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Representation of odd integers as the sum of one prime, two squares of primes and powers of 2

by

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1. Introduction. In 1999, Liu–Liu–Zhan [6, Theorem 4] proved that every large odd N can be represented as the sum of one prime, two squares of primes and k powers of 2. Let $r_k(N)$ denote the number of solutions of the equation

$$N = p_1 + p_2^2 + p_3^2 + 2^{\nu_1} + 2^{\nu_2} + \cdots + 2^{\nu_k}. \tag{1.1}$$

One may anticipate that a small k in (1.1) is not sufficient to give the positivity of $r_k(N)$. Since it is well known that one of the fundamental problem on the solubility of additive equations is to determine a lower bound for the number of variables in the equation, an interesting question arises of how many powers of 2 are needed to ensure $r_k(N) > 0$.

In this paper, we shall show that $k \geq 22000$, so a not very large number of variables in the equation (1.1) is sufficient to ensure $r_k(N) > 0$. More precisely, we have the following result:

THEOREM 1. *Let $r_k(N)$ be as defined above. Then there exists a constant $k_0 \geq 22000$ and a constant N_k depending on k only such that if $N \geq N_k$, $k \geq k_0$, then $r_k(N) > 0$.*

2. The circle method. First, we give some notations. Define

$$P = N^{2/15-\varepsilon}, \quad Q = N/PL^{14}, \quad L = \log N.$$

For $\alpha \in [1/Q, 1 + 1/Q]$, from Dirichlet's lemma on rational approximation, we have

$$\alpha = a/q + \lambda, \quad |\lambda| \leq 1/qQ,$$

where $1 \leq a \leq q \leq Q$, $(a, q) = 1$. Take

$$M(a, q) = [a/q - 1/qQ, a/q + 1/qQ],$$

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$$\mathbf{M} = \bigcup_{q \le P} \bigcup_{a=1,\,(a,q)=1}^{q} M(a,q),$$
$$C(\mathbf{M}) = [1/Q, 1+1/Q]/\mathbf{M}.$$

Since $2P \le Q$, the sets $M(a,q)$ are mutually disjoint.

For χ mod q, define

$$\mathtt{C}(\chi,a) = \sum_{h=1}^{q} \overline{\chi}(h) e\biggl(\frac{ah^2}{q}\biggr), \qquad \mathtt{C}(\chi^0,a) = C(a,q),$$
$$\mathtt{D}(\chi,a) = \sum_{h=1}^{q} \overline{\chi}(h) e\biggl(\frac{ah}{q}\biggr), \qquad \mathtt{D}(\chi^0,a) = D(a,q).$$

If χ_1, χ_2, χ_3 are characters mod q, we define

$$B(n,q,\chi_1,\chi_2,\chi_3) = \sum_{a=1,\,(a,q)=1}^{q} e\biggl(-\frac{an}{q}\biggr)\mathtt{C}(\chi_1,a)\mathtt{C}(\chi_2,a)\mathtt{D}(\chi_3,a),$$

$$B(n,q) = B(n,q,\chi_1^0,\chi_2^0,\chi_3^0), \tag{2.1}$$

$$\sigma_1(n) = \sum_{q \le P} \frac{B(n,q)}{\phi^3(q)}.$$

Define

$$T_1(\alpha) = \sum_{p \le N} \log p\, e(\alpha p), \qquad S_1(\alpha) = \sum_{p^2 \le N} \log p\, e(\alpha p^2),$$
$$T(\alpha) = \sum_{M < p \le N} \log p\, e(\alpha p), \qquad S(\alpha) = \sum_{M < p^2 \le N} \log p\, e(\alpha p^2),$$

where $M = NL^{-14}$.

Theorem 1 depends mainly on Theorem 2:

THEOREM 2. *For $N/2 \le n \le N$, we have*

$$\int_{\mathbf{M}} S_1^2(\alpha) T_1(\alpha) e(-n\alpha)\, d\alpha = \frac{\pi}{4}\,\sigma_1(n) n + O(NL^{-1}).$$

LEMMA 2.1. *Let χ_j, $j = 1,2,3$, be primitive characters mod r_j, $r_0 = [r_1, r_2, r_3]$, and χ^0 a principal character mod q. Then for any $\varepsilon > 0$ there exists $c(\varepsilon)$ such that*

$$\sum_{q \le x,\, r_0 | q} \frac{1}{\phi^3(q)} |B(n,q,\chi_1\chi^0,\chi_2\chi^0,\chi_3\chi^0)| \le_{c(\varepsilon)} r_0^{-1/2+\varepsilon} L^{10}.$$

Proof. The proof is similar to that of Lemma 7 of [2], so we leave it to the reader as an exercise. Define

$$V_1(\lambda) = \sum_{M<m^2\le N} e(m^2\lambda), \qquad V_2(\lambda) = \sum_{M<m\le N} e(m\lambda),$$

$$W(\chi,\lambda) = \sum_{M<p^2\le N} \log p\,\chi(p)e(p^2\lambda) - \delta_\chi \sum_{M<m^2\le N} e(m^2\lambda), \tag{2.2}$$

$$U(\chi,\lambda) = \sum_{M<p\le N} \log p\,\chi(p)e(\alpha p) - \delta_\chi \sum_{M<m\le N} e(m\lambda),$$

where

$$\delta_\chi = \begin{cases} 1 & \text{if } \chi = \chi^0, \\ 0 & \text{if } \chi \ne \chi^0. \end{cases}$$

Obviously, we have

$$S(\alpha) = \frac{C(a,q)}{\phi(q)} V_1(\lambda) + \frac{1}{\phi(q)} \sum_{\chi \bmod q} \mathtt{C}(\chi,a)W(\chi,\lambda),$$

$$T(\alpha) = \frac{D(a,q)}{\phi(q)} V_2(\lambda) + \frac{1}{\phi(q)} \sum_{\chi \bmod q} \mathtt{D}(\chi,a)U(\chi,\lambda).$$

Then

$$\int_{\mathbf{M}} S^2(\alpha)T(\alpha)e(-n\alpha)\,d\alpha = I_1 + I_2 + 2I_3 + I_4 + 2I_5 + I_6,$$

where

$$I_1 = \sum_{q\le P} \frac{\mu(q)}{\phi^3(q)} \sum_{a=1,\,(a,q)=1}^{q} e\Big(-\frac{an}{q}\Big) C^2(a,q)$$
$$\times \int_{-1/qQ}^{1/qQ} \Big(\sum_{M\le m^2\le N} e(m^2\lambda)m^{-1}\Big)^2 \Big(\sum_{M\le m\le N} e(m\lambda)\Big) e(-n\lambda)\,d\lambda,$$

$$I_2 = \sum_{q\le P} \frac{1}{\phi^3(q)} \sum_{a=1,\,(a,q)=1}^{q} \mu(q)e\Big(-\frac{an}{q}\Big)$$
$$\times \int_{-1/qQ}^{1/qQ} V_2(\lambda)\Big(\sum_{\chi \bmod q} \mathtt{C}(\chi,a)W(\chi,\lambda)\Big)^2 e(-n\lambda)\,d\lambda,$$

