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Lower bounds for the number of integral polynomials with given order of discriminants

by

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Dedicated to Professor Wolfgang Schmidt

1. Introduction. The discriminant of a polynomial is a vital characteristic that crops up in various problems of number theory. For example, they play an important role in Diophantine equations, Diophantine approximation and algebraic number theory [3–7].

Let

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = a_n(x-\alpha_1)\cdots(x-\alpha_n)$$

be a polynomial. By definition,

$$D(P) = a_n^{2n-2} \prod_{1\le i<j\le n} (\alpha_i - \alpha_j)^2. \tag{1}$$

It is also well known that

$$D(P) = (-1)^{\binom{n}{2}} \begin{vmatrix} 1 & a_{n-1} & a_{n-2} & \ldots & a_0 & 0 & \ldots \\ 0 & a_n & a_{n-1} & \ldots & a_1 & a_0 & \ldots \\ & & & \ldots & & & \\ 0 & \ldots & 0 & a_n & \ldots & a_1 & a_0 \\ n & (n-1)a_{n-1} & (n-2)a_{n-2} & \ldots & 0 & 0 & \ldots \\ 0 & na_n & (n-1)a_{n-1} & \ldots & a_1 & 0 & \ldots \\ & & & \ldots & & & \\ 0 & \ldots & \ldots & 0 & na_n & \ldots & a_1 \end{vmatrix}. \tag{2}$$

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Trivially, by (2), whenever P has rational integral coefficients the discriminant $D(P)$ is also an integer. Furthermore, if $D(P) \neq 0$ then

$$|D(P)| \geq 1. \tag{3}$$

Also (1) obviously implies that $D(P) \neq 0$ if and only if $P(x)$ has no multiple roots.

Fix $n \in \mathbb{N}$. Let $Q > Q_0(n)$, where $Q_0(n)$ is a sufficiently large number. Throughout $\mathcal{P}_n(Q)$ will denote the class of non-zero polynomials $P(x)$ with $\deg P \leq n$ and $H(P) \leq Q$. Furthermore, $\mathcal{P}'_n(H)$ will be the subclass of $\mathcal{P}_n(H)$ consisting of polynomials P with $H(P) = H$. In what follows $c(n)$ and c_j, $j = 0, 1, \ldots$, will stand for some positive constants depending on n only. Also we will use the Vinogradov symbol $A \ll B$ meaning that $A \leq c_0 B$. The notation $A \asymp B$ means $B \ll A \ll B$.

Using the representation (2) for $D(P)$ one readily verifies that $|D(P)| < c(n)Q^{2n-2}$. Thus, by (3), we have

$$1 \leq |D(P)| < c(n)Q^{2n-2} \tag{4}$$

for integral polynomials P with no multiple roots.

The number of polynomials in the class $\mathcal{P}_n(Q)$ is finite and is easily verified to satisfy

$$\#\mathcal{P}_n(Q) < 2^{2n+2}Q^{n+1}.$$

The latter together with (4) shows that $[1, c(n)Q^{2n-2}]$ contains intervals of length $c(n)Q^{n-3}$ free from values of discriminants of $P \in \mathcal{P}_n(Q)$. For $n \geq 4$ these intervals can be arbitrarily large. Thus, the discriminants $D(P)$ are rather sparse in the interval $[1, c(n)Q^{2n-2}]$ they belong to. In this paper we establish a sharp lower bound for the number of polynomials $P \in \mathcal{P}_n(Q)$ with relatively small discriminants. To the best of our knowledge this is the first result of this kind.

2. Main theorems. In this paper we prove two theorems. The first one deals with the distribution of discriminants of integral polynomials.

THEOREM 1. *Let $v \in [0, 1/2]$. Then there are at least $c(n)Q^{n+1-2v}$ polynomials P in $\mathcal{P}_n(Q)$ with*

$$|D(P)| < Q^{2n-2-2v}. \tag{5}$$

In all likelihood the lower bound on the number of polynomials satisfying (5) is best possible up to a constant multiple. Establishing this would also give a lower bound in (5).

The following is an effective metrical result that represents the key to establishing Theorem 1. Let Q denote a sufficiently large number, $v \in [0, 1/2]$ and let c_1 and c_2 be positive constants. By Minkowski's theorem on linear forms, for any $x \in [-1/2, 1/2]$ the system of inequalities

$$(6) \qquad \begin{cases} |P(x)| < c_1 Q^{-n+v}, \\ |P'(x)| < c_2 Q^{1-v} \end{cases}$$

has a solution in polynomials $P \in \mathcal{P}_n(Q)$ whenever $c_1 c_2 > 1$. Our next result shows that the condition $c_1 c_2 > 1$ cannot be substantially relaxed. To state the result we introduce further notation. Let $\mathcal{L}_{n,Q}(c_1, c_2)$ denote the set of $x \in I \subset [-1/2, 1/2]$ such that (6) has a solution in $P \in \mathcal{P}_n(Q)$.

THEOREM 2. *Let Q denote a sufficiently large number*, $v \in [0, 1/2]$ *and let c_1 and c_2 be positive constants such that $c_1 c_2 < n^{-1} 2^{-n-11}$. Then*

$$\mu \mathcal{L}_{n,Q}(c_1, c_2) < |I|/2,$$

where μ denotes the Lebesgue measure on the real axis.

