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On polynomials with flat squares

by

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1. Introduction. Let A be an infinite set of nonnegative integers. By squaring the infinite series $f(z) := \sum_{i\in A} z^i$ with $0,1$ coefficients, we obtain

$$f(z)^2 = \sum_{n=0}^{\infty} R(A,n)z^n,$$

where $R(A,n)$ is the number of representations of n in the form $n = a_1 + a_2$ with $a_1, a_2 \in A$. An old conjecture of Erdős and Turán [6] asserts that for any such infinite set A and any nonnegative integer n_0 the coefficients $R(A,n)$, $n \geq n_0$, cannot all lie in the interval $[1, C]$ with some constant $C = C(A, n_0)$. This deep USD 500 problem [5] remains open, although some progress has been made in [1], [2], [7], [10]. It was proved, for instance, that the numbers $R(A,n)$, $n \geq 0$, cannot all lie in the interval $[1,7]$ (see [1]). See also [4], [9], [11] for some constructions of A such that $R(A,n) \geq 1$ for all $n \geq 0$, but $R(A,n)$ is "small" in terms of n. Nathanson [8] describes this Erdős–Turán problem as "one of the most famous and tantalizing unsolved problems in additive number theory".

Given any polynomial $p(z) = \sum_{j=0}^{d} a_j z^j$ with nonnegative coefficients, let us denote the largest quotient between pairs of its coefficients by $q(p) = \max_{0\leq i,j\leq d} a_i/a_j$. In particular, we set $q(p) = +\infty$ if the polynomial $p(z)$ is not the zero polynomial, but has at least one coefficient equal to zero. Clearly, $q(p) \geq 1$ with equality if and only if all the coefficients of $p(z)$ are equal. We say that the polynomial p is "flat" if this quotient $q(p)$ is "small". Recently, the first named author [3] considered the following polynomial version of the Erdős–Turán problem. Let $p(z)$ be a polynomial of degree d with nonnegative real coefficients. In particular, it can be a *Newman polynomial*, i.e. a polynomial with coefficients $0, 1$. As above for the series $f(z)$, consider

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the square of $p(z)$, namely,

$$p(z)^2 = r_0 + r_1 z + \cdots + r_{2d} z^{2d}.$$

Then, for each positive integer $d \geq 1$, we are interested in the smallest positive number $\kappa(d)$ for which there is a polynomial $p(z)$ of degree d with nonnegative real coefficients such that the coefficients $r_0, r_1, \ldots, r_{2d}$ of $p(z)^2$ all lie in the interval $[1, \kappa(d)]$. Similarly, let $\kappa_{\rm rec}(d)$ be the smallest positive number such that all coefficients of the square $p(z)^2$ of a *reciprocal* polynomial $p(z)$ of degree d (i.e. satisfying $p(z) = z^d p(1/z)$) with nonnegative real coefficients lie in the interval $[1, \kappa_{\rm rec}(d)]$. It is easy to see that the numbers $\kappa(d)$ and $\kappa_{\rm rec}(d)$ exist for every $d \in \mathbb{N}$. Indeed, by compactness, the infimum $\inf q(p^2)$, where p runs through polynomials with nonnegative coefficients of degree d, is attained and is equal to $\kappa(d)$. Also, $\inf q(p^2)$, where p runs through reciprocal polynomials with nonnegative coefficients of degree d, is attained and is equal to $\kappa_{\rm rec}(d)$.