$$I_3 = \sum_{q\le P} \frac{1}{\phi^3(q)} \sum_{a=1,\,(a,q)=1}^{q} \mu(q)C(a,q)e\Big(-\frac{an}{q}\Big)$$
$$\times \int_{-1/qQ}^{1/qQ} V_1(\lambda)V_2(\lambda)\Big(\sum_{\chi \bmod q} \mathtt{C}(\chi,a)W(\chi,\lambda)\Big) e(-n\lambda)\,d\lambda,$$

$$I_4 = \sum_{q\le P}\frac{1}{\phi^3(q)}\sum_{a=1,\,(a,q)=1}^{q} C^2(a,q)e\Big(-\frac{an}{q}\Big)$$
$$\times \int_{-1/qQ}^{1/qQ} V_1^2(\lambda)\Big(\sum_{\chi \bmod q}\mathtt{D}(\chi,a)U(\chi,\lambda)\Big)e(-n\lambda)\,d\lambda,$$
$$I_5 = \sum_{q\le P}\frac{1}{\phi^3(q)}\sum_{a=1,\,(a,q)=1}^{q} C(a,q)e\Big(-\frac{an}{q}\Big)$$
$$\times \int_{-1/qQ}^{1/qQ} V_1(\lambda)\Big(\sum_{\chi \bmod q}\mathtt{C}(\chi,a)W(\chi,\lambda)\Big)\Big(\sum_{\chi \bmod q}\mathtt{D}(\chi,a)U(\chi,\lambda)\Big)e(-n\lambda)\,d\lambda,$$
$$I_6 = \sum_{q\le P}\frac{1}{\phi^3(q)}\sum_{a=1,\,(a,q)=1}^{q} e\Big(-\frac{an}{q}\Big)$$
$$\times \int_{-1/qQ}^{1/qQ}\Big(\sum_{\chi \bmod q}\mathtt{C}(\chi,a)W(\chi,\lambda)\Big)^2\Big(\sum_{\chi \bmod q}\mathtt{D}(\chi,a)U(\chi,\lambda)\Big)e(-n\lambda)\,d\lambda.$$

We also define

$$J' = \sum_{r_1\le P} r_1^{-1/4+\varepsilon}\sideset{}{^*}\sum_{\chi_1 \bmod r_1}\max_{|\lambda|\le 1/r_1Q}|W(\chi_1,\lambda)|,$$
$$K = \sum_{r_2\le P} r_2^{-1/4+\varepsilon}\sideset{}{^*}\sum_{\chi_2 \bmod r_2}\Big(\int_{-1/r_2Q}^{1/r_2Q}|W(\chi_2,\lambda)|^2\,d\lambda\Big)^{1/2},$$
$$J = \sum_{r_3\le P} r_3^{-\varepsilon}\sideset{}{^*}\sum_{\chi_3 \bmod r_3}\max_{|\lambda|\le 1/r_3Q}|U(\chi_3,\lambda)|,$$
$$K_1 = \Big(\int_{-1/Q}^{1/Q}|V_2(\lambda)|^2\,d\lambda\Big)^{1/2},\qquad K_2 = \Big(\int_{-1/Q}^{1/Q}|V_1(\lambda)|^2\,d\lambda\Big)^{1/2}.$$

First, we estimate I_6:

$$|I_6| = \bigg|\sum_{q\le P}\frac{1}{\phi^3(q)}\sum_{\chi_1 \bmod q}\cdots\sum_{\chi_3 \bmod q}B(n,q,\chi_1,\chi_2,\chi_3)$$
$$\times\int_{-1/qQ}^{1/qQ}W(\chi_1,\lambda)W(\chi_2,\lambda)U(\chi_3,\lambda)e(-n\lambda)\,d\lambda\bigg|$$
$$\le \sum_{r_1\le P}\cdots\sum_{r_3\le P}\sideset{}{^*}\sum_{\chi_1 \bmod r_1}\cdots\sideset{}{^*}\sum_{\chi_3 \bmod r_3}\sum_{q\le P,\,r_0|q}\bigg|\frac{B(n,q,\chi_1\chi^0,\chi_2\chi^0,\chi_3\chi^0)}{\phi^3(q)}\bigg|$$
$$\times\int_{-1/qQ}^{1/qQ}|W(\chi_1\chi^0,\lambda)W(\chi_2\chi^0,\lambda)W(\chi_3\chi^0,\lambda)|\,d\lambda,$$

where χ^0 is a principal character mod q, $r_0 = [r_1, r_2, r_3]$, and the notation $\sum^*$ refers to primitive characters. For $q \le P$ and $M \le p \le N$, we have $(q,p) = 1$. Using this and (2.2), for the primitive character χ_3, we get $W(\chi_3\chi^0, \lambda) = W(\chi_3, \lambda)$. Thus by Lemma 2.1 we obtain

$$\begin{aligned}
|I_6| \le & \sum_{r_1 \le P} \cdots \sum_{r_3 \le P} \sideset{}{^*}\sum_{\chi_1 \bmod r_1} \cdots \sideset{}{^*}\sum_{\chi_3 \bmod r_3} \\
& \times \int_{-1/r_0Q}^{1/r_0Q} |W(\chi_1, \lambda)|\, |W(\chi_2, \lambda)|\, |W(\chi_3, \lambda)|\, d\lambda \\
& \times \sum_{q \le P,\, r_0 | q} \left| \frac{B(n, q, \chi_1\chi^0, \chi_2\chi^0, \chi_3\chi^0)}{\phi^3(q)} \right| \\
\ll & L^{10} \sum_{r_1 \le P} \sum_{r_2 \le P} \sum_{r_3 \le P} r_0^{-1/2+\varepsilon} \sideset{}{^*}\sum_{\chi_1 \bmod r_1} \sideset{}{^*}\sum_{\chi_2 \bmod r_2} \sideset{}{^*}\sum_{\chi_3 \bmod r_3} \\
& \times \int_{-1/r_0Q}^{1/r_0Q} |W(\chi_1, \lambda)|\, |W(\chi_2, \lambda)|\, |U(\chi_3, \lambda)|\, d\lambda \\
\ll & L^{10} \sum_{r_3 \le P} r_3^{-\varepsilon} \sideset{}{^*}\sum_{\chi_3 \bmod r_3} \max_{|\lambda| \le 1/r_3Q} |U(\chi_3, \lambda)| \\
& \times \sum_{r_1 \le P} r_1^{-1/4+\varepsilon} \sideset{}{^*}\sum_{\chi_1 \bmod r_1} \left(\int_{-1/r_1Q}^{1/r_1Q} |W(\chi_1, \lambda)|^2\, d\lambda \right)^{1/2} \\
& \times \sum_{r_2 \le P} r_2^{-1/4+\varepsilon} \sideset{}{^*}\sum_{\chi_2 \bmod r_2} \left(\int_{-1/r_2Q}^{1/r_2Q} |W(\chi_2, \lambda)|^2\, d\lambda \right)^{1/2} \\
= & L^{10} J K^2.
\end{aligned}$$