3. Auxiliary lemmas. This section contains several lemmas that will be used in the course of establishing Theorem 2.

For each polynomial $P(x) \in \mathbb{Z}[x]$ of degree n with roots $\alpha_1, \ldots, \alpha_n$, that is,

$$P(x) = a_n (x - \alpha_1) \cdots (x - \alpha_n),$$

we pick a root, say α_1, and consider only those $x \in I$ with $\min_{1 \le i \le n} |x - \alpha_i| = |x - \alpha_1|$. Furthermore, order the other roots of P according to the distance from α_1 so that

$$|\alpha_1 - \alpha_2| \le |\alpha_1 - \alpha_3| \le \cdots \le |\alpha_1 - \alpha_n|.$$

Define vectors $(\mu_2, \mu_3, \ldots, \mu_n)$ and $(l_2, l_3, \ldots, l_n)$ by setting

$$|\alpha_1 - \alpha_j| = H^{-\mu_j}, \qquad l_j - 1 = [\mu_j T], \qquad j = \overline{2, n},$$

where $T = [n/\varepsilon] + 1$ and ε is a small positive number. It is readily seen that $(l_j - 1) T^{-1} \le \mu_j < l_j T^{-1}$. Further, define

$$p_j = \frac{l_{j+1} + \cdots + l_n}{T}, \qquad j = \overline{1, n-1}.$$

The polynomials $P \in \mathcal{P}'_n(H)$ that have the same vector $\bar{s} = (l_2, \ldots, l_n)$ form a subclass which will be denoted by $\mathcal{P}_n(H, \bar{s})$.

Given $P \in \mathcal{P}'_n(H)$, let

$$S(\alpha_i) = \{x \in \mathbb{R} : |x - \alpha_i| = \min_{1 \le j \le n} |x - \alpha_j|\}.$$

LEMMA 1 (see [1]). *If P is a polynomial and $x \in S(\alpha_1)$, then*

$$|x - \alpha_1| \le 2^n |P(x)|\, |P'(\alpha_1)|^{-1},$$

$$|x - \alpha_1| \le \min_{2 \le j \le n} \Big(2^{n-j} |P(x)|\, |P'(\alpha_1)|^{-1} \prod_{k=2}^{j} |\alpha_1 - \alpha_k| \Big)^{1/j}.$$

Lemma 2. *If $x \in S(\alpha_1)$, then*

$$|x-\alpha_1| \le n\,\frac{|P(x)|}{|P'(x)|}.$$

Proof. Using the representation $P(x) = a_n(x-\alpha_1)\cdots(x-\alpha_n)$ we obtain

$$\frac{|P'(x)|}{|P(x)|} \le \sum_{j=1}^{n} \frac{1}{|x-\alpha_j|} \le \frac{n}{|x-\alpha_1|},$$

whence the lemma readily follows.

Lemma 3 (see [7]). *If $|a_n| \gg H(P)$, then all roots α_j of P satisfy*

$$|\alpha_j| \ll 1.$$

In the next lemma we consider polynomials of fixed height H only.

Lemma 4 (see [7]). *Let $k, m \in \mathbb{Z}$ and $P \in \mathcal{P}'_n(H)$. Then*

$$\max_{k\le m\le k+n} |P(m)| > c(n)H.$$

Lemma 5 (see [2]). *For any $n \in \mathbb{N}$ with $n > 1$ and real $\delta > 0$ there is an effectively computable bound $K_0(\delta, n)$ such that for any $K > K_0$ and positive real ς, τ, η the following holds. If $P_1(x), P_2(x) \in \mathbb{Z}[x]$ are coprime and*

$$\max(\deg P_1, \deg P_2) \le n, \qquad \max(H(P_1), H(P_2)) \le K^{\varsigma},$$

and if there is an interval $I \subset \mathbb{R}$ with $|I| = K^{-\eta}$ such that

$$\max(|P_1(x)|, |P_2(x)|) < K^{-\tau} \quad \text{for all } x \in I,$$

then

$$\tau + \varsigma + 2\max\{\tau+\varsigma-\eta, 0\} < 2n\varsigma + \delta.$$

Lemma 6. *Let $P \in \mathcal{P}'_n(H)$ with $|D(P)| < Q^{2n-2-2v}$. Then there is a polynomial $T(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0$ satisfying*

$$|D(T)| = |D(P)|, \qquad H(T) \ll H \quad \text{and} \quad |b_n| \gg H.$$

Proof. Let $\alpha_1, \ldots, \alpha_n$ be the roots of $P(x) = a_n x^n + \cdots + a_1 x + a_0$. By Lemma 4, there is an integer m_0 with $1 \le m_0 \le n+1$ such that

$$|P(m_0)| > c(n)H.$$

Consider the polynomial $P_1(x) = P(x+m_0) = a_n x^n + a'_{n-1}x^{n-1} + \cdots + a'_1 x + P(m_0)$. Its roots are $\beta_j = \alpha_j - m_0$, $1 \le j \le n$, and

$$|D(P_1)| = a_n^{2n-2}\Big|\prod_{1\le i<j\le n}(\beta_i-\beta_j)^2\Big| = a_n^{2n-2}\Big|\prod_{1\le i<j\le n}(\alpha_i-\alpha_j)^2\Big| = |D(P)|.$$

The polynomial $T(x) = x^n P_1(1/x) = P(m_0)x^n + a''_{n-1}x^{n-1} + \cdots + a''_1 x + a_n$ has roots $\gamma_j = 1/\beta_j = 1/(\alpha_j - m_0)$, and satisfies

$$|D(T)| = P(m_0)^{2n-2}\Big|\prod_{1\le i<j\le n}(\beta_i-\beta_j)^2\beta_i^{-2}\beta_j^{-2}\Big|.$$

But $|\prod_{1\le i<j\le n}\beta_i^{-2}\beta_j^{-2}| = (P(m_0)a_n^{-1})^{2n-2}$, therefore $|D(T)| = |D(P)|$. The condition $H(T) < c(n)H$ is obviously satisfied because $H(T) = H(P_1)$ and $H(P_1) \asymp H(P)$.

LEMMA 7 (see [1]). *Let $P \in \mathcal{P}'_n(H)$. Then*

$$|P^{(l)}(\alpha_1)| \ll H^{1-p_l}, \quad 1 \le l \le n-1.$$

LEMMA 8 (see [2]). *The measure of the set of those x such that the inequality*

$$|P(x)| < H^{-w}$$

for $w > n-1$ and $H > H_0$ has infinitely many solutions in reducible polynomials $P(x)$, tends to zero as $H_0 \to \infty$.

4. Proof of Theorem 1 modulo Theorem 2. We shall show how Theorem 1 follows from Theorem 2.

Suppose $P(x) \in \mathbb{Z}[x]$, $\deg P \le n$, $|a_n| > cH$.

