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Solving an exponential Diophantine equation

by

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Introduction. Let $k > 1$, $L > 0$, $k, L \in \mathbb{Z}$, and let $p_1 < \cdots < p_L$ be rational primes. We are going to show that if $(k, L, p_1, \ldots, p_L)$ satisfies

$$k \prod_{i=1}^{L} p_i^{k-1} = \prod_{i=1}^{L} \frac{p_i^k - 1}{p_i - 1}, \tag{1}$$

then $k = 2$, $L = 2$, $p_1 = 2$ and $p_2 = 3$.

If we look carefully, the equation (1) with $k = 2$ is a special case of the perfect numbers problem. In this paper, we solve (1) with elementary and analytic methods. The author was motivated by similar problems which appear in [G, Ch. B].

Preliminaries. We denote by $\omega(n)$ the number of different primes which divide n, and by $\pi(n)$ the number of primes which are at most n.

First note that if $(k, L, p_1, \ldots, p_L)$ solves (1) we have $L \neq 1$, because $\left(p_1, \frac{p_1^k - 1}{p_1 - 1}\right) = 1$.

PROPOSITION 1. *If* $(k, L, p_1, \ldots, p_L)$ *is a solution of* (1) *and* $2 \mid k$ *then* $k = 2$, $L = 2$, $p_1 = 2$ *and* $p_2 = 3$.

Proof. Let $k = 2^\alpha r$ with $2 \nmid r$. For all $p_i \neq 2$ we have

$$\frac{p_i^k - 1}{p_i - 1} = \frac{p_i^r - 1}{p_i - 1} \prod_{j=0}^{\alpha-1} \frac{p_i^{2^{j+1} r} - 1}{p_i^{2^j r} - 1}.$$

Also, for all $0 \le j \le \alpha - 1$ we have $2 \mid \frac{p_i^{2^{j+1} r} - 1}{p_i^{2^j r} - 1}$ so

$$\frac{p_i^k - 1}{p_i - 1} = 2^\alpha r_i \quad \text{with } r_i \in \mathbb{N}.$$

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If $p_1 > 2$ then

$$2^{\alpha} r \prod_{i=1}^{L} p_i^{k-1} = k \prod_{i=1}^{L} p_i^{k-1} = \prod_{i=1}^{L} \frac{p_i^k - 1}{p_i - 1} = 2^{\alpha L} m \quad \text{with } m \in \mathbb{N},$$

which contradicts $L \neq 1$.

Thus, we have $p_1 = 2$. Substituting $p_1 = 2$ in (1) gives

$$2^{\alpha+k-1} r \prod_{i=2}^{L} p_i^{k-1} = k \prod_{i=1}^{L} p_i^{k-1} = \prod_{i=1}^{L} \frac{p_i^k - 1}{p_i - 1} = 2^{\alpha(L-1)} \widetilde{m} \quad \text{with } \widetilde{m} \in \mathbb{N}.$$

The last equality yields

$$\alpha(L-1) \leq \alpha + k - 1. \tag{2}$$

As $(k, L, p_1, \ldots, p_L)$ is a solution of (1),

$$k = \prod_{i=1}^{L} \frac{p_i^k - 1}{p_i^{k-1}(p_i - 1)} < \prod_{i=1}^{L} \frac{p_i}{p_i - 1} \leq \prod_{i=1}^{L} \frac{i+1}{i} = L + 1. \tag{3}$$

By (2) and (3),

$$\alpha(L-1) \leq \alpha + k - 1 < \alpha + L. \tag{4}$$

Suppose $L = 2$. By (3), $k = 2$ so

$$4p_2 = \frac{2^2 - 1}{2 - 1} \frac{p_2^2 - 1}{p_2 - 1} = 3 \frac{p_2^2 - 1}{p_2 - 1}, \tag{5}$$

which gives the solution $k = 2$, $L = 2$, $p_1 = 2$ and $p_2 = 3$.

If $L = 3$, by (3), $k = 2$, and if $L \geq 4$, by (4), $\alpha < L/(L-2) \leq 2$, thus from now on we assume $\alpha = 1$.