For I_5, we have the following estimate:

$$\begin{aligned}
|I_5| \le & L^{10} \sum_{r_2 \le P} \sum_{r_3 \le P} [r_2, r_3]^{-1/2+\varepsilon} \sideset{}{^*}\sum_{\chi_2 \bmod r_2} \sideset{}{^*}\sum_{\chi_3 \bmod r_3} \\
& \times \int_{-1/r_0Q}^{1/r_0Q} |W(\chi_2, \lambda)|\, |V_1(\lambda)|\, |U(\chi_3, \lambda)|\, d\lambda \\
\le & L^{10} \sum_{r_3 \le P} r_3^{-\varepsilon} \sideset{}{^*}\sum_{\chi_3 \bmod r_3} \max_{|\lambda| \le 1/r_3Q} |U(\chi_3, \lambda)| \left(\int_{-1/Q}^{1/Q} |V_1(\lambda)|^2\, d\lambda \right)^{1/2} \\
& \times \sum_{r_2 \le P} r_2^{-1/4+\varepsilon} \sideset{}{^*}\sum_{\chi_2 \bmod r_2} \left(\int_{-1/r_2Q}^{1/r_2Q} |W(\chi_2, \lambda)|^2\, d\lambda \right)^{1/2} \\
= & L^{10} J K K_2.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
|I_4| &\le L^{10} \sum_{r_3 \le P} r_3^{-\varepsilon} \sideset{}{^*}\sum_{\chi_3 \bmod r_3} \max_{|\lambda| \le 1/r_3 Q} |U(\chi_3, \lambda)| \\
&\qquad \times \Big(\int\limits_{-1/Q}^{1/Q} |V_1(\lambda)|^2 \, d\lambda \Big)^{1/2} \Big(\int\limits_{-1/Q}^{1/Q} |V_1(\lambda)|^2 \, d\lambda \Big)^{1/2} \\
&= L^{10} J K_2^2, \\
|I_3| &\ll L^{10} \sum_{r_2 \le P} r_2^{-1/4+\varepsilon} \sideset{}{^*}\sum_{\chi_2 \bmod r_2} \max_{|\lambda| \le 1/r_2 Q} |W(\chi_2, \lambda)| \\
&\qquad \times \Big(\int\limits_{-1/Q}^{1/Q} |V_1(\lambda)|^2 \, d\lambda \Big)^{1/2} \Big(\int\limits_{-1/Q}^{1/Q} |V_2(\lambda)|^2 \, d\lambda \Big)^{1/2} \\
&= L^{10} J' K_1 K_2, \\
|I_2| &\ll L^{10} \sum_{r_1 \le P} r_1^{-1/4+\varepsilon} \sideset{}{^*}\sum_{\chi_1 \bmod r_1} \max_{|\lambda| \le 1/r_1 Q} |W(\chi_1, \lambda)| \\
&\qquad \times \sum_{r_2 \le P} r_2^{-1/4+\varepsilon} \sideset{}{^*}\sum_{\chi_2 \bmod r_2} \Big(\int\limits_{-1/r_2 Q}^{1/r_2 Q} |W(\chi_2, \lambda)|^2 \, d\lambda \Big)^{1/2} \Big(\int\limits_{-1/Q}^{1/Q} |V_2(\lambda)|^2 \, d\lambda \Big)^{1/2} \\
&= L^{10} J' K K_1.
\end{aligned}$$

Now, we estimate J', J, K, K_1 and K_2. Since

$$V_2(\lambda) = \sum_{M < m \le N} e(m\lambda) \ll \min(N, 1/|\lambda|), \tag{2.3}$$

we have

$$K_1 \le \Big(\int\limits_0^{1/N} N^2 \, d\lambda + \int\limits_{1/N}^{\infty} \lambda^{-2} \, d\lambda \Big)^{1/2} \ll \sqrt{N}.$$

Since

$$\begin{aligned}
V_1(\lambda) &= \sum_{M < m^2 \le N} e(m^2 \lambda) = \int\limits_{\sqrt{M}}^{\sqrt{N}} e(\lambda u^2) \, du + O(1 + |\lambda| N) \\
&= \frac{1}{2} \sum_{M < m \le N} m^{-1/2} e(m\lambda) + O(1 + |\lambda| N),
\end{aligned} \tag{2.4}$$

we obviously have

$$V_1(\lambda) \ll \min\Big(\sqrt{N}, \frac{1}{\sqrt{M}} |\lambda|^{-1} \Big) + O(|\lambda| N + 1).$$

Thus

$$|V_1(\lambda)|^2 \ll \left(\min\left(\sqrt{N}, \frac{1}{\sqrt{M}}\,|\lambda|^{-1}\right)\right)^2 + O(1+|\lambda|^2N^2).$$

Next,

$$\begin{aligned} K_2 &= \Big(\int_{-1/Q}^{1/Q} |V_1(\lambda)|^2\,d\lambda\Big)^{1/2} \\ &\ll \Big(\int_{-1/Q}^{1/Q} \min\Big(N, \frac{1}{M}\,|\lambda|^{-2}\Big)\,d\lambda\Big)^{1/2} + \Big(\int_{-1/Q}^{1/Q} (1+|\lambda|^2N^2)\,d\lambda\Big)^{1/2}. \end{aligned}$$

The first term on the right hand side is

$$\ll \Big(\int_0^{1/\sqrt{MN}} N\,d\lambda + \int_{1/\sqrt{MN}}^{\infty} \frac{1}{M}\,\lambda^{-2}\,d\lambda\Big)^{1/2} \ll (L^7+L^7)^{1/2} \ll L^4,$$

while the second is $\ll 1$. Therefore,

$$K_2 \ll L^4.$$

For J' and K, we have the following lemmas:

LEMMA 2.2 (see [4, Lemma 5.1]). *Let $A > 0$ be arbitrary. Then*

$$J' \ll N^{1/2}L^{-A},$$

where the O-constant depends at most on A.

LEMMA 2.3 (see [4, Lemma 6.1]). *We have*

$$K \ll L^c,$$

where $c > 0$ is an absolute constant.

Now it remains to estimate J. This is similar to Lemma 5.1 of [4]. For completeness, we write it out in detail.

LEMMA 2.4. *Let $A > 0$ be arbitrary. Then there exists a constant $B = B(A) > 0$ such that*

$$J \ll NL^{-A},$$

where the O-constant depends at most on A.

Proof. Since

$$J \ll \max_{R\le P} J_R L,$$

where the definition of J_R is similar to that of J except that the sum is over $r \sim R$, we only need to prove $J_R \ll NL^{-A}$. We divide the proof into two cases: $L^B < R \le P$ and $R \le L^B$. So we need to prove the following three lemmas.

LEMMA 2.5. *We have*

$$\sum_{r\sim R}\ \sideset{}{^*}\sum_{\chi \bmod r}\ \max_{|\lambda|\le 1/rQ}|U(\chi,\lambda)| \ll R^{\varepsilon}NL^{-A}. \tag{2.5}$$

Let

$$\widehat{U}(\chi,\lambda)=\sum_{M<m\le N}(\Lambda(m)\chi(m)-\delta_\chi)e(m\lambda).$$

Then

$$U(\chi,\lambda)-\widehat{U}(\chi,\lambda)=-\sum_{j\ge 2}\ \sum_{M<p^j\le N}\log p\,\chi(p)e(p^j\lambda)\ll N^{1/2},$$

hence (2.5) becomes a consequence of the following lemma:

LEMMA 2.5′. *We have*

$$\sum_{r\sim R}\ \sideset{}{^*}\sum_{\chi \bmod r}\ \max_{|\lambda|\le 1/rQ}|\widehat{U}(\chi,\lambda)| \ll R^{\varepsilon}NL^{-A}$$

where $L^B\le R\le P$ and A is a positive number.