Lemma 6 shows that the last inequality does not impose a restriction: if $|a_n| \le cH$ then the polynomial can be transformed into a polynomial with a large highest coefficient without changing the value of its discriminant.

Using Dirichlet's principle we shall prove that for any $x \in I \subset \mathbb{R}$ and $Q > 1$ there are two real positive numbers c_3 and c_4 with $\min(c_3, c_4) \le 1$ and $c_3c_4 > 8n$ such that the system of inequalities

$$\begin{cases} |P(x)| < c_3Q^{-n+v}, \\ |P'(x)| < c_4Q^{1-v}, \quad H(P) \le Q, \end{cases} \tag{7}$$

holds for some polynomials $P \in \mathcal{P}_n(Q)$.

Let $c_3 = 1$, $c_4 = 8n$. Then system (7) may be rewritten as

$$\begin{cases} |P(x)| < Q^{-n+v}, \\ |P'(x)| < 8nQ^{1-v}. \end{cases} \tag{8}$$

The existence of solutions to (8) and Theorem 2 implies that for $\gamma = n^{-1}2^{-n-12}$ the system of inequalities

$$\begin{cases} \gamma Q^{-n+v} < |P(x)| < Q^{-n+v}, \\ \gamma Q^{1-v} < |P'(x)| < 8nQ^{1-v}, \end{cases} \tag{9}$$

has solutions in $P \in \mathcal{P}_n(Q)$ for all $x \in B_1$ with $\mu B_1 \ge |I|/2$. Indeed, if one of the inequalities in (9) does not hold then $|P(x)| \le \gamma Q^{-n+v}$, $|P'(x)| < 8nQ^{-1-v}$ and $8n\gamma < 2^{-n-9}$. If $|P'(x)| < \gamma Q^{1-v}$, $|P(x)| \le Q^{-n+v}$ then $c_1c_2 < n^{-1}2^{-n-12}$. The claim reduces to the fact that the system of inequalities does not hold on the set B with measure $\mu B < |I|/2$ and holds for all $x \in B_1 = I \setminus B$ with $\mu B_1 \ge |I|/2$.

Let us choose $x_1 \in B_1$. Then we can find a polynomial $P_1(x)$ for which system (9) holds for $x = x_1$. For all x in the interval $|x - x_1| < Q^{-2/3}$, the

Mean Value Theorem gives

$$(10)\qquad P_1'(x) = P_1'(x_1) + P_1''(\xi_1)(x - x_1) \quad \text{for some } \xi_1 \in [x, x_1].$$

The obvious estimate $|P''(\xi_2)| < n^3 Q$ implies $|P''(\xi_1)(x - x_1)| < n^3 Q^{1/3}$. But $|P_1'(x_1)| \gg Q^{1/2}$ for $v \le 1/2$ and therefore for sufficiently large Q it follows from (10) and the second inequality in (9) that

$$\tfrac{1}{2}\gamma Q^{1-v} < \tfrac{1}{2}|P_1'(x_1)| < |P_1'(x)| < 2|P_1'(x_1)| < 16nQ^{1-v}.$$

In view of the values of $P(x_1)$ and $P'(x_1)$ given by (9) we can distinguish four possible combinations of signs. We will consider the case when $P_1(x_1) < 0$ and $P_1'(x_1) > 0$. The remaining ones can be dealt with in a similar way. Again we use the Mean Value Theorem:

$$(11)\qquad P_1(x) = P_1(x_1) + P_1'(\xi_2)(x - x_1) \quad \text{for some } \xi_2 \in [x_1, x].$$

Write $x = x_1 + \Delta$ and suppose that $\Delta > 2\gamma^{-1}Q^{-n-1+2v}$. If $P_1(x_1) < P_1(x_1 + \Delta) < 0$ then the first inequality of (9) implies

$$0 < P_1(x_1 + \Delta) - P_1(x_1) < Q^{-n+v}.$$

On the other hand, we have

$$|P'(\xi_2)\Delta| > \tfrac{1}{2}\gamma Q^{1-v} 2\gamma^{-1} Q^{-n-1+2v} = Q^{-n+v}.$$

We thus obtain a contradiction to (11). This means that $P_1(x_1 + \Delta) > 0$ and there is a real root α of $P_1(x)$ between x_1 and $x_1 + \Delta$.

At the same time

$$(12)\qquad |x_1 - \alpha| < 2\gamma^{-1}Q^{-n-1+2v} = n2^{n+13}Q^{-n-1+2v}.$$

Now we shall obtain a lower bound for $|x_1 - \alpha|$. Again we consider only one of four possibilities, $P_1(x_1) > 0$, $P_1'(x_1) < 0$. At $x = x_1 + \Delta_1$ we have

$$(13)\qquad P_1(x) = P_1(x_1) + P_1'(\xi_3)\Delta_1 \quad \text{for some } \xi_3 \in [x_1, x].$$

If $\Delta_1 < 2^{-4}n^{-1}\gamma Q^{-n-1+2v}$ then $|P_1(x_1)| > \gamma Q^{-n+v}$ and $|P'(\xi_3)\Delta_1| < \gamma Q^{-n+v}$. Then (13) implies that $P_1(x)$ cannot have any root in $[x_1, x_1 + \Delta_1]$ and therefore for any root α, we have

$$n^{-1}2^{-n-13}Q^{-n-1+2v} < |x - \alpha|.$$

Let α be the root of $P_1(x)$ closest to x_1. Using the representation

$$P_1'(\alpha) = P_1'(x_1) + P_1''(\xi_4)(x_1 - \alpha) \quad \text{for some } \xi_4 \in [x, \alpha],$$

the estimate $|P_1''(\xi)| < n^3 Q$ and (12) for sufficiently large Q we get

$$n^{-1}2^{-n-13}Q^{1-v} < |P_1'(\alpha)| < 16nQ^{1-v}.$$

The square of derivative is a factor of the discriminant of P. Taking into account that for $|a_n| \asymp H(P)$ all roots of the polynomial are bounded (see