If $L \geq 10$ we proceed as in (3):

$$k = \prod_{i=1}^{L} \frac{p_i^k - 1}{p_i^{k-1}(p_i - 1)} < \prod_{i=1}^{L} \frac{p_i}{p_i - 1} \leq \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \prod_{i=5}^{L+1} \frac{i+1}{i} = \frac{3(L+2)}{4}. \tag{6}$$

Thus, by (2) and (6), $\alpha(L-1) \leq \alpha + k - 1 < \alpha + (3L+2)/4$. In particular, $1 = \alpha < \frac{3L+2}{4(L-2)} \leq 1$, which is a contradiction.

It is sufficient to analyze those pairs which satisfy $2 < L < 10$, $k - 1 < L$, $2 \mid k$ and $4 \nmid k$. These conditions imply $k \in \{2, 6\}$.

If $k = 2$ and $L > 2$ then, as in (5), $p_2 = 3$ and $2^2 \cdot 3 = \frac{2^2-1}{2-1} \cdot \frac{3^2-1}{3-1}$. So (1) is equivalent to

$$\prod_{i=3}^{L} p_i = \prod_{i=3}^{L} \frac{p_i^2 - 1}{p_i - 1},$$

which is impossible since $p_i < \frac{p_i^2-1}{p_i-1}$.

Finaly, if $k = 6$ we have

$$2^6 \cdot 3 \prod_{i=2}^{L} p_i^5 = \frac{2^6-1}{2-1} \prod_{i=2}^{L} \frac{p_i^6-1}{p_i-1} = 3^2 \cdot 7 \prod_{i=2}^{L} \frac{p_i^6-1}{p_i-1}.$$

Thus, $p_2 = 3$ and $p_{i_1} = 7$. This yields

$$2^6 \cdot 3^6 \cdot 7^5 \prod_{\substack{i=3\\ i\neq i_1}}^{L} p_i^5 = 2^5 \cdot 3^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 43 \prod_{\substack{i=3\\ i\neq i_1}}^{L} \frac{p_i^6-1}{p_i-1}. \tag{7}$$

Hence, there exists $1 \leq i_2 \leq L$ such that $p_{i_2} = 19$ and $2^2 \mid \frac{19^6-1}{19-1}$. Consequently, by (7), 2^7 divides the right hand side but not the left hand side of (1), which is impossible. ■

Lemma 1. *If $(k, L, p_1, \ldots, p_L)$ is a solution of (1) with p prime and $k = p^\alpha$ with $\alpha > 0$ then $k = 2$, $L = 2$, $p_1 = 2$ and $p_2 = 3$.*

Proof. By Proposition 1, if $p = 2$ then $k = 2$, $L = 2$, $p_1 = 2$ and $p_2 = 3$.

Suppose $p \neq 2$. For each p_i there exists $1 \leq j \leq L$ such that $p_i \mid \frac{p_j^{p^\alpha}-1}{p_j-1}$. By Fermat's Little Theorem,

$$p_i \mid p_j^{p_i-1} - 1 \quad \text{and} \quad p_i \mid p_j^{p^\alpha} - 1. \tag{8}$$

If $p_i \mid p_j - 1$ then $p_j = gp_i + 1$ with $g \in \mathbb{N}$. So

$$p_i \mid \sum_{l=1}^{p^\alpha} \binom{p^\alpha}{l} (gp_i)^{l-1} \quad \text{implies} \quad p_i \mid p^\alpha \quad \text{and} \quad p_i = p.$$

If $p_i \nmid p_j - 1$ then, by (8), $(p^\alpha, p_i - 1) > 1$. In particular, $p \mid p_i - 1$ so $p_i = \widetilde{g}p + 1$ with $\widetilde{g} \in \mathbb{N}$. Since

$$\frac{p_i^k-1}{p_i-1} = \sum_{l=1}^{p^\alpha} \binom{p^\alpha}{l} (\widetilde{g}p)^{l-1} \quad \text{and} \quad p^\alpha \mid \sum_{l=1}^{p^\alpha} \binom{p^\alpha}{l} (\widetilde{g}p)^{l-1},$$

we conclude that

$$p^\alpha \mid \frac{p_i^k-1}{p_i-1} \quad \text{but} \quad p^{\alpha+1} \nmid \frac{p_i^k-1}{p_i-1}.$$

.

If $L = 2$ we have two possibilities:

$(*)$ $p \in \{p_1, p_2\}$:

Without loss of generality suppose $p = p_1$. Then

$$p^{\alpha+p^\alpha-1} p_2^{p^\alpha-1} = \frac{p^{p^\alpha}-1}{p-1} \frac{p_2^{p^\alpha}-1}{p_2-1}.$$

Hence, $p^{p^\alpha-1} \mid \frac{p^{p^\alpha}-1}{p-1}$, which is impossible.