This is proved by a slight modification of the proof of Lemma 5.1 of [4] (replace $N^{1/2}$ by N).

For $R\le L^B$, we have the following result:

LEMMA 2.6. *Let $A>0$, $B>0$ be arbitrary numbers. Then for $R\le L^B$, we have $J_R\ll NL^{-A}$, where the implied constant depends at most on A and B.*

The proof is similar to that of Lemma 5.5 of [4] (replace $T=N^{1/6}$ by $T=N^{1/3}$).

Now, we can estimate $I_1,\dots,I_6$. We have

$$\begin{aligned}
|I_6| &\ll L^{10}JK^2\ll L^{10}NL^{2c}L^{-A}\ll NL^{-A},\\
|I_5| &\ll L^{10}JKK_2\ll L^{10}NL^{-A}L^{4+c}\ll NL^{-A},\\
|I_4| &= L^{10}JK_2^2\ll L^{10}NL^{-A}L^{8},\\
|I_3| &\ll L^{10}J'K_1K_2\ll L^{10}N^{1/2}L^{-A}N^{1/2}L^{4}\ll NL^{-A},\\
|I_2| &\ll L^{10}J'KK_1\ll L^{10}NL^{-A}L^{c}\ll NL^{-A}.
\end{aligned}$$

Thus, we have

$$\int_{\mathbf{M}} T(\alpha)S^2(\alpha)e(-n\alpha)\,d\alpha = I_1+I_2+2I_3+I_4+2I_5+I_6=I_1+O(NL^{-A}).$$

It remains to deal with I_1. From (2.3) and (2.4), we have

$$I_1=\frac14\sum_{q\le P}\frac{\mu(q)}{\phi^3(q)}\sum_{a=1,\,(a,q)=1}^{q}e\Big(-\frac{an}{q}\Big)C^2(a,q) \tag{$*$}$$

$$\times \int_{-1/qQ}^{1/qQ} \Big(\sum_{M\le m\le N} e(m\lambda)m^{-1/2}\Big)^2 \Big(\sum_{M\le m\le N} e(m\lambda)\Big) e(-n\lambda)\,d\lambda$$

$$+ O\bigg(\sum_{q\le P} \frac{1}{\phi^3(q)}\,\phi(q)\cdot\phi^2(q) \int_{-1/qQ}^{1/qQ} (1+\lambda N)^2 \min(N,1/\lambda)\,d\lambda\bigg)$$

$$= \frac{1}{4}\sum_{q\le P} \frac{\mu(q)}{\phi^3(q)} \sum_{a=1,\,(a,q)=1}^{q} e\Big(-\frac{an}{q}\Big) C^2(a,q)$$

$$\times \bigg(\int_{-1/2}^{1/2} \Big(\sum_{M\le m\le N} e(m\lambda)m^{-1/2}\Big)^2 \Big(\sum_{M\le m\le N} e(m\lambda)\Big) e(-n\lambda)\,d\lambda$$

$$- 2 \int_{1/qQ}^{1/2} \Big(\sum_{M\le m\le N} e(m\lambda)m^{-1/2}\Big)^2 \Big(\sum_{M\le m\le N} e(m\lambda)\Big) e(-n\lambda)\,d\lambda\bigg)$$

$$+ O(N^2/Q^2).$$

By Vinogradov's upper bound (see [8, Ch. VI, Problem $14b(\alpha)$])

$$|\mathsf{C}(\chi,a)| \le 2q^{1/2}d(q)$$

and $M^{-1}P^2Q^2 = NL^{-14}$, the second sum in $(*)$ is, according to the earlier estimate for $V_1(\lambda)$ and $V_2(\lambda)$,

$$\ll \sum_{q\le P} \frac{\mu^2(q)}{\phi^3(q)} \sum_{\substack{a\le q\\(a,q)=1}} |C^2(a,q)| \int_{1/qQ}^{1/2} M^{-1}\lambda^{-3}\,d\lambda$$

$$\ll \sum_{q\le P} \frac{\mu^2(q)}{\phi^2(q)}\, qd^2(q) M^{-1}P^2Q^2 \ll NL^{-1}.$$

Hence

$$I_1 = \frac{1}{4}\sum_{q\le P} \frac{\mu(q)}{\phi^3(q)} \sum_{\substack{a\le q\\(a,q)=1}} e\Big(-\frac{an}{q}\Big) C^2(a,q)$$

$$\times \sum_{M<m_1\le N}\ \sum_{M<m_2<n-M-m_1} (m_1m_2)^{-1/2} + O(NL^{-1})$$

$$= \frac{1}{4}\,\pi\sigma_1(n)n + O(NL^{-1}),$$

where $\sigma_1(n)$ is defined in (2.1). Therefore,

$$\int_{\mathbf{M}} S^2(\alpha)T(\alpha)e(-n\alpha)\,d\alpha = \frac{1}{4}\,\pi\sigma_1(n)n + O(NL^{-1}).$$

In the following, we want to show that $\int_{\mathbf{M}} S_1^2(\alpha)T_1(\alpha)e(-n\alpha)\,d\alpha$ can be replaced by $\int_{\mathbf{M}} S^2(\alpha)T(\alpha)e(-n\alpha)\,d\alpha$. Indeed,

$$\begin{aligned}&\int_{\mathbf{M}} (S^2(\alpha)T(\alpha) - S_1^2(\alpha)T_1(\alpha))e(-n\alpha)\,d\alpha\\ &\qquad \ll \int_{\mathbf{M}} |S_1^2(\alpha)|\,|T(\alpha) - T_1(\alpha)|\,d\alpha + \int_0^1 |S^2(\alpha) - S_1^2(\alpha)|\,|T(\alpha)|\,d\alpha\\ &\qquad =: H_1 + H_2.\end{aligned}$$

From Cauchy's inequality, we have

$$H_1 \le \Big(\int_0^1 \Big|\sum_{p\le M} \log p\, e(\alpha p)\Big|^2\,d\alpha\Big)^{1/2}\Big(\int_0^1 |S_1(\alpha)|^4\,d\alpha\Big)^{1/2} =: H_{11}^{1/2}H_{12}^{1/2},$$

where

$$H_{11} = \int_0^1 \Big|\sum_{p\le M} \log p\, e(\alpha p)\Big|^2\,d\alpha \ll L^2M,$$

$$H_{12} = \int_0^1 |S_1(\alpha)|^4\,d\alpha \ll L^4 Z(N),$$

where $Z(N)$ is the number of solutions of the equation

$$p_1^2 + p_2^2 = p_3^2 + p_4^2 \tag{2.6}$$

and $p_j \le N$, $1 \le j \le 4$, are all primes. From [7, Satz 3], the number of p_1, p_2, p_3, p_4 satisfying (2.6) with $p_1p_2 \ne p_3p_4$ is $O(NL^{-3})$. From the prime number theorem, (2.6) has $O(NL^{-2})$ trivial solutions. Hence