Lemma 3) we can estimate the differences $|\alpha_i - \alpha_j|$, $2 \le i < j \le n$, by a constant $c(n)$. Thus, for $x_1 \in B_1$ we can construct a polynomial $P_1(x)$ with

$$|D(P_1)| \ll Q^{2n-2-2v}.$$

Define $x_{01} = \inf\{x : x \in I \cap B_1\}$. Clearly $x_1 \in B_1$ can be taken from the interval $J_1 = [x_{01}, x_{01} + Q^{-n-1}]$. Set $J_1' = [x_{01}, x_{01} + Q^{-n-1} + 4\gamma^{-1}Q^{-n-1+2v}]$ and $x_{02} = \inf\{x : x \in (I \setminus J_1') \cap B_1\}$. Choose $x_2 \in J_2 = [x_{02}, x_{02} + Q^{-n-1}] \cap B_1$. By construction, we have

$$|x_2 - x_1| > 4\gamma^{-1}Q^{-n-1+2v}. \tag{14}$$

For this point we can construct a polynomial $P_2(x)$ again satisfying the system of inequalities (9) at x_2. We will show that $P_2(x) \ne P_1(x)$. Consider the value of the polynomial $P_1(x)$ at $x = x_2$. Then

$$P_1(x_2) = P_1(x_1) + P_1'(\xi_5)(x_2 - x_1) \quad \text{for some } \xi_5 \in [x_1, x_2].$$

Using $|P_1(x_1)| < Q^{-n+v}$, $|P_1'(\xi_5)| > (\gamma/2)Q^{1-v}$ and (14) we obtain

$$|P_1(x_2)| > Q^{-n+v},$$

so P_1 does not satisfy the first inequality of (9). Thus, $P_2(x)$ is different from $P_1(x)$ at x_2. The discriminant $D(P_2)$ also satisfies (5). Moreover, for a point $x_3 \in B_1$ with $x_3 - x_2 > 4\gamma^{-1}Q^{-n-1+2v}$, we construct a polynomial $P_3(x)$ different from $P_1(x)$ and $P_2(x)$ that satisfies conditions (5) and (9). It is clear that repeating the described procedure we can construct $c(n)Q^{n+1-2v}$ polynomials $P(x)$ with discriminants satisfying (5).

5. Proof of Theorem 2. We start by estimating the measure of the set of those x such that the system

$$\begin{cases} |P(x)| < c_1 Q^{-n+v}, \\ Q^{1-v_1} < |P'(x)| < c_2 Q^{1-v} \end{cases} \tag{15}$$

is solvable in P, where v_1 with $v < v_1 < 1$ will be specified later.

We shall show that $P'(x)$ in the second inequality of (15) can be replaced by $P'(\alpha)$, where α denotes the root of P nearest to x. Using the Mean Value Theorem gives

$$P'(x) = P'(\alpha) + P''(\xi_1)(x - \alpha) \quad \text{for some } \xi_1 \in (\alpha, x).$$

By Lemma 2,

$$|x - \alpha| < n\,\frac{|P(x)|}{|P'(x)|}.$$

Then

$$|P'(\alpha)| = |P'(x) - P''(\xi_1)(x - \alpha)|.$$

As

$$|P''(\xi_1)(x - \alpha)| \le n^3 Q c_1 n Q^{-n-1+v+v_1} = c_1 n^4 Q^{-n+v+v_1}$$

for sufficiently large Q we obtain

$$\tfrac{3}{4}Q^{1-v_1} \le \tfrac{3}{4}|P'(x)| \le |P'(\alpha)| \le \tfrac{4}{3}|P'(x)| \le \tfrac{4}{3}c_2Q^{1-v}$$

and

$$\tfrac{3}{4}|P'(\alpha)| \le |P'(x)| \le \tfrac{4}{3}|P'(\alpha)|.$$

Therefore for sufficiently large Q, (15) implies

$$\begin{cases} |P(x)| < c_1Q^{-n+v}, \\ \tfrac{3}{4}Q^{1-v_1} < |P'(\alpha)| < \tfrac{4}{3}c_2Q^{1-v}, \\ |a_j| \le Q. \end{cases} \tag{16}$$

Let $\mathcal{L}'_n(v)$ denote the set of x for which the system (16) is solvable in P. Now we will prove that $\mu\mathcal{L}'_n(v) < \frac{3}{8}|I|$.

Consider the intervals

$$\sigma_1(P): \quad |x-\alpha| < \tfrac{4}{3}c_1nQ^{-n+v}|P'(\alpha)|^{-1}$$

and

$$\sigma_2(P): \quad |x-\alpha| < c_5Q^{-1+v}|P'(\alpha)|^{-1}.$$

The value of c_5 will be specified below. Obviously

$$|\sigma_1(P)| \le \tfrac{4}{3}c_1c_5^{-1}nQ^{-n+1}|\sigma_2(P)|. \tag{17}$$

Fix the vector $\overline{b} = (a_n,\dots,a_2)$ of coefficients of $P(x)$. The polynomials $P \in \mathcal{P}_n(Q)$ with the same vector $\overline{b}$ form a subclass of $\mathcal{P}_n(Q)$ which will be denoted by $\mathcal{P}(\overline{b})$.

The interval $\sigma_2(P_1)$ with $P_1 \in \mathcal{P}(\overline{b})$ is called *inessential* if there is another interval $\sigma_2(P_2)$ with $P_2 \in \mathcal{P}(\overline{b})$ such that

$$|\sigma_2(P_1) \cap \sigma_2(P_2)| \ge 0.5|\sigma_2(P_1)|.$$

Otherwise for any $P_2 \in \mathcal{P}(\overline{b})$ different from P_1,

$$|\sigma_2(P_1) \cap \sigma_2(P_2)| < 0.5|\sigma_2(P_1)|$$

and the interval $\sigma_2(P_2)$ is called *essential.*

The case of essential intervals. In this case every point $x \in I$ belongs to no more than two essential intervals $\sigma_2(P)$. Hence for any vector $\overline{b}$,

$$\sum_{P\in\mathcal{P}_1(\overline{b})} |\sigma_2(P)| \le 2|I|. \tag{18}$$

We have to sum over the lengths of the essential intervals $\sigma_1(P)$ inside the class $\mathcal{P}(\overline{b})$ with fixed vector $\overline{b}$, and then over all classes. We can estimate the number of classes as the number of all possible vectors $\overline{b}$,

$$(2Q+1)^{n-1} = (2Q)^{n-1}\left(1+\frac{1}{2Q}\right)^{n-1} \le 2^{n-1}Q^{n-1}e^{(n-1)/2Q} < 2^nQ^{n-1}.$$