(**) $p \notin \{p_1, p_2\}$:

We know that $r_i p^\alpha = \frac{p_i^k - 1}{p_i - 1}$ with $p \nmid r_i$ for $i \in \{1, 2\}$. Thus

$$p^\alpha p_1^{p^\alpha} p_2^{p^\alpha} = \frac{p_1^{p^\alpha} - 1}{p_1 - 1} \frac{p_2^{p^\alpha} - 1}{p_2 - 1} = r_1 r_2 p^{2\alpha},$$

which contradicts the assumption $p \notin \{p_1, p_2\}$.

If $L = 3$ then

$$k = \prod_{i=1}^{3} \frac{p_i^k - 1}{p_i^{k-1}(p_i - 1)} < \prod_{i=1}^{3} \frac{p_i}{p_i - 1} \le \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} = \frac{105}{48} < 3,$$

which is impossible since $2 \nmid k$.

From now on we will assume $L > 3$; then there are two primes p_{i_1}, p_{i_2} which are not p. By the work above, $p^\alpha \mid \frac{p_{i_1}^k - 1}{p_{i_1} - 1}$ and $p^\alpha \mid \frac{p_{i_2}^k - 1}{p_{i_2} - 1}$; thus in (1) $p^{2\alpha}$ divides the right hand side so there exists $1 \le i_0 \le L$ such that $p_{i_0} = p$. With this observation, (1) implies

$$p^{\alpha + p^\alpha - 1} \prod_{\substack{i=1 \\ i \ne i_0}}^{L} p_i^{p^\alpha - 1} = p^{\alpha(L-1)} \frac{p^{p^\alpha} - 1}{p - 1} \prod_{\substack{i=1 \\ i \ne i_0}}^{L} \frac{p_i^{p^\alpha} - 1}{p^\alpha (p_i - 1)} = p^{\alpha(L-1)} m$$

with $m \in \mathbb{N}$ and $p \nmid m$. Hence, $\alpha + p^\alpha - 1 = \alpha(L - 1)$. Also

$$k = \prod_{i=1}^{L} \frac{p_i^k - 1}{p_i^{k-1}(p_i - 1)} < \prod_{i=1}^{L} \frac{p_i}{p_i - 1} \le \prod_{i=2}^{L+1} \frac{i+1}{i} = \frac{L+2}{2}.$$

Thus,

$$\alpha(L - 1) = \alpha + k - 1 < \alpha + \frac{L+2}{2} - 1 = \alpha + L,$$

in particular

$$\alpha < \frac{L}{2(L-2)} \quad \text{and then} \quad \alpha < 1.$$

This yields $\alpha = 0$, which contradicts the assumption. ∎

Lemma 2. *If* $(k, L, p_1, \dots, p_L)$ *solves* (1) *for* k *with prime decomposition* $k = \prod_{i=1}^{\omega(k)} q_i^{\alpha_i}$ *where* $\alpha_i > 0$ *and* $2 < q_1 < \dots < q_{\omega(k)}$, *then* $L < \omega(k)(k+1)$.

Proof. Define $Q_i = q_i^{\alpha_i}$, $\widetilde{Q}_0 = 1$,

$$\mathcal{M}_i = \left\{ p_j : \text{there exists } p_l \text{ such that } p_j \mid \frac{p_l^{\widetilde{Q}_i} - 1}{p_l^{\widetilde{Q}_{i-1}} - 1} \right\}$$

and $\widetilde{Q}_i = \prod_{l=1}^{i} Q_l$. Note that

$$\frac{p_i^k - 1}{p_i - 1} = \prod_{i=1}^{\omega(k)} \frac{p_i^{\widetilde{Q}_i} - 1}{p_i^{\widetilde{Q}_{i-1}} - 1}.$$

Hence, there is i_0 such that $|\mathcal{M}_{i_0}| \geq L/\omega(k)$.