In [3] the first named author introduced the sequence $y_0 = 1$, $y_1 = 1/2$, $y_2 = 3/8, \ldots$, where each y_n, $n \geq 1$, is defined by the recurrence formula

$$2y_{2k}y_0 + 2y_{2k-1}y_1 + \cdots + 2y_{k+1}y_{k-1} + y_k^2 = 1 \tag{1}$$

for n even, i.e. $n = 2k$, $k \in \mathbb{N}$, and

$$2y_{2k+1}y_0 + 2y_{2k}y_1 + \cdots + 2y_{k+2}y_{k-1} + 2y_{k+1}y_k = 1 \tag{2}$$

for n odd, i.e. $n = 2k+1$, $k \geq 0$. We define the following reciprocal polynomial:

$$p_d(z) := y_0 + y_1 z + y_2 z^2 + \cdots + y_2 z^{d-2} + y_1 z^{d-1} + y_0 z^d. \tag{3}$$

Note that, by Theorem 2 below, $p_d(z)$ has positive coefficients. The quotients $q(p_d^2)$ between the largest and the smallest coefficients of $p_d(z)^2$ for $d = 1, \ldots, 12$ have been calculated in [3]:

$$q(p_1^2) = 2, \quad q(p_2^2) = 2.25, \quad q(p_3^2) = 2.5, \quad q(p_4^2) = \frac{169}{64} = 2.640625, \quad \ldots,$$
$$q(p_{11}^2) = \frac{106405}{32768} = 3.24722290\ldots, \quad q(p_{12}^2) = \frac{3458321}{1048576} = 3.2981191\ldots.$$

Clearly,

$$q(p_d^2) \geq \kappa_{\rm rec}(d) \geq \kappa(d)$$

for each $d \geq 1$, because the polynomial $p_d(z)$ is reciprocal, by (3), and has positive coefficients, by Theorem 2. The calculations with small d show that

$$\kappa(d) = \kappa_{\rm rec}(d) = q(p_d^2) \tag{4}$$

for $d = 1$ and $d = 2$. Since from (1)–(3) it is not even clear whether $q(p_d^2)$ is bounded or unbounded as $d \to \infty$ the first named author asked in [3] if (4) holds for all d or not and if $\kappa(d)$, $q(p_d^2)$ are bounded or unbounded as $d \to \infty$.

In this paper, we are able to prove the second equality in (4):

THEOREM 1. *We have* $\kappa_{\mathrm{rec}}(d) = q(p_d^2)$ *for each* $d \geq 1$.

This shows that the polynomial (3) is optimal among all reciprocal polynomials in the sense that it has the "flattest" square. The sequence of rational numbers y_n defined in (1), (2) can be determined explicitly in terms of central binomial coefficients:

THEOREM 2. *We have* $y_n = 2^{-2n}\binom{2n}{n}$ *for each* $n \geq 0$.

The next theorem shows that $q(p_d^2)$ is unbounded and answers a corresponding question raised in [3]:

THEOREM 3. *We have*

$$q(p_d^2) = 2(y_0^2 + y_1^2 + \cdots + y_{(d-1)/2}^2)$$

for each odd positive integer d *and*

$$q(p_d^2) = 2(y_0^2 + y_1^2 + \cdots + y_{d/2-1}^2) + y_{d/2}^2$$

for each even positive integer d. *Here,* $y_n = 2^{-2n}\binom{2n}{n}$ *and* $q(p_d^2) \sim (2/\pi)\log d$ *as* $d \to \infty$.

We say that a subset A of $\{0, 1, \ldots, d\}$ is *symmetric* if for any i in the range $0 \leq i \leq [d/2]$ the integers i and $d - i$ either both lie in A or both do not lie in A. Here and below, $[\cdot]$ stands for the integral part of a number. Assume that $A + A = \{0, 1, \ldots, 2d\}$. By Theorems 1 and 3, we have $\kappa_{\mathrm{rec}}(d) \sim (2/\pi)\log d$ as $d \to \infty$. Applying this result to the reciprocal Newman polynomial $p(z) = \sum_{j\in A} z^j$ of degree d whose square has positive (integer) coefficients we obtain the following corollary:

COROLLARY 4. *Let* A *be a symmetric subset of the set* $\{0, 1, \ldots, d\}$ *such that* $A + A = \{0, 1, \ldots, 2d\}$. *Then there is an element* $a \in \{0, 1, \ldots, 2d\}$ *which has at least* $c \log d$ *representations in the form* $a = a_1 + a_2$, $a_1, a_2 \in A$. *Here,* c *is an absolute positive constant.*

We next prove Theorem 2, then, using it, Theorem 3 and, finally, using both, we establish Theorem 1.