From (17) and (18) we obtain

$$\sum_{\bar{b},\,|a_j|\le Q}\ \sum_{P\in\mathcal{P}(\bar{b})} |\sigma_1(P)| < \tfrac{4}{3}c_1c_5^{-1}nQ^{-n+1}2|I|2^nQ^{n-1} = n2^{n+2}c_1c_5^{-1}.$$

Thus for $c_5 = n2^{n+5}c_1$ the measure will be no larger than $|I|/8$.

The case of inessential intervals. Let us estimate the values of $|P_j(x)|$, $j=1,2$, on the intersection $\sigma_2(P_1,P_2)$ of the intervals $\sigma_2(P_1)$ and $\sigma_2(P_2)$. By the Mean Value Theorem,

$$P_j(x) = P_j'(\alpha)(x-\alpha) + \tfrac{1}{2}P_j''(\xi_2)(x-\alpha)^2 \quad \text{for some } \xi_2 \in (\alpha, x),$$

and

$$P_j'(x) = P_j'(\alpha) + P_j''(\xi_3)(x-\alpha) \quad \text{for some } \xi_3 \in (\alpha, x).$$

The second summand is estimated by

$$|P''(\xi_2)(x-\alpha)^2| \le 2n^3c_5^2Q^{-3+2v+2v_1},$$

while

$$|P'(\alpha)(x-\alpha)| < c_5Q^{-1+v}.$$

As $2v_1 < 2-v$ for an appropriate choice of $v_1 < 3/4$ we obtain

$$|P_j(x)| \le \tfrac{4}{3}c_5Q^{-1+v}, \quad j=1,2. \tag{19}$$

Similarly we obtain the following estimates:

$$|P_j'(\alpha)| < \tfrac{4}{3}c_2Q^{1-v}, \quad |P_j''(\xi_3)(x-\alpha)| < 2n^3c_5Q^{-1+v+v_1}$$

and for $v_1 \le 1$,

$$|P_j'(x)| \le \tfrac{4}{3}c_2Q^{1-v}, \quad j=1,2. \tag{20}$$

Define $K(x) = P_2(x) - P_1(x)$. Obviously $K(x)$ is not identically zero and has the form $K(x) = b_1x + b_0$. Moreover, (19) and (20) imply

$$|b_1x + b_0| < \tfrac{8}{3}c_5Q^{-1+v} \tag{21}$$

and

$$|b_1| = |K'(x)| < \tfrac{8}{3}c_2Q^{1-v}.$$

For fixed b_0 and b_1 the measure of those $x \in I$ that satisfy (21) does not exceed $\frac{16}{3}c_5Q^{-\lambda}b_1^{-1}$. Given that $x \in I$ and (21) is satisfied we find that b_0 can have not more than $|I|\,|b_1| + 2$ values. Summing over all b_0 we obtain an estimate for the measure when b_1 is fixed,

$$\tfrac{16}{3}c_5Q^{-1+v}b_1^{-1}(|I|\,|b| + 2) < 6c_5Q^{-1+v}. \tag{22}$$

After summing (22) over all $|b_1|$ we have

$$2^5c_2c_5Q^{1-v-\lambda}|I| = n2^{n+8}c_1c_2|I| = \tfrac{1}{8}|I|.$$

From $c_1c_2 < n^{-1}2^{-n-11}$ we can estimate the total measure for both essential and inessential intervals by $|I|/4$.

Now we consider the remaining case. Our task is to estimate the measure of $\mathcal{L}_n''(v)$, the set of all x such that the system

(23) $$\begin{cases} |P(x)| < Q^{-n+v}, \\ |P'(x)| < Q^{1-v_1}, \\ |a_j| \le Q, \end{cases}$$

is solvable in $P \in \mathcal{P}_n(Q)$.

To prove Theorem 2 it remains to show that

$$\mu\mathcal{L}''_n(v) \ll \tfrac{1}{4}|I|.$$

The proof splits into the following cases:

1. $l_2T^{-1} + p_1 \ge n + 1 - v$,
2. $n + 0.1 \le l_2T^{-1} + p_1 < n + 1 - v$,
3. $7/4 \le l_2T^{-1} + p_1 < n + 0.1$,
4. $l_2T^{-1} + p_1 < 7/4$.

Case 1:

(24) $$l_2T^{-1} + p_1 \ge n + 1 - v.$$

Consider the class $\mathcal{P}_t(\bar{s}) = \bigcup_{2^t \le H < 2^{t+1}} \mathcal{P}_n(H, \bar{s})$. Since Q is a sufficiently large number and $H \le Q$, we have $t_0 < t \ll \log Q$. We are going to compare two estimates for $|x - \alpha_1|$ obtained from (23) and Lemma 1 for $x \in S(\alpha_1)$,

(25) $$|x - \alpha_1| \le 2^n \frac{|P(x)|}{|P'(\alpha_1)|} \ll 2^{t(-n+v-1+p_1+(n-1)\varepsilon)}$$

and

(26) $$|x - \alpha_1| \le \left(2^{n-1} \frac{|P(x)|\,|\alpha_1 - \alpha_2|}{|P'(\alpha_1)|}\right)^{1/2} \ll 2^{t(-n+v-1+p_2+(n-2)\varepsilon)/2}.$$

In the case (24) we use the estimate (26). Let us divide the interval I into smaller parts I_j with $\mu I_j = 2^{-t((n+1-v-p_2)/2-\gamma)}$, where γ is a positive constant.