Note that if $p_j \mid \frac{p_l^{\widetilde{Q}_i} - 1}{p_l^{\widetilde{Q}_{i-1}} - 1}$ then we have two possibilities:

$(*)$ $p_j \mid p_l^{\widetilde{Q}_{i-1}} - 1$:

Let $w \in \mathbb{N}$ be such that $wp_j + 1 = p_l^{\widetilde{Q}_{i-1}}$. Then

$$\frac{p_l^{\widetilde{Q}_i} - 1}{p_l^{\widetilde{Q}_{i-1}} - 1} = \frac{(wp_j + 1)^{Q_i} - 1}{wp_j} = \sum_{l=1}^{Q_i} \binom{Q_i}{l} (wp_j)^{l-1},$$

thus $p_j \mid Q_i$ and $p_j = q_i$.

$(**)$ $p_j \nmid p_l^{\widetilde{Q}_{i-1}} - 1$:

By Fermat's Little Theorem, $p_j \mid (p_l^{\widetilde{Q}_{i-1}})^{p_j - 1} - 1$ and $p_j \mid (p_l^{\widetilde{Q}_{i-1}})^{Q_i} - 1$. Thus, $(p_j - 1, Q_i) > 1$ and $q_i \mid p_j - 1$. Let $p_j = vq_i + 1$ with $v \in \mathbb{N}$. Then

$$\frac{p_j^{Q_i} - 1}{p_j - 1} = \frac{(vq_i + 1)^{Q_i} - 1}{vq_i} = \sum_{l=1}^{Q_i} \binom{Q_i}{l} (vq_i)^{l-1},$$

and thereby $Q_i \mid \frac{p_j^k - 1}{p_j - 1}$.

In particular, for all $p_j \in \mathcal{M}_{i_0} \setminus \{q_{i_0}\}$ we have $Q_{i_0} \mid \frac{p_j^k - 1}{p_j - 1}$. Thus, $Q_{i_0}^{|M_{i_0}| - 1} = q_{i_0}^{\alpha_{i_0}(|M_{i_0}| - 1)}$ divides the left hand side in (1), and the right hand side is equal to

$$q_{i_0}^{\alpha_0} \frac{k}{q_{i_0}^{\alpha_{i_0}}} \prod_{\substack{i=1 \\ p_i \neq q_{i_0}}}^{L} p_i^{k-1} \qquad \text{with } \alpha_0 \in \{\alpha_{i_0}, \alpha_{i_0} + k - 1\}.$$

Moreover, if all $p_i \neq q_{i_0}$ then $\alpha_0 = \alpha_{i_0}$, and if there is j_0 such that $p_{j_0} = q_{i_0}$ then $\alpha_0 = \alpha_{i_0} + k - 1$. Consequently, we have the inequality

$$\alpha_{i_0}\left(\frac{L}{\omega(k)} - 1\right) \leq \alpha_{i_0}(|\mathcal{M}_{i_0}| - 1) \leq \alpha_0 \leq \alpha_{i_0} + k - 1.$$

Finally,

$$L \leq \omega(k)\left(\frac{\alpha_{i_0} + k - 1}{\alpha_{i_0}} + 1\right) \leq \omega(k)(k + 1). \ \blacksquare$$

Main theorem. We order the prime numbers as $P_1 = 2 < P_2 = 3 < P_3 = 5 < \cdots$.

THEOREM 1. *If* $k = \prod_{i=1}^{\omega(k)} q_i^{\alpha_i}$ *with* $\alpha_i > 0$ *and* $2 < q_1 < \cdots < q_{\omega(k)}$ *primes then there is no solution* $(k, L, p_1, \ldots, p_L)$ *of* (1).

Proof. Suppose that such a solution does exist. We have the inequality

$$k = \prod_{i=1}^{L} \frac{p_i^k - 1}{p_i^{k-1}(p_i - 1)} < \prod_{i=1}^{L} \frac{p_i}{p_i - 1} \le \prod_{i=2}^{L+1} \frac{P_i}{P_i - 1} = \frac{\prod_{i=1}^{L+1} \frac{P_i}{P_i - 1}}{2},$$

so by [RS],

$$k < \frac{e^C}{2}(\log P_{L+1})\left(1 + \frac{1}{\log\log P_{L+1}}\right), \tag{9}$$

where $e^C = 1.78107\ldots$ is Euler's constant.