$$H_{12} \le NL^2. \tag{2.7}$$

Therefore

$$H_1 \le (ML^2)^{1/2}(NL^2)^{1/2} \ll NL^{-5}.$$

We have

$$\begin{aligned}H_2 &\le \Big(\int_0^1 |S(\alpha) - S_1(\alpha)|^4\,d\alpha\Big)^{1/4}\Big(\int_0^1 |S(\alpha) + S_1(\alpha)|^4\,d\alpha\Big)^{1/4}\\ &\qquad \times \Big(\int_0^1 |T(\alpha)|^2\,d\alpha\Big)^{1/2}\\ &=: H_{21}^{1/2}H_{22}^{1/4}H_{23}^{1/4},\end{aligned}$$

$$H_{21} = \int_0^1 |T(\alpha)|^2 \, d\alpha \ll L^2 N,$$
$$H_{22} = \int_0^1 |S(\alpha) - S_1(\alpha)|^4 \, d\alpha \ll L^4 Z(M),$$

where $Z(M)$ is defined in (2.6). From (2.7) we have

$$H_{22} \ll ML^2 = NL^{-12},$$
$$H_{23} = \int_0^1 |S(\alpha) + S_1(\alpha)|^4 \, d\alpha \ll \int_0^1 |S(\alpha)|^4 \, d\alpha + \int_0^1 |S_1(\alpha)|^4 \, d\alpha.$$

Since

$$\int_0^1 |S(\alpha)|^4 \, d\alpha = \sum_{\substack{p_1^2+p_2^2=p_3^2+p_4^2 \\ M<p_i^2\le N}} \log p_1 \cdots \log p_4 \le \sum_{\substack{p_1^2+p_2^2=p_3^2+p_4^2 \\ p_i^2\le N}} \log p_1 \cdots \log p_4$$
$$= \int_0^1 |S_1(\alpha)|^4 \, d\alpha,$$

we have

$$H_{23} \ll \int_0^1 |S_1(\alpha)|^4 \, d\alpha = \sum_{\substack{p_1^2+p_2^2=p_3^2+p_4^2 \\ p_i^2\le N}} \log p_1 \cdots \log p_4 \ll L^4 Z(N) \ll NL^2.$$

Hence

$$H_2 \le H_{21}^{1/2} H_{22}^{1/4} H_{23}^{1/4} \ll (NL^2)^{1/2} (NL^{-12})^{1/4} (NL^2)^{1/4} \ll NL^{-1},$$
$$\int_{\mathbf{M}} (S^2(\alpha)T(\alpha) - S_1^2(\alpha)T_1(\alpha)) e(-n\alpha) \, d\alpha = O(NL^{-1}).$$

Therefore we have

$$\int_{\mathbf{M}} S_1^2(\alpha)T_1(\alpha)e(-n\alpha) \, d\alpha = \int_{\mathbf{M}} S^2(\alpha)T(\alpha)e(-n\alpha) \, d\alpha + O(NL^{-1}).$$

Let

$$\Xi(N,k) = \{n : n = N - (2^{\nu_1} + \cdots + 2^{\nu_k})\},$$
$$A_1 = \{n = N - 2^{\nu_1} - 2^{\nu_2} - \cdots - 2^{\nu_k} : \nu_i \le \log_2(N/kL),\ 1 \le i \le k\}.$$

We have

$$\sum_{n\in\Xi(N,k)} \int_{\mathbf{M}} T_1(\alpha)S_1^2(\alpha)e(-n\alpha) \, d\alpha = \frac{\pi}{4} \sum_{n\in\Xi(N,k)} \sigma_1(n)n + O(NL^{k-1}).$$

Therefore

$$\sum_{n\in\Xi(N,k)}\int_{\mathbf{M}} T_1(\alpha)S_1^2(\alpha)e(-n\alpha)\,d\alpha \tag{2.8}$$
$$\begin{aligned}
&= \frac{\pi}{4}\sum_{n\in\Xi(N,k)}\sigma_1(n)n + O(NL^{k-1})\\
&\ge \frac{\pi}{4}\sum_{n\in A_1}\sigma_1(n)n + O(NL^{k-1})\\
&= \frac{\pi}{4}\sum_{n\in A_1}\sigma_1(n)(N-2^{\nu_1}-\cdots-2^{\nu_k}) + O(NL^{k-1})\\
&\ge \frac{\pi}{4}N(1-L^{-1})\sum_{n\in A_1}\sigma_1(n) + O(NL^{k-1})\\
&\ge \frac{\pi}{4}N(1-\delta)\sum_{n\in A_1}\sigma_1(n) + O(NL^{k-1}),
\end{aligned}$$

where δ is a sufficiently small positive number. Now, we will mainly deal with $\sigma_1(n)$.

3. The singular series. We need the following lemmas:

Lemma 3.1 (see [5, Lemma 3]). *Let $\eta < 1/7e$. The set $\mathcal{E}$ of $\alpha\in[0,1]$ for which $|G(\alpha)|\ge(1-\eta)L$ has measure $\le L^{5/2}N^{\Theta-1}$, where*

$$\begin{aligned}\Theta = \Theta(\eta) &= \frac{1}{\log 2}\,\eta\csc^2\Big(\frac{\pi}{8}\Big)\log\Big(\frac{1}{\eta\csc^2(\pi/8)}\Big)\\
&\quad+\frac{1}{\log 2}\Big(1-\eta\csc^2\Big(\frac{\pi}{8}\Big)\Big)\log\Big(\frac{1}{1-\eta\csc(\pi/8)}\Big).\end{aligned}$$

Lemma 3.2 (see [5, Lemma 4]). *If α is a rational number with odd denominator q satisfying $1<\xi(q)<L$, then*

$$|G(\alpha)|\le\Big(1-\frac{1}{\xi(q)\csc^2(\pi/8)}+\frac{2}{L}\Big)L,$$

where $\xi(q)$ denotes the least positive integer ξ satisfying

$$2^{\xi}\equiv 1 \pmod q$$

for odd q.

Lemma 3.3. *Let $A(q)=\prod_{p\mid q}A(p)$, where p is prime and $A(p)$ satisfies*

$$A(p)=\begin{cases}\sqrt{p}+1 & \text{if } p\equiv 1 \pmod 4,\\ \sqrt{p+1} & \text{if } p\equiv -1 \pmod 4.\end{cases}$$

Then

$$\sum_{\xi(q)\le x} \frac{\mu^2(q)}{\phi^3(q)} A^2(q)q \le c_1 \log^2 x$$

and

$$\sum_{\xi(q)\le x} \frac{\mu^2(q)}{\phi^2(q)} A^2(q) \le c_2 \log^{1.5} x,$$

where $c_1 = 5.287076611,\ c_2 = 3.803.$

Proof. Let

$$X = \prod_{\xi\le x}(2^{\xi}-1).$$

If $\xi(q)\le x$, then $q\,|\,X$, and obviously $2\nmid X$, $X\le 2^{x^2}$. We have

$$\sum_{\xi(q)\le x} \frac{\mu^2(q)}{\phi^3(q)} A^2(q)q \le \sum_{q|X} \frac{\mu^2(q)}{\phi^3(q)} A^2(q)q \le \prod_{p|X}\left(1+\frac{A^2(p)}{\phi^3(p)}\,p\right).$$