For an integral polynomial $P(x)$ and an interval I_j we shall write "$P(x)$ belongs to I_j" or "I_j contains $P(x)$" if there is a point $x \in I_j$ that satisfies the system (23). Let $\sigma(P)$ denote the measure of $x \in S(\alpha_1)$ satisfying (23).

(a) Assume that there is at most one polynomial $P \in \mathcal{P}_t(\bar{s})$ that belongs to every I_j. Then for every polynomial the measure of the set of those x that satisfy (23) does not exceed $c(n)2^{-t(n+1-v-p_2-(n-2)\varepsilon)/2}$ and the number of I_j is less than $2^{t((n+1-v-p_2)/2-\gamma)}|I|$. Therefore

(27) $$\sum_{P \in \mathcal{P}_t(\bar{s})} \sigma(P) \ll \sum_{P \in \mathcal{P}_t(\bar{s})} 2^{t((n+1-v-p_2)/2-\gamma)}|I| \cdot c(n)2^{-t(n+1-v-p_2-(n-2)\varepsilon)/2} \ll 2^{-t\gamma_1},$$

where $\gamma_1 = \gamma - (n-2)\varepsilon/2$.

The sum (27) extends over all $t \ge t_0$. Since $\sum_{t>t_0} 2^{-t\gamma_1} \ll 2^{-t_0\gamma_1}$, for sufficiently large t_0 the measure of the set of those x such that the system (23) holds and polynomials $P(x)$ satisfy Case 1(a) does not exceed $|I|/32$.

(b) Suppose the contrary, that there are intervals I_j that contain at least two polynomials, i.e. we can find polynomials P_1 and P_2 from the class $\mathcal{P}_t(\bar{s})$, and points x_1 and x_2 from I_j, that satisfy the system of inequalities

$$\begin{cases} |P_1(x_1)| \ll 2^{t(-n+v)}, \\ |P_1'(x_1)| \ll 2^{t(1-v_1)}, \end{cases} \qquad \begin{cases} |P_2(x_2)| \ll 2^{t(-n+v)}, \\ |P_2'(x_2)| \ll 2^{t(1-v_1)}. \end{cases}$$

Let us estimate the value of $P_1(x)$ and $P_2(x)$ at points of the interval I_j. Using Taylor's expansion for $P_i(x)$ at α_1,

$$P_i(x) = \sum_{j=1}^{n} \frac{P_i^{(j)}(\alpha_1)(x-\alpha_1)^j}{j!},$$

and estimates $|P^j(\alpha_1)|$ from Lemmas 1 and 7 we get

$$|P_j(x)| \ll 2^{t(1-p_j+j((-n+v-1+p_j+(n-j)\varepsilon)/j+\gamma))} \ll 2^{t(-n+v+n\gamma_1)}.$$

Now for polynomials P_1 and P_2 without common roots we can apply Lemma 5.

Since we have $\tau = n - v - n\gamma_1$, $\varsigma = 1$, $\eta = (n+1-v-p_2)/2 - \gamma$, it follows that

$$n - v - n\gamma_1 + 1 + 2\left(n - v - n\gamma_1 + 1 - \frac{n+1-v-p_2}{2} + \gamma\right) < 2n + \delta.$$

Hence

$$2 - 2v < \delta + (3n-2)\gamma_1.$$

The latter leads to a contradiction for $v \le 1/2$ and sufficiently small γ, ε and δ.

Case 2:

$$n + 0.1 \le l_2 T^{-1} + p_1 < n + 1 - v. \tag{28}$$

Let us divide the interval I into intervals I_j, where $|I_j| = 2^{t(-l_2/T+\gamma)}$.

(a) Assume that no more than one polynomial $P \in \mathcal{P}_t(\bar{s})$ belongs to every I_j. We use inequality (25). For every polynomial the measure of the set of x's satisfying (23) does not exceed $c(n)2^{-t(n+1-v-p_1-(n-1)\varepsilon)}$. Further, the number of I_j is less than $2^{t(n+1-v-p_1-\gamma)}|I|$. Therefore

$$\sum_{P\in\mathcal{P}_t(\bar{s})} \sigma(P) \ll \sum_{P\in\mathcal{P}_t(\bar{s})} 2^{t(l_2/T-\gamma)} \cdot 2^{-t(n+1-v-p_1-(n-1)\varepsilon)} \ll 2^{-t\gamma_2}, \tag{29}$$

where $\gamma_2 = \gamma - (n-1)\varepsilon$. Again we sum the estimate (29) over all $t > t_0$ as in formula (27). It is clear that the total sum is less than $|I|/32$.

(b) Assuming, as in Case 1 above, the existence of an interval I_j that contains at least two different polynomials $P_1(x)$ and $P_2(x)$, for any $x \in I_j$ by Taylor's expansion we get

$$|P_i(x)| \ll 2^{-t(l_2T^{-1}+p_1-1-2\gamma)}, \qquad i = 1,2.$$

For $P_1(x)$ and $P_2(x)$ which have no common roots, on I_j we may apply Lemma 5 with $\varsigma = 1$, $\eta = l_2T^{-1} - \gamma$, $\tau + 1 = l_2T^{-1} + p_1 - 2\gamma$. Note that $l_2T^{-1} \le p_1$. Then

$$l_2T^{-1} + 3p_1 - 4\gamma < 2n + \delta.$$

This together with (28) implies the inequalities

$$2n + \tfrac{1}{5} - 4\gamma \le l_2T^{-1} + 3p_1 - 4\gamma < 2n - \delta,$$

so

$$\tfrac{1}{5} < \delta + 4\gamma,$$

which are contradictory for small δ and γ.