2. Proof of Theorem 2. Consider the function

$$g(z) := y_0 + y_1 z + y_2 z^2 + \cdots.$$

From (1) and (2), we deduce that

$$g(z)^2 = 1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z}.$$

On the other hand, let

$$g_2(z) := (1-z)^{-1/2} = \sum_{n=0}^{\infty} (-z)^n \binom{-1/2}{n}.$$

Note that

$$(-1)^n\binom{-1/2}{n} = \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{2^n n!} = \frac{(2n)!}{2^{2n} n!^2} = 2^{-2n}\binom{2n}{n}.$$

Setting $t_n := 2^{-2n}\binom{2n}{n}$, we obtain

$$g_2(z) = t_0 + t_1 z + t_2 z^2 + \cdots.$$

But $g_2(z)^2 = g(z)^2 = 1/(1-z)$, so the sequence t_n, $n = 0, 1, \ldots$, satisfies the same recurrence formulas (1), (2). Since each y_n is uniquely determined by $y_0, \ldots, y_{n-1}$ and $y_0 = t_0 = 1$, this implies $y_n = t_n = 2^{-2n}\binom{2n}{n}$ for each $n \geq 0$, as claimed. This completes the proof of Theorem 2.

Similarly, for each integer $k \geq 2$, the kth power of the series

$$g_k(z) := \sum_{n=0}^{\infty} (-1)^n \binom{-1/k}{n} z^n = (1-z)^{-1/k}$$

with positive coefficients $(-1)^n\binom{-1/k}{n}$ is equal to the series $g_k(z)^k = 1/(1-z) = \sum_{n=0}^{\infty} z^n$ with coefficients $1, 1, 1, \ldots$. This shows that the Erdős–Turán problem for the kth power of the series with nonnegative real (instead of $0, 1$) coefficients has a trivial answer: such a power can have all coefficients equal.

3. Proof of Theorem 3. Write

$$p_d(z)^2 = (y_0 + y_1 z + \cdots + y_1 z^{d-1} + y_0 z^d)^2 = s_0 + s_1 z + \cdots + s_d z^d + \cdots + s_0 z^{2d}.$$

By (1), (2), we have $s_0 = s_1 = \cdots = s_{[d/2]} = 1$. Set

$$y_i^* := y_{\min\{i, d-i\}} = \begin{cases} y_i & \text{for } 0 \leq i \leq [d/2], \\ y_{d-i} & \text{for } [d/2]+1 \leq i \leq d, \end{cases}$$

and $y_i^* = y_i := 0$ for $i \notin \mathbb{Z}$. Then $p_d(z) = \sum_{i=0}^{d} y_i^* z^i$, so

$$s_\ell = \sum_{i=0}^{\ell} y_i^* y_{\ell-i}^* = 2\sum_{i=0}^{[\ell/2]} y_i^* y_{\ell-i}^* - (y_{\ell/2}^*)^2 \tag{5}$$

for each integer ℓ satisfying $0 \leq \ell \leq d$. Also, as $p_d(z)$ is reciprocal, $s_\ell = s_{2d-\ell}$ for $d+1 \leq \ell \leq 2d$. We claim that

$$1 < s_\ell < s_d \tag{6}$$

for each ℓ in the range $[d/2]+1 \leq \ell \leq d-1$.