When $p\ge 16$,

$$1+\frac{A^2(p)}{(p-1)^3}\,p < 1+\frac{2}{p-1}.$$

By Lemma 3.3, we have

$$\begin{aligned}\prod_{3\le p|X}\left(1+\frac{A^2(p)p}{(p-1)^3}\right) &\le \frac{5}{4}\prod_{3\le p|X}\left(1+\frac{2}{p-1}\right)\\ &= \prod_{p|2X}\left(1+\frac{2}{p-1}\right)\cdot\frac{5}{12} \le \prod_{p|2X}\left(1+\frac{1}{p-1}\right)^2\cdot\frac{5}{12}\\ &= \frac{(2X)^2}{\phi^2(2X)}\cdot\frac{5}{12} \le \frac{5}{12}\cdot 4e^{2\gamma}\log^2 x =: c_1\log^2 x,\end{aligned}$$

since by Lemma 5 of [5], we have

$$\frac{2X}{\phi(2X)} < 2e^{\gamma}\log x. \tag{3.1}$$

Note that

$$1.7810 < e^{\gamma} < 1.78108. \tag{3.2}$$

So we have

$$\sum_{\xi(q)\le x} \frac{\mu^2(q)A^2(q)q}{\phi^3(q)} < c_1\log^2 x,$$

where

$$c_1 = 5.287076611.$$

We have

$$\sum_{\xi(q)\le n} \frac{\mu^2(q)}{\phi^2(q)}\, A^2(q) \le \sum_{q|X} \frac{\mu^2(q)}{\phi^2(q)}\, A^2(q) = \prod_{p|X}\left(1+\frac{A^2(p)}{\phi^2(p)}\right),$$

and for $p \ge 25$,

$$1+\frac{A^2(p)}{(p-1)^2} \le 1+\frac{1.5}{p-1}.$$

Thus

$$\begin{aligned}\prod_{3\le p|X}\left(1+\frac{A^2(p)}{\phi^2(p)}\right) &\le \prod_{3\le p|X}\left(1+\frac{1.5}{p-1}\right)\cdot 1.413867968\\ &= \prod_{p|2X}\left(1+\frac{1.5}{p-1}\right)\cdot\frac{1}{2.5}\cdot 1.413867968\\ &\le \prod_{p|2X}\left(1+\frac{1}{p-1}\right)^{1.5}\cdot\frac{1}{2.5}\cdot 1.413867968\\ &< 3.803\log^{1.5} x,\end{aligned}$$

where we have used (3.1) and (3.2).

LEMMA 3.4 (see [5, Lemma 6]). *For odd q and $k \ge 2$, we have*

$$r_{k,k}(0) \le 2L^{2k-2}, \qquad \sum_{q|n} r_{k,k}(n) \le L^{2k-1}\left(1+\frac{L}{\xi(q)}\right),$$

where $r_{k,k}(n)$ is the number of ways to write n as

$$n = 2^{\nu_1}+\dots+2^{\nu_k}-2^{\mu_1}-\dots-2^{\mu_k} \qquad (1\le \nu_i,\mu_i \le L).$$

Consider the sum

$$\sum = \sum_{\substack{3\le q\le R\\ 2\nmid q}} \frac{\mu^2(q)}{\phi^3(q)} \sum_{\substack{a=1\\ (a,q)=1}}^{q} |C^2(a,q)|\,|G^k(a/q)|.$$

According to the length of $\xi(q)$, we obtain

$$\sum = \Bigg(\sum_{\substack{3\le q\le R\\ \xi(q)\le E}} + \sum_{\substack{3\le q\le R\\ \xi(q)> E}}\Bigg)\frac{\mu^2(q)}{\phi^3(q)} \sum_{\substack{a=1\\ (a,q)=1}}^{q} |C^2(a,q)|\,|G^k(a/q)|, \tag{3.3}$$

where E is a constant $\le L$.

By Lemmas 3.2 and 3.3, the first sum on the right hand side of (3.3) is

$$\begin{aligned}\sum_{\substack{3\le q\le R\\ \xi(q)\le E}} &\le L^k\left(1-\frac{1}{E\csc^2(\pi/8)}\right)^k \sum_{\substack{3\le q\le R\\ \xi(q)\le E}} \frac{\mu^2(q)}{\phi^2(q)}\,A^2(q)\\ &\le c_2\log^{1.5}E\left(1-\frac{1}{E\csc^2(\pi/8)}\right)^k L^k.\end{aligned}$$

Remark. $C(a,q)$ is multiplicative, so for $q = p_1p_2\cdots p_s$, $C(a,q) = C(a_1,p_1)\cdots C(a_s,p_s)$, with $a_i = aq/p_i$. It is easy to see that when $p_i \equiv 1 \pmod 4$, we have $|C(a_i,p_i)| = |\pm\sqrt{p_i}-1| \le \sqrt{p_i}+1$, while when $p_i \equiv -1 \pmod 4$, $|C(a_i,p_i)| = |\sqrt{-p_i}-1| \le \sqrt{p_i}+1$. Thus we have

$$|C(a_i,p_i)| \le A(p_i), \qquad |C(a,q)| \le \prod_{p|q} A(p) = A(q).$$

For the last sum in (3.3), we use Lemma 3.4. When $k = 2m$,

$$\begin{aligned}\sum_{\substack{a=1\\ (a,q)=1}}^{q} |G(a/q)|^k = \sum_{\substack{a=1\\ (a,q)=1}}^{q} |G(a/q)|^{2m} &\le \sum_{a=1}^{q} |G(a/q)|^{2m} = q\sum_{q|n} r_{m,m}(n)\\ &\le qL^{2m-1}\left(1+\frac{L}{\xi(q)}\right) = qL^{k-1} + \frac{qL^k}{\xi(q)}.\end{aligned}$$

When $k = 2m+1$,

$$\begin{aligned}\sum_{\substack{a=1\\ (a,q)=1}}^{q} |G(a/q)|^k = \sum_{\substack{a=1\\ (a,q)=1}}^{q} |G(a/q)|^{2m+1} &\le L \sum_{\substack{a=1\\ (a,q)=1}}^{q} |G(a/q)|^{2m}\\ &\le L\left(qL^{2m-1}\left(1+\frac{L}{\xi(q)}\right)\right) = qL^{k-1} + \frac{qL^k}{\xi(q)}.\end{aligned}$$

Thus we have

$$\sum_{\substack{a=1\\ (a,q)=1}}^{q} |G(a/q)|^k \le qL^{k-1} + \frac{qL^k}{\xi(q)},$$

and so

$$\begin{aligned}\sum_{\substack{3\le q\le R\\ \xi(q)>E}} &\le \sum_{\substack{3\le q\le R\\ \xi(q)>E}} \frac{\mu^2(q)}{\phi^3(q)} \sum_{\substack{a=1\\ (a,q)=1}}^{q} |C^2(a,q)|\,|G^k(a/q)|\\ &\le \sum_{\substack{3\le q\le R\\ \xi(q)>E}} \frac{\mu^2(q)}{\phi^3(q)}\,A^2(q) \sum_{\substack{a=1\\ (a,q)=1}}^{q} |G^k(a/q)|\end{aligned}$$

$$\leq L^{k-1} \sum_{3\leq q\leq R} \frac{\mu^2(q)}{\phi^3(q)} A^2(q)q + L^k \sum_{\substack{3\leq q\leq R\\ \xi(q)>E}} \frac{\mu^2(q)}{\phi^3(q)} A^2(q) \frac{q}{\xi(q)}.$$