Case 3:

$$\tfrac{7}{4} \le l_2T^{-1} + p_1 < n + \tfrac{1}{10}. \tag{30}$$

This case represents the largest interval for $l_2T^{-1} + p_1$ and is the most difficult. We divide I into intervals I_j of length $2^{-tl_2T^{-1}}$. First let us estimate the value of a polynomial $P \in \mathcal{P}_n$ and its derivative on I_j. For this purpose expand by Taylor's formula, in the neighborhood of α_1,

$$P(x) = \sum_{j=1}^{n} \frac{P^{(j)}(\alpha_1)(x-\alpha_1)^j}{j!},$$

$$|P'(\alpha_1)(x-\alpha_1)| \ll 2^{t(1-p_1-l_2T^{-1})},$$

$$|P''(\alpha_1)(x-\alpha_1)^2| \ll 2^{t(1-p_2-2l_2T^{-1})} \ll 2^{t(1-p_1-l_2T^{-1})},$$

$$|P^{(i)}(\alpha_1)(x-\alpha_1)^i| \ll 2^{t(1-p_i-il_2T^{-1})} \ll 2^{t(1-p_1-l_2T^{-1})}, \qquad 3 \le i \le n.$$

Similarly we treat the derivative:

$$P'(x) = \sum_{j=0}^{n-1} \frac{P^{(j+1)}(\alpha_1)(x-\alpha_1)^j}{j!},$$

$$|P'(\alpha_1)| \asymp 2^{t(1-p_1)},$$

$$|P^{(i)}(\alpha_1)(x-\alpha_1)^{i-1}| \ll 2^{t(1-p_i-(i-1)l_2T^{-1})} \ll 2^{t(1-p_1)}, \qquad 2 \le i \le n.$$

Thus, if the polynomial $P(x)$ belongs to the interval I_j it should satisfy the system

$$\begin{cases} |P(x)| \ll 2^{t(1-p_1-l_2t^{-1})}, \\ |P'(x)| \asymp 2^{t(1-p_1)}. \end{cases} \tag{31}$$

Consider those intervals that contain $c(n)2^{t\varrho}$ polynomials. Then the measure of the set of $x \in I$ that satisfy (23) is

$$2^{t(-n+v-1+p_1+(n-1)\varepsilon)}c(n)2^{t\varrho}2^{tl_2T^{-1}}.$$

If $\varrho < n+1-v-(p_1+l_2T^{-1})$ and $t > t_0$ the measure can be estimated by $|I|/32$.

To simplify subsequent computations we introduce

$$u := n+1-v-p_1-l_2T^{-1}.$$

It follows from (30) and $v \le 1/2$ that $u \ge 2/5$. Let $u_1 = u - 1/5 \ge 1/5$ and represent u_1 as the sum $u_1 = [u_1] + \{u_1\}$.

Let $n+1-v-p_1-\varrho-l_2T^{-1} \le 0$, i.e. $\varrho \ge u$. By Dirichlet's principle, there exist at least $c(n)2^{t(\{u_1\}+0.2)}$ polynomials $P_1(x), \dots, P_k(x)$, where $k \gg 2^{t(\{u_1\}+0.2)}$ such that the first $[u_1]$ coefficients are identical.

Consider the polynomials $R_j(x) = P_{j+1}(x) - P_1(x)$, which obviously satisfy

$$\deg R_j(x) \le n - [u_1], \qquad H(R_j) \ll 2^t.$$

From (31) we get

$$\begin{cases} |R_j(x)| \ll 2^{t(1-p_1-l_2T^{-1})}, & j = 1, \dots, k, \\ |R_j'(x)| \ll 2^{t(1-p_1)}. \end{cases} \tag{32}$$

Every coefficient of the polynomial R_j ranges in the interval $[-2^{t+1}, 2^{t+1}]$. We divide all intervals into equal parts of length $2^{t\vartheta}$, where $\vartheta = 1 - \{u_1\}/(n-[u_1])$.

Then at least $c(n)2^{t/5}$ polynomials fall into the same intervals. Hence the height of their differences $R_j(x)$ will be less than

$$c(n)2^{t\vartheta} = c(n)2^{t(1-\{u_1\}/(n-[u_1]))}.$$

Define $S_j(x) = R_{j+1}(x) - R_1(x)$ and rewrite (32) as follows:

$$\begin{cases} |S_j(x)| \ll 2^{t(1-p_1-l_2\cdot T^{-1})}, & j = 1, \dots, k-1, \\ |S_j'(x)| \ll 2^{t(1-p_1)}, & j = 1, \dots, k-1, \\ \deg S_j \le n - [u_1], \\ H(S_i) < 2^{t(1-\{u_1\}/(n-[u_1]-1))}. \end{cases} \tag{33}$$

(a) Suppose there are coprime polynomials of type $S_i(x)$. Then applying Lemma 5 with $I = I_j$ and

$$\tau = p_1 + l_2T^{-1} - 1, \qquad \vartheta = 1 - \frac{\{u_1\}}{n-[u_1]},$$

$$\max\{\deg S_1, \deg S_2\} \le \deg S = n - [u_1], \qquad \eta = l_2T^{-1},$$

we get

$$p_1 + l_2T^{-1} - \frac{\{u_1\}}{n-[u_1]} + 2\Big(p_1 + l_2T^{-1} - \frac{\{u_1\}}{n-[u_1]} - l_2T^{-1}\Big) \le 2(n-[u_1])\Big(1 - \frac{\{u_1\}}{n-[u_1]}\Big) + \delta.$$

This implies

$$3p_1 - l_2T^{-1} - \frac{3\{u_1\}}{n-[u_1]} \le 2p_1 + 2l_2T^{-1} + 2v + 0.4 - 2 + \delta.$$

Replacing p_1 by l_2T^{-1} and representing $n-[u_1]-1$ as $p_1 + l_2T^{-1} + 0.2 + \{u_1\} - 1$ we obtain

$$\frac{8}{5} - 2v < \frac{3\{u_1\}}{p_1 + l_2T^{-1} + v + 0.2 + \{u_1\} - 1} + \delta. \tag{34}$$

By writing the right hand side of the inequality as a function of $\{u_1\}$ and v in $[0;1)\times[0;1/4)$, we show that it does not exceed $0.4 + 3/(p_1 + l_2T^{-1} + 0.2) + \delta$, but our assumption $p_1 + l_2T^{-1} > 1$ leads to a contradiction to (34) for small enough δ.