Note that $y_i^* = y_i$ for $i \leq \ell/2 \leq [d/2]$. Similarly, $y_{\ell-i}^* = y_{d-\ell+i}$ for $i \leq \ell - [d/2] - 1$ and $y_{\ell-i}^* = y_{\ell-i}$ for $i \geq \ell - [d/2]$. Hence, by (5),

$$s_\ell = 2\sum_{i=0}^{[\ell/2]} y_i y_{\ell-i}^* - y_{\ell/2}^2 = 2\sum_{i=0}^{\ell-[d/2]-1} y_i y_{d-\ell+i} + 2\sum_{i=\ell-[d/2]}^{[\ell/2]} y_i y_{\ell-i} - y_{\ell/2}^2. \tag{7}$$

Inserting $\ell = d$ into (5) we find that

$$s_d = 2\sum_{i=0}^{[d/2]} y_i^2 - y_{d/2}^2 = 2\sum_{i=0}^{d-[d/2]-1} y_i^2 + y_{d/2}^2. \tag{8}$$

By Theorem 2,

$$\frac{y_{s-1}}{y_s} = \frac{2^{2s}(s!)^2(2s-2)!}{2^{2s-2}(s-1)!^2(2s)!} = \frac{4s^2}{2s(2s-1)} = \frac{2s}{2s-1} > 1 \tag{9}$$

for each $s \in \mathbb{N}$. Thus $y_i > y_{d-\ell+i}$, because $i < d-\ell+i$. Similarly, $y_i \geq y_{\ell-i}$, because $i \leq \ell/2$. Thus, using

$$[\ell/2] \leq [(d-1)/2] = d - [d/2] - 1,$$

from (7) and (8) we obtain

$$s_d - y_{d/2}^2 = 2\sum_{i=0}^{d-[d/2]-1} y_i^2 > 2\sum_{i=0}^{\ell-[d/2]-1} y_i y_{d-\ell+i} + 2\sum_{i=\ell-[d/2]}^{[\ell/2]} y_i y_{\ell-i} = s_\ell + y_{\ell/2}^2.$$

Hence $s_d > s_\ell + y_{d/2}^2 + y_{\ell/2}^2 \geq s_\ell$, giving the second inequality in (6).

The proof of the first inequality in (6) is simpler. Fix an integer ℓ in the range $[d/2]+1 \leq \ell \leq d$. Observe that, by (1), (2), $\sum_{i=0}^{\ell} y_i y_{\ell-i} = 1$. By (9), we find that $y_i \leq y_i^* = y_{\min\{i,d-i\}}$ and $y_{\ell-i} \leq y_{\ell-i}^*$ for $i \leq \ell \leq d$. So $y_i y_{\ell-i} \leq y_i^* y_{\ell-i}^*$ for each $i = 0, 1, \ldots, \ell$. Moreover, at least one inequality is strict, because $\ell > [d/2]$. So (5) yields

$$1 = \sum_{i=0}^{\ell} y_i y_{\ell-i} < \sum_{i=0}^{\ell} y_i^* y_{\ell-i}^* = s_\ell.$$

This completes the proof of (6).

Now, from (6) it follows that all s_j, where $j = 0, 1, \ldots, 2d$, belong to the interval $[s_0, s_d]$. Here $s_0 = 1$. It is easily seen that $s_d = 2(y_0^2 + y_1^2 + \cdots + y_{(d-1)/2}^2)$ for odd positive integer d and $s_d = 2(y_0^2 + y_1^2 + \cdots + y_{d/2-1}^2) + y_{d/2}^2$ for even positive integer d. This proves the formulas for $q(p_d^2) = s_d$ as stated in the theorem.

We next find an asymptotical formula for $q(p_d^2)$. Fix $\varepsilon > 0$. By Theorem 2 and Stirling's formula,

$$y_n = \frac{(2n)!}{2^{2n} n!^2} \sim \frac{(2n/e)^{2n}\sqrt{2\pi 2n}}{2^{2n}(n/e)^{2n} 2\pi n} = \frac{1}{\sqrt{\pi n}}$$

as $n \to \infty$. So there is a positive integer $d_0(\varepsilon)$ such that

$$\frac{1-\varepsilon}{\pi n} < y_n^2 < \frac{1+\varepsilon}{\pi n} \tag{10}$$

for each $n \geq d_0(\varepsilon)$. Thus, in both cases (even and odd d), we have

$$\left| q(p_d^2) - 2\sum_{n=d_0(\varepsilon)}^{[d/2]} y_n^2 \right| \leq 2d_0(\varepsilon) + 1. \tag{11}$$

