of $\mathcal{F}^\circ_{n,m}(\Psi)$, $\mathcal{W}^\circ_{n,m}(\Psi)$ and $\mathcal{B}^\circ_{n,m}(\Psi)$.

Without much effort, it is possible to establish the multi-error analogue of Theorem 2—the proof is practically unchanged.

Theorem 4. *For any n, m and $\Psi_1, \ldots, \Psi_m$ we have*

$$|\mathcal{F}^\circ_{n,m}(\Psi_1, \ldots, \Psi_m)| \in \{0, 1\}.$$

If the statement of Lemma 4 can be generalized to the multi-error framework the above theorem would answer the following question and thereby yield the analogue of Theorem 1.

Question 2. *Is it true that $|\mathcal{F}^\circ_{n,m}(\Psi_1, \ldots, \Psi_m)| = |\mathcal{W}^\circ_{n,m}(\Psi_1, \ldots, \Psi_m)|$?*

Note that if the answer to Question 2 is yes, then so is the answer to our next question.

QUESTION 3. *Is it true that* $|\mathcal{B}^{\circ}_{n,m}(\Psi_1,\dots,\Psi_m)|=0$*?*

Multiplicative approximation. Given $\Psi:\mathbb{Z}^n\to[0,+\infty)$, let $\mathcal{A}^{\times}_{n,m}(\Psi)$ be the set of $\mathbf{X}\in[0,1]^{nm}$ such that

$$\prod_{i=1}^{m}\|\mathbf{q}\mathbf{X}^{(j)}\|<\Psi(\mathbf{q}) \tag{19}$$

for infinitely many $\mathbf{q}\in\mathbb{Z}^n$. Here $\|\cdot\|$ denotes the distance to the nearest integer. Naturally, this enables us to define the associated multiplicative sets $\mathcal{F}^{\times}_{n,m}(\Psi)$, $\mathcal{W}^{\times}_{n,m}(\Psi)$ (of multiplicatively Ψ-well approximable points) and $\mathcal{B}^{\times}_{n,m}(\Psi)$ (of multiplicatively Ψ-badly approximable points). Clearly, if $\Psi:=\Psi_1\cdots\Psi_m$ then

$$\begin{aligned}\mathcal{A}(\Psi_1,\dots,\Psi_m)&\subset\mathcal{A}^{\times}_{n,m}(\Psi),\\ \mathcal{F}(\Psi_1,\dots,\Psi_m)&\subset\mathcal{F}^{\times}_{n,m}(\Psi),\\ \mathcal{W}(\Psi_1,\dots,\Psi_m)&\subset\mathcal{W}^{\times}_{n,m}(\Psi).\end{aligned}$$

However, it is easily seen that

$$\mathcal{B}(\Psi_1,\dots,\Psi_m)\not\subset\mathcal{B}^{\times}_{n,m}(\Psi).$$

QUESTION 4. *Is it true that* $\mathcal{A}^{\times}_{n,m}(\Psi)$, $\mathcal{F}^{\times}_{n,m}(\Psi)$ *and* $\mathcal{W}^{\times}_{n,m}(\Psi)$ *are of measure* 0 *or* 1*?*

QUESTION 5. *Is it true that* $|\mathcal{B}^{\times}_{n,m}(\Psi)|=0$*?*

Note that when $n=1$, $m=2$ and $\Psi(q):=q^{-1}$, the answer to Question 5 is yes for obvious reasons. Indeed, it is conjectured that

$$\mathcal{B}^{\times}_{1,2}(q\mapsto q^{-1})=\emptyset.$$

This is Littlewood's famous conjecture in the theory of Diophantine approximation.

Approximation by rational planes. The inequality (2) takes on two "extreme" forms of rational approximation depending on whether $n=1$ or $m=1$. When $m=1$, it corresponds to approximating arbitrary points by $(n-1)$-dimensional rational planes (i.e. rational hyperplanes) and gives rise to the dual theory of Diophantine approximation. When $n=1$, it corresponds to approximating arbitrary points by 0-dimensional rational planes (i.e. rational points) and gives rise to the simultaneous theory of Diophantine approximation. For $d\in\{0,\dots,n-1\}$, it is natural to consider the Diophantine approximation theory in which points in $\mathbb{R}^n$ are approximated by d-dimensional rational planes—the dual and simultaneous theories just represent the extreme. The foundations have been developed in some depth by W. M. Schmidt [21] in the sixties and more recently by M. Laurent [16]. However, apart from the extreme cases, there appears to be no analogue of Theorem 1 within the theory of approximation by d-dimensional rational planes.

Approximation by algebraic numbers. Sprindžuk's [22] celebrated proof of Mahler's conjecture [17] led Baker [1] to make the following stronger conjecture that was eventually established by Bernik [6]. *Let $n \in \mathbb{N}$ and $\psi : \mathbb{N} \to (0, +\infty)$ be monotonic. Then for almost every real x the inequality*

$$|P(x)| < H(P)^{-n+1}\psi(H(P)) \tag{20}$$

holds for finitely many $P \in \mathbb{Z}[x]$ with $\deg P \leq n$ if

$$\sum_{r=1}^{\infty} \psi(r) < \infty. \tag{21}$$

Here $H(P)$ is the height of P, i.e. the maximum of the absolute values of the coefficients of P. The case $\psi(h) := h^{-1-\varepsilon}$ corresponds to Mahler's conjecture. In [2] it has been shown that if the sum in (21) diverges and ψ is monotonic, then for almost every real x inequality (20) holds infinitely often. More recently [3], the monotonicity assumption in Bernik's convergence result has been removed. However, removing the monotonicity assumption from the divergence result remains an open problem akin to the Duffin–Schaeffer conjecture. In the first instance, it would be natural and desirable to ask for a zero-one law.

QUESTION 6. *Is it true that the set of $x \in [0, 1]$ such that* (20) *holds for infinitely many $P \in \mathbb{Z}[x]$ with $\deg P \leq n$ is of measure* 0 *or* 1*?*

The following is a related question concerning explicit approximation by algebraic numbers.

QUESTION 7. *Is it true that the set of real $x \in [0, 1]$ such that*

$$|x - \alpha| < H(\alpha)^{-n}\psi(H(\alpha)) \tag{22}$$

holds for infinitely many real algebraic α of $\deg \alpha \leq n$ is of measure 0 *or* 1*?*

Here $H(\alpha)$ stands for the height of the minimal defining polynomial of α. If (21) is satisfied, a simple application of the Borel–Cantelli lemma shows that the set under consideration is of measure zero. On the other hand, if the sum in (21) diverges and ψ is monotonic, the set under consideration is known to have measure one (see [2]). The upshot is that Question 7 only needs to be considered when ψ is non-monotonic and the sum in (21) diverges.

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