(b) If all the polynomials $S_j(x)$ are of the type $lS_0(x)$ then $|2^{0.4t}S_0(x)| \ll 2^{t(1-p_1-l_2T^{-1})}$ and

$$|S_0(x)| \ll H(S_0)^{\frac{1-p_1-l_2T^{-1}-0.2}{1-\{u_1\}/(n-[u_1])-0.2}}.$$

If the inequality

$$p_1 + l_2T^{-1} + 0.2 - 1 > (n-[u_1])\Big(1 - \frac{\{u_1\}}{n-[u_1]} - 0.2\Big) \tag{35}$$

is false then Sprindžuk's theorem [7] implies that in Case 2(b) we can estimate the measure by $|I|/32$.

Since the product on the right side of (35) is $p_1 + l_2T^{-1} - 0.8$, the inequality (35) leads to a contradiction.

(c) If there is a reducible polynomial $S_i(x)$ decomposing into the product $S_i(x) = T_1(x)T_2(x)$ then system (33) implies that

$$\begin{cases} |T_1(x)| \ll H(T_1)^{(1-p_1-l_2T^{-1})(1-\{u_1\}/(n-[u_1]))^{-1}}, \\ \deg T \le n - [u_1] - 1. \end{cases}$$

To apply Sprindžuk's theorem we verify the inequality

$$p_1 + l_2T^{-1} - 1 > (n-[u_1]-1)\Big(1 - \frac{\{u_1\}}{n-[u_1]}\Big). \tag{36}$$

To this end rewrite the right hand side of (36) as

$$p_1 + l_2T^{-1} + 0.2 - 2 + \frac{1}{p_1 + l_2T^{-1} + 0.2}.$$

Determining the maximum of this expression as the function of $\{u_1\}$ we find that the right side does not exceed

$$p_1 + l_2T^{-1} + d_2 - 2 + \frac{1}{p_1 + l_2T^{-1} + 0.2}.$$

Thus again the relevant measure in case (c) does not exceed $|I|/32$.

CASE 4:

$$l_2T^{-1} + p_1 < 7/4.$$

Let us estimate the expression $l_2T^{-1} + p_1$ from below. To do this we have to prove that $|P'(x)| \asymp 2^{t(1-p_1)}$. By Taylor's formula for $P'(x)$ at α, we get

$$P'(x) = \sum_{j=0}^{n-1} \frac{P^{(j+1)}(\alpha)(x-\alpha)^j}{j!}.$$

Clearly $|P'(\alpha)| \asymp 2^{t(1-p_1)}$. The remaining terms of the sum satisfy

$$|P^{(i)}(\alpha)(x-\alpha)^{i-1}| \ll 2^{t(1-p_1)}, \qquad 2 \le i \le n.$$

Since $|P'(x)| \ll 2^{t/3}$, we have $1 - p_1 \le 1/3$ or equivalently $2/3 \le p_1$. Thus we need to consider the system

$$(37) \qquad \begin{cases} |P(x)| < 2^{t(-n+v)}, \\ |P'(x)| < 2^{t/3}, \\ 2/3 < l_2T^{-1} + p_1 < 7/4. \end{cases}$$

All solutions of (37) with α_1 being the closest root to x are contained in the interval

$$(38) \qquad \sigma(P) = \{x \in I : |x - \alpha_1| < 2^{t(-n+v)}|P'(\alpha_1)|^{-1}\}.$$

Apart from $\sigma(P)$ we also consider the following interval $\sigma_1(P)$, which contains $\sigma(P)$:

$$(39) \qquad \sigma_1(P) = \{x \in I : |x - \alpha_1| < 2^{t(v-0.9)}|P'(\alpha_1)|^{-1}\}.$$

From (38) and (39) we get

$$\mu\sigma(P) \ll 2^{t(-n+v+1-v)}\mu\sigma_1(P) = 2^{t(-n+0.9)}\mu\sigma_1(P).$$

Divide all polynomials in $\mathcal{P}_n$ into classes $\mathcal{P}_{\bar{b}}$ according to the $n-1$ first coefficients $\bar{b} = (a_n, a_{n-1}, \ldots, a_2)$. Obviously $\#\bar{b} \asymp 2^{t(n-1)}$.

(a) If $\mu\sigma_1(P_1) \cap \mu\sigma_1(P_2) < \frac{1}{2}\mu\sigma_1(P_1)$ then $\sum_{P\in\mathcal{P}_{\bar{b}}} \mu\sigma_1(P) \ll |I|$. Summing over all classes we obtain

$$\sum_{\bar{b}} \sum_{P\in\mathcal{P}_{\bar{b}}} \mu\sigma(P) \le 2^n 2^{t(n-1)} n 2^{t(-n+0.9)} 2|I| \le n2^{n+1}2^{-0.1t}|I|.$$

(b) If $\mu\sigma_1(P_1) \cap \mu\sigma_1(P_2) \ge \frac{1}{2}\mu\sigma_1(P_1)$ we denote $R(x) = P_1(x) - P_2(x)$. Since P_1 and P_2 belong to the same class $\mathcal{P}_{\bar{b}}$, $R(x)$ is of the type $ax + b$.

Moreover, taking into account the estimates of the polynomials and their derivatives we obtain

$$\begin{cases} |ax+b| \ll 2^{t(-0.9+v)}, \\ |a| \ll 2^{t(1-p_1)}, \end{cases}$$

so

$$\left| x + \frac{b}{a} \right| \ll 2^{t(-0.9+v)} |a|^{-1}. \tag{40}$$

It is clear that inequality (40) holds for the whole essential interval. Summing estimates (40) first over all b which do not exceed $c(n)|a|\,|I|$, and then over all a, we obtain $c(n)2^{t(-0.9+v+1-p_1)}|I| = c(n)2^{t(v-p_1+0.1)}|I| \ll 2^{-0.1t}|I|$. Let us sum the estimates of cases (a) and (b) over all $t > t_0$. We deduce that in Case 4 the measure of the set of those x that satisfy (23) does not exceed $|I|/32$. Altogether for Cases 1–4 the measure of the set $\mathcal{L}''_n(v)$ does not exceed $|I|/4$, thus proving the theorem.

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