By Lemma 3.3, the first sum on the right hand side above is $\ll \log^2 R$. The second sum is

$$\leq \sum_{m>E} \frac{1}{m} \sum_{\xi(q)=m} \frac{\mu^2(q)}{\phi^3(q)} A^2(q)q \leq \int_E^\infty \frac{1}{t^2}\bigg(\sum_{\xi(q)\leq t} \frac{\mu^2(q)}{\phi^3(q)} A^2(q)q\bigg)\, dt$$
$$\leq c_1 \int_E^\infty \frac{\log^2 t}{t^2}\, dt = c_1\bigg(\frac{\log^2 E}{E} + \frac{2\log E}{E} + \frac{2}{E}\bigg).$$

Hence

$$\sum_{\substack{3\leq q\leq R\\ \xi(q)>E}} \leq c_1 L^k\bigg(\frac{\log^2 E}{E} + \frac{2\log E}{E} + \frac{2}{E}\bigg) + O(L^{k-1}\log^2 R).$$

Therefore

$$\sum \leq c_1 L^k\bigg(\frac{\log^2 E}{E} + \frac{2\log E}{E} + \frac{2}{E}\bigg) \tag{3.4}$$
$$+ c_2 \log^{1.5} E\bigg(1 - \frac{1}{E\csc^2(\pi/8)}\bigg)^k L^k + O(L^{k-1}\log^2 R).$$

Define

$$A(n,q) := \frac{\mu(q)}{\phi^3(q)} \sum_{\substack{a=1\\ (a,q)=1}}^{q} C^2(a,q)e\bigg(-\frac{an}{q}\bigg),$$

$$\sigma_1(n) = \sum_{q\leq P} A(n,q), \qquad \sigma_0(n) = \sum_{q=1}^{\infty} A(n,q), \qquad \sigma(n) = \sum_{q\leq R} A(n,q).$$

We will prove that

$$|A(n,p)| \leq \begin{cases} \dfrac{3p+1}{(p-1)^3} & \text{if } p\nmid n,\\ \dfrac{p^2-1}{(p-1)^3} & \text{if } p\mid n. \end{cases}$$

We have

$$C(a,p) = \chi(a)S(p,1) - 1,$$

where $(a,p)=1$, $\chi(a) = \big(\frac{a}{p}\big)$ is the Legendre symbol and

$$S(p,1) = \sum_{h=1}^{p} e\bigg(\frac{h^2}{p}\bigg)$$

is the Gauss sum. Here $S(p,1)$ satisfies

$$S(p,1)=\begin{cases}\sqrt{p} & \text{if } p\equiv 1 \pmod 4,\\ i\sqrt{p} & \text{if } p\equiv -1 \pmod 4.\end{cases}\tag{3.5}$$

Therefore, we have

$$\begin{aligned}\sum_{a=1}^{p-1} C^2(a,p)e\Big(-\frac{an}{p}\Big) &= \sum_{a=1}^{p-1}(\chi(a)S(p,1)-1)^2 e\Big(-\frac{an}{p}\Big)\\ &= (S^2(p,1)+1)\sum_{a=1}^{p-1} e\Big(-\frac{an}{p}\Big) - 2S(p,1)\sum_{a=1}^{p-1}\chi(a)e\Big(-\frac{an}{p}\Big)\\ &=: \mathbf{S}_1-\mathbf{S}_2.\end{aligned}\tag{3.6}$$

For $\mathbf{S}_1$, we need the following result:

$$\sum_{a=1}^{p-1} e\Big(-\frac{an}{p}\Big)=\begin{cases} p-1 & \text{if } p\,|\,n,\\ -1 & \text{if } p\nmid n.\end{cases}$$

By this result and (3.5), we obtain

$$\mathbf{S}_1=\begin{cases} p^2-1, & p\equiv 1 \pmod 4,\ p\,|\,n,\\ -(p-1)^2, & p\equiv -1 \pmod 4,\ p\,|\,n,\\ -(p+1), & p\equiv 1 \pmod 4,\ p\nmid n,\\ p-1, & p\equiv -1 \pmod 4,\ p\nmid n.\end{cases}\tag{3.7}$$

For $\mathbf{S}_2$, if $p\,|\,n$, then $\sum_{a=1}^{p-1}\chi(a)=0$; if $p\nmid n$, let

$$F(n)=\sum_{a=1}^{p}\Big(\frac{a}{p}\Big)e\Big(\frac{an}{p}\Big).$$

Then obviously we have

$$F(n)=\Big(\frac{n}{p}\Big)F(1)$$

and

$$\begin{aligned}S(p,1) &= \sum_{m=1}^{p} e\Big(\frac{m^2}{p}\Big)=\sum_{m=1}^{p-1} e\Big(\frac{m^2}{p}\Big)+1\\ &= \sum_{a=1}^{p-1}\Big(1+\Big(\frac{a}{p}\Big)\Big)e\Big(\frac{a}{p}\Big)+1=\sum_{a=1}^{p}\Big(\frac{a}{p}\Big)e\Big(\frac{a}{p}\Big)=F(1).\end{aligned}$$

Hence

$$\mathbf{S}_2=2S^2(p,1)\Big(-\frac{n}{p}\Big).\tag{3.8}$$

From (3.5), (3.6) and (3.8), we get

$$\sum_{a=1}^{p-1} C^2(a,p)e\left(-\frac{an}{p}\right)$$

$$= \begin{cases} \mathbf{S}_1, & p \mid n, \\ -(p+1) - 2p = -3p - 1, & \left(-\frac{n}{p}\right) = 1,\ p \equiv 1 \pmod 4, \\ (p-1) + 2p = 3p - 1, & \left(-\frac{n}{p}\right) = 1,\ p \equiv -1 \pmod 4, \\ -(p+1) + 2p = p - 1, & \left(-\frac{n}{p}\right) = -1,\ p \equiv 1 \pmod 4, \\ (p-1) - 2p = -(p+1), & \left(-\frac{n}{p}\right) = -1,\ p \equiv -1 \pmod 4. \end{cases}$$

Therefore, we have

$$|A(n,p)| \le \begin{cases} \dfrac{3p+1}{(p-1)^3} & \text{if } p \nmid n, \\ \dfrac{p^2-1}{(p-1)^3} & \text{if } p \mid n. \end{cases}$$

Thus

$$|A(n,q)| \le 2 \prod_{\substack{p \mid q \\ p \nmid n}} \frac{25}{p^2} \prod_{\substack{p \mid q \\ p \mid n}} \frac{25}{p} = 2 \prod_{p \mid q} \frac{25}{p^2} \prod_{\substack{p \mid q \\ p \mid n}} p \ll q^{-2+\varepsilon}(q,n).$$

Therefore

$$\sum_{q>x} |A(n,q)| \ll \sum_{s \mid n} s \sum_{st>x} (st)^{-2+\varepsilon} \ll x^{-1+\varepsilon} d(n).$$

Thus,

$$\sigma_1(n) = \sigma_0(n) + O(P^{-1+\varepsilon}), \qquad \sigma_1(n) = \sigma(n) + O(R^{-1+\varepsilon}).$$