By Lemma 1, $\omega(k) > 1$ and thus $k \ge 15 = 3 \cdot 5$. Therefore

$$14 < \frac{e^C}{2}(\log P_{L+1})\left(1 + \frac{1}{\log\log P_{L+1}}\right) \quad \text{so} \quad P_{L+1} > 17,$$

thus $1 + \frac{1}{\log\log P_{L+1}} > 2$, which implies

$$7 < \frac{e^C}{2} \log P_{L+1} < \log P_{L+1} \quad \text{so} \quad e^7 < P_{L+1}. \tag{10}$$

We have $k = \prod_{i=1}^{\omega(k)} q_i^{\alpha_i} \ge 3^{\sum_{i=1}^{\omega(k)} \alpha_i} \ge 3^{\omega(k)}$ and Lemma 2 yields $L < \omega(k)(k+1)$. By Table 1 and [R] (1),

$$\frac{P_{L+1}}{2 + \log P_{L+1}} \le \pi(P_{L+1}) = L+1 < \omega(k)(k+1)+1 \le \frac{(k+1)\log k}{\log 3} + 1. \tag{11}$$

Our next step is to prove the following two claims.

CLAIM 1. *If* $x > 4$ *then*

$$\frac{(x+1)\log x}{\log 3} + 1 < x^{3/2}.$$

Proof. Define

$$f : \mathbb{R}^+ \to \mathbb{R}, \qquad f(x) = x^{3/2} - \frac{(x+1)\log x}{\log 3} - 1.$$

Then

$$f'(x) = \frac{3}{2}x^{1/2} - \frac{\frac{x+1}{x} + \log x}{\log 3} \quad \text{and} \quad f''(x) = \frac{3}{4}x^{-1/2} - \frac{\frac{1}{x} - \frac{1}{x^2}}{\log 3}.$$

Note that $x > \left(\frac{4}{3\log 3}\right)^2$ implies $f''(x) > 0$. Thus, if $x > 4 > \left(\frac{4}{3\log 3}\right)^2$ we have $f'(x) > f'(4) > 0$, and hence $f(x) > f(4) > 0$. ■

(1) For all $x \ge 55$ prime, [R] gives the inequality $\frac{x}{2+\log x} \le \pi(x)$; for $x < 55$ prime, this follows from Table 1.

Claim 2. *If* $x > e^6$ *then* $x^2 > 8(1 + \log x)^5$.

Proof. Let

$$f : \mathbb{R}^+ \to \mathbb{R}, \qquad f(x) = x^2 - 8(1 + \log x)^5.$$

Then

$$f'(x) = 2x - 40\frac{(1 + \log x)^4}{x} \quad \text{and} \quad f''(x) = 2 - \frac{40(1 + \log x)^3(3 - \log x)}{x^2}.$$

Note that if $x > e^3$ then $f''(x) > 2$. Hence, if $x > e^6$ then $f'(x) > f'(e^6) > 0$, and so $f(x) > f(e^6) > 0$. ∎

Now we return to the proof of our theorem. Since $P_{L+1} > e^7$, Claim 1 and the inequalities (9), (11) give us

$$\frac{P_{L+1}}{2 + \log P_{L+1}} < \frac{(k+1)\log k}{\log 3} + 1 < k^{3/2}$$

and

$$k^{3/2} < \left(\frac{e^C}{2}(\log P_{L+1})\left(1 + \frac{1}{\log\log P_{L+1}}\right)\right)^{3/2} < (e^C \log P_{L+1})^{3/2}.$$

In particular

$$\begin{aligned} P_{L+1}^2 &< (2 + \log P_{L+1})^2 (e^C \log P_{L+1})^3 < e^{3C}(1 + \log P_{L+1})^5 \\ &< 8(1 + \log P_{L+1})^5. \end{aligned}$$

But this contradicts Claim 2, showing there is no such solution. ∎

Table 1

P_L	L	$\frac{P_L}{2+\log P_L} \approx$
2	1	0.74262
3	2	0.96817
5	3	1.38525
7	4	1.77398
11	5	2.50911
13	6	2.84778
17	7	3.51732
19	8	3.84270
23	9	4.47863
29	10	5.40309
31	11	5.70483
37	12	6.59428
41	13	7.17589
43	14	7.46372
47	15	8.03398
53	16	8.87728

References

[G] R. K. Guy, *Unsolved Problems in Number Theory*, Springer, New York, 2004.

[R] J. B. Rosser, *Explicit bounds for some functions of prime numbers*, Amer. J. Math. 63 (1941), 211–232.

[RS] J. B. Rosser and L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math. 6 (1962), 64–94.

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