Using $\sum_{n=d_0(\varepsilon)}^{[d/2]}(1/n)\sim\log d$ as $d\to\infty$ and (10), we deduce that

$$\sum_{n=d_0(\varepsilon)}^{[d/2]} y_n^2 \in \left[\frac{(1-\varepsilon)^2}{\pi}\log d, \frac{(1+\varepsilon)^2}{\pi}\log d\right]$$

for $d\ge d_1(\varepsilon)$. Thus, by (11),

$$\frac{2(1-\varepsilon)^3}{\pi}\log d < q(p_d)^2 < \frac{2(1+\varepsilon)^3}{\pi}\log d$$

for $d\ge d_2(\varepsilon)$. It follows that $q(p_d^2)\sim\frac{2}{\pi}\log d$ as $d\to\infty$.

4. Proof of Theorem 1. Let V_n be a subset of vectors $(x_0,\dots,x_{n-1})$ in $\mathbb{R}^n$ determined by the inequalities

$$\begin{gathered} x_0,x_1,\dots,x_{n-1}\ge 0,\\ x_0^2\ge 1,\\ 2x_0x_1\ge 1,\\ 2x_0x_2+x_1^2\ge 1,\\ 2x_0x_3+2x_1x_2\ge 1,\\ \vdots\\ \sum_{i=0}^{n-1}x_ix_{n-1-i}=2x_0x_{n-1}+2x_1x_{n-1}+\cdots\ge 1. \end{gathered}$$

The key element in the proof of the theorem is the following:

Lemma 5. *Let* $\mathbf{v}\in V_n$. *Then* $|\mathbf{v}|^2\ge y_0^2+\cdots+y_{n-1}^2$, *where equality holds if and only if* $\mathbf{v}=(y_0,\dots,y_{n-1})$.

Proof. Suppose that $\mathbf{v}=(x_0,\dots,x_{n-1})\in V_n$. By Theorem 2, $y_n>0$ for each $n\ge 0$. So, for every pair i,j satisfying $0\le i<j\le n-1$, we have

$$\frac{x_i^2y_j}{y_i}+\frac{x_j^2y_i}{y_j}\ge 2x_ix_j,$$

where equality holds if and only if $x_jy_i=x_iy_j$. Fix an integer ℓ in $[0,n-1]$. Replacing each double product $2x_ix_{\ell-i}$ in this way and leaving $x_{\ell/2}^2$ as it is (if ℓ is even), we obtain

$$\begin{aligned} 1&\le\sum_{i=1}^{\ell}x_ix_{\ell-i}=2x_0x_\ell+2x_1x_{\ell-1}+\cdots\\ &\le\frac{x_0^2y_\ell}{y_0}+\frac{x_\ell^2y_0}{y_\ell}+\frac{x_1^2y_{\ell-1}}{y_1}+\frac{x_{\ell-1}^2y_1}{y_{\ell-1}}+\cdots=\sum_{i=0}^{\ell}\frac{x_i^2y_{\ell-i}}{y_i}. \end{aligned}$$

Here, the second inequality becomes equality if and only if $(x_0,\dots,x_\ell)=\lambda_\ell(y_0,\dots,y_\ell)$ with a scalar multiple $\lambda_\ell>0$. For such a vector $(x_0,\dots,x_\ell)$, the first inequality,

$$1 \le \sum_{i=1}^{\ell} x_i x_{\ell-i} = \lambda_\ell^2 \sum_{i=1}^{\ell} y_i y_{\ell-i} = \lambda_\ell^2$$

(see (1), (2)), is equality if and only if $\lambda_\ell = 1$. Hence $1 = \sum_{i=0}^{\ell} x_i^2 y_{\ell-i}/y_i$ for $\ell = 0, 1, \ldots, n-1$ if and only if $\mathbf{v} = (x_0, \ldots, x_{n-1}) = (y_0, \ldots, y_{n-1}) \in V_n$.