Obviously, the function of q

$$\frac{\mu(q)}{\phi^3(q)} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} C^2(a,q)e\left(-\frac{an}{q}\right)$$

is multiplicative, therefore we have

$$\sigma_0(n) = \sum_{q=1}^{\infty} \frac{\mu(q)}{\phi^3(q)} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} C^2(a,q)e\left(-\frac{an}{q}\right)$$

$$= \prod_{p} \left(1 + \frac{1}{\phi^3(p)} \sum_{a=1}^{p-1} C^2(a,p)e\left(-\frac{an}{p}\right)\right).$$

This infinite series is positive and has positive partial sums, it must converge to a positive constant, say c, i.e.

$$\sigma_0(n) = \prod_p (1 + A(n,p)) = c > 0.$$

When n is sufficiently large, we have $\sigma_1(n) > 0$, $\sigma(n) > 0$. Thus, we could replace $\sigma_1(n)$ by $\sigma(n)$. Take

$$\varepsilon = \frac{\log\log\log R}{\log N}, \qquad R = o(N).$$

We have

$$\sum_{n\in A_1} \sigma_1(n) = \sum_{n\in A_1} \sigma(n) + O(L^k R^{-1+\varepsilon}).$$

Define

$$\begin{aligned}\sum_{n\in A_1} \sigma(n) &= \sum_{q\le R} \frac{\mu^2(q)}{\phi^3(q)} \sum_{\substack{a=1\\(a,q)=1}}^{q} C^2(a,q) e\Big(-\frac{aN}{q}\Big)\Bigg(\sum_{\nu=1}^{\log_2(N/kL)} e\Big(\frac{a}{q}2^\nu\Big)\Bigg)^k \\ &=: \sum_{q\le R} F(q,N).\end{aligned}$$

When $q = 1, 2$, $F(q,N)$ contributes $2L^k$ to the above sum. Thus,

$$\begin{aligned}\sum_{n\in A_1} \sigma(n) &= 2L^k + \sum_{3\le q\le R} F(q,N) \\ &= 2L^k + \sum_{\substack{3\le q\le R\\ 2\nmid q}} F(q,N) + \sum_{\substack{3\le q\le R/2\\ 2\nmid q}} F(q,N) \\ &=: 2L^k + \textstyle\sum_1 + \sum_2.\end{aligned} \tag{3.9}$$

From (3.4), we have

$$\begin{aligned}|\textstyle\sum_1 + \sum_2| \le{}& 2c_1 L^k \Big(\frac{\log^2 E}{E} + \frac{2\log E}{E} + \frac{2}{E}\Big) \\ &+ 2c_2 \log^{1.5} E \Big(1 - \frac{1}{E\csc^2(\pi/8)}\Big)^k L^k \\ &+ O(L^{k-1}\log^2 R),\end{aligned} \tag{3.10}$$

where $c_1 = 5.287076611$, $c_2 = 3.803$. If we take $R = \exp(\sqrt{\log N}/\log\log N)$, the O-term above is $O(L^k(\log\log N)^{-2})$.

4. The proof of Theorem 1

LEMMA 4.1 (see [3, Lemma 5.1]). *For p a prime,*

$$T(\alpha)=\sum_{p\le N}\log p\, e(\alpha p),\qquad S(\alpha)=\sum_{p^2\le N}\log p\, e(\alpha p^2),$$

$$G(\alpha)=\sum_{\nu\le\log_2 N} e(\alpha 2^\nu),$$

we have

$$\int_0^1 |T(\alpha)G(\alpha)|^4\,d\alpha \le c_5\,\frac{\pi^2}{16}\,NL^4,$$

where

$$c_5\le\left(\frac{11^4\cdot 43\cdot\pi^{24}}{2^{24}\cdot 5^2}+\frac{8}{\pi^2}\log^2 2\right)(1+\varepsilon)^9.$$

LEMMA 4.2 (see [5, Lemma 9]). *We have*

$$\int_0^1 |S(\alpha)G(\alpha)|^2\,d\alpha \le \frac{2}{\log^2 2}\,c_3NL^2,$$

where $c_3<17.2435$.

REMARK. It follows from the proofs of the above lemmas that they remain true if $T(\alpha)$ and $S(\alpha)$ are replaced by $T_1(\alpha)$ and $S_1(\alpha)$ respectively.

From (2.8), (3.9) and (3.10), we have

$$\int_{\mathbf{M}} \ge \left(\frac{\pi}{2}-\delta\right)NL^k+\left(\sum\nolimits_1+\sum\nolimits_2\right), \tag{4.1}$$

where

$$\left|\sum\nolimits_1+\sum\nolimits_2\right|\le 2c_1NL^k\left(\frac{\log^2 E}{E}+2\,\frac{\log E}{E}+\frac{2}{E}\right) + 2c_2\log^{1.5}E\left(1-\frac{1}{E\csc^2(\pi/8)}\right)^k. \tag{4.2}$$

Let

$$r_k(N)=\int_0^1 T_1(\alpha)S_1^2(\alpha)G^k(\alpha)e(-N\alpha)\,d\alpha=\int_{\mathbf{M}}+\int_{C(\mathbf{M})\cap\mathcal{E}}+\int_{C(\mathbf{M})\cap C(\mathcal{E})}$$

with the set $\mathcal{E}$ from Lemma 3.1.

In order to estimate the minor arcs, we first estimate the second integral. For $\alpha\in C(\mathbf{M})$, $\alpha=a/q+\lambda$, $1\le a\le q$, $(a,q)=1$, where $P\le q\le Q$, we use Theorem 3 of [1] to obtain

$$S_1(\alpha)\ll N^{1/2+\varepsilon}(P^{-1}+N^{-1/4}+QN^{-1})^{1/4}\ll N^{1/2-1/30}.$$

Take $\eta = 1/868$, so Θ in Lemma 3.1 is $< 1/15$. Thus the second integral can be estimated as

$$\int\limits_{C(\mathbf{M})\cap\mathcal{E}} \le N^{\Theta-1}N^{1-1/15+4\varepsilon}L^kN \ll NL^{-1+k}. \tag{4.3}$$

The last integral can be estimated as

$$\int\limits_{C(\mathbf{M})\cap C(\mathcal{E})} \le ((1-\eta)L)^{k-3}\Big(\int\limits_0^1 |T_1(\alpha)G(\alpha)|^2\,d\alpha\Big)^{1/2}\Big(\int\limits_0^1 |S_1(\alpha)G(\alpha)|^4\,d\alpha\Big)^{1/2}. \tag{4.4}$$

From (4.4), Lemma 4.1 and Lemma 4.2, we have

$$\int\limits_{C(\mathbf{M})\cap C(\mathcal{E})} \le 23667416(1-\eta)^{k-3}(NL^k). \tag{4.5}$$

Take $E = 400$; when $k \ge 22000$, from (4.1)–(4.3) and (4.5), we have

$$\begin{aligned} r_k(N) &\ge \Big(\frac{\pi}{2} - 2\delta\Big)NL^k - 23667416(1-\eta)^{k-3}NL^k \\ &\quad - NL^k\bigg(2c_1NL^k\Big(\frac{\log^2 E}{E} + 2\,\frac{\log E}{E} + \frac{2}{E}\Big) \\ &\quad + 2c_2\log^{1.5} E\Big(1 - \frac{1}{E\csc^2(\pi/8)}\Big)^k\bigg) > 0. \end{aligned}$$

Thus the proof of Theorem 1 is complete.

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