Let $\mu_0, \ldots, \mu_{n-1}$ be some positive constants to be chosen later. Multiplying the ℓth inequality, $1 \le \sum_{i=1}^{\ell} x_i x_{\ell-i}$, by μ_ℓ and adding them for $\ell = 0, 1, \ldots, n-1$, we find that

$$\sum_{\ell=0}^{n-1} \mu_\ell \le \sum_{\ell=0}^{n-1} \mu_\ell \sum_{i=0}^{\ell} x_i x_{\ell-i} \le \sum_{\ell=0}^{n-1} \mu_\ell \sum_{i=0}^{\ell} \frac{x_i^2 y_{\ell-i}}{y_i} = \sum_{i=0}^{n-1} \frac{x_i^2}{y_i} \sum_{\ell=i}^{n-1} \mu_\ell y_{\ell-i}. \tag{12}$$

We next show that positive numbers $\mu_0, \ldots, \mu_{n-1}$ can be chosen so that all coefficients $a_i := y_i^{-1} \sum_{\ell=i}^{n-1} \mu_\ell y_{\ell-i}$ for x_i^2 in the inequality (12), i.e. $\sum_{\ell=0}^{n-1} \mu_\ell \le \sum_{i=0}^{n-1} a_i x_i^2$, are equal: $a_{n-1} = \cdots = a_0$, namely,

$$\begin{aligned}\frac{\mu_{n-1} y_0}{y_{n-1}} &= \frac{\mu_{n-1} y_1}{y_{n-2}} + \frac{\mu_{n-2} y_0}{y_{n-2}} = \frac{\mu_{n-1} y_2}{y_{n-3}} + \frac{\mu_{n-2} y_1}{y_{n-3}} + \frac{\mu_{n-3} y_0}{y_{n-3}} = \cdots \\ &= \frac{\mu_{n-1} y_{n-1}}{y_0} + \cdots + \frac{\mu_1 y_1}{y_0} + \mu_0.\end{aligned}$$

Indeed, set $\mu_{n-1} := 1$ and then, step by step left to right, determine $\mu_{n-2}, \mu_{n-3}, \ldots, \mu_0$. We claim that $\mu_{n-1}, \ldots, \mu_0$ are all positive. For a contradiction assume that $\mu_{n-1} = 1 > 0, \ldots, \mu_{n-i+1} > 0$, but $\mu_{n-i} \le 0$ for some i satisfying $2 \le i \le n$. Since

$$\frac{\mu_{n-i} y_0}{y_{n-i}} = \sum_{j=1}^{i-1} \mu_{n-j} \left(\frac{y_{i-j-1}}{y_{n-i+1}} - \frac{y_{i-j}}{y_{n-i}} \right)$$

and $\mu_{n-1}, \ldots, \mu_{n-i+1} > 0$, this can happen only if some difference

$$\frac{y_{i-j-1}}{y_{n-i+1}} - \frac{y_{i-j}}{y_{n-i}}$$

is at most 0. Hence $y_{i-j-1} y_{n-i} \le y_{n-i+1} y_{i-j}$ for some i, j satisfying $1 \le j \le i-1 \le n-1$. However, by (9), $y_{i-j-1} > y_{i-j}$ and $y_{n-i} > y_{n-i+1}$, giving $y_{i-j-1} y_{n-i} > y_{n-i+1} y_{i-j}$, a contradiction.

Now, since all μ_i are positive and all a_i, $i = 0, 1, \ldots, n-1$, are equal, we must have

$$\sum_{\ell=0}^{n-1} \mu_\ell \le \sum_{i=0}^{n-1} a_i x_i^2 = a_{n-1} \sum_{i=0}^{n-1} x_i^2 = \frac{\mu_{n-1} y_0}{y_{n-1}} \sum_{i=0}^{n-1} x_i^2. \tag{13}$$

As we already observed, for $(x_0, \ldots, x_{n-1}) = (y_0, \ldots, y_{n-1})$ (and only for this vector), we have equality in (12) and so in (13). Thus

$$\sum_{\ell=0}^{n-1} \mu_\ell = \frac{\mu_{n-1} y_0}{y_{n-1}} \sum_{i=0}^{n-1} y_i^2.$$

Hence, by (13), we find that

$$|\mathbf{v}|^2 = \sum_{i=0}^{n-1} x_i^2 \geq \frac{y_{n-1}}{\mu_{n-1} y_0} \sum_{\ell=0}^{n-1} \mu_\ell = \sum_{i=0}^{n-1} y_i^2.$$

This proves the lemma.

For the proof of Theorem 1, fix $d \in \mathbb{N}$ and assume that $p(z) = x_0 + x_1 z + \dots + x_1 z^{d-1} + x_0 z^d$ is a reciprocal polynomial of degree d with nonnegative coefficients such that the coefficients of its square $p(z)^2 = r_0 + r_1 z + \dots + r_1 z^{2d-1} + r_0 z^{2d}$ are all greater than or equal to 1. Then

$$r_0 = x_0^2 \geq 1, \quad r_1 = 2x_0 x_1 \geq 1, \quad \dots, \quad r_{[d/2]} = \sum_{i=0}^{[d/2]} x_i x_{[d/2]-i} \geq 1,$$

and so $(x_0, \dots, x_{[d/2]}) \in V_{[d/2]+1}$. The coefficient r_d for z^d in $p(z)^2$ is equal to

$$2(x_0^2 + \dots + x_{(d-1)/2}^2)$$

for d odd and to

$$2(x_0^2 + \dots + x_{d/2-1}^2) + x_{d/2}^2$$

for d even.

For d odd, by Lemma 5, we have $r_d = 2(x_0^2 + \dots + x_{(d-1)/2}^2) \geq 2(y_0^2 + \dots + y_{(d-1)/2}^2)$. Moreover, if $x_i \neq y_i$ for at least one $i \in \{0, \dots, (d-1)/2\}$, then this inequality is strict. This implies that the polynomial $p(z)^2$ has at least one coefficient greater than $2(y_0^2 + \dots + y_{(d-1)/2}^2)$, unless $x_0 = y_0, \dots, x_{(d-1)/2} = y_{(d-1)/2}$. So $q(p^2) \geq q(p_d^2) = 2(y_0^2 + \dots + y_{(d-1)/2}^2)$ for every reciprocal polynomial p with nonnegative coefficients. On the other hand, the example $p(z) = p_d(z)$ shows that all coefficients of $p_d(z)^2$ lie in the interval $[1, 2(y_0^2 + \dots + y_{(d-1)/2}^2)]$ (see Theorem 2 and, more precisely, inequality (6)).

For d even, applying Lemma 5 to $n = d/2$ and to $n = d/2 + 1$, we find that

$$\begin{aligned} r_d = 2(x_0^2 + \dots + x_{d/2-1}^2) + x_{d/2}^2 &= \sum_{i=0}^{d/2-1} x_i^2 + \sum_{i=0}^{d/2} x_i^2 \\ &\geq \sum_{i=0}^{d/2-1} y_i^2 + \sum_{i=0}^{d/2} y_i^2 = 2(y_0^2 + \dots + y_{d/2-1}^2) + y_{d/2}^2. \end{aligned}$$

Consequently, $q(p^2) \geq q(p_d^2) = 2(y_0^2 + \dots + y_{d/2-1}^2) + y_{d/2}^2$ for every reciprocal polynomial p with nonnegative coefficients. The proof of Theorem 1 can now be concluded as above with the same example $p(z) = p_d(z)$.

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