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Pair correlation of the zeros of the Riemann zeta function in longer ranges

by

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1. Introduction. We assume the Riemann Hypothesis (RH) for the Riemann zeta function $\zeta(s)$ throughout this paper, thus $\varrho = 1/2+i\gamma$ denotes a non-trivial zero of the Riemann zeta function.

In the early 1970s, Hugh Montgomery considered the pair correlation function

$$F(x,T) = \sum_{0<\gamma,\gamma'\le T} x^{i(\gamma-\gamma')} w(\gamma-\gamma') \quad \text{with } w(u) = \frac{4}{4+u^2}.$$

Here the sum is a double sum over the imaginary parts of the non-trivial zeros of $\zeta(s)$. He proved in [9] that, as $T\to\infty$,

$$F(x,T) \sim \frac{T}{2\pi}\log x + \frac{T}{2\pi x^2}\log^2 T$$

for $1\le x\le T$ (actually he only proved this for $1\le x\le o(T)$ and the full range was done by Goldston [5]). He conjectured that

$$F(x,T) \sim \frac{T}{2\pi}\log T$$

for $T\le x\le T^M$, M fixed, which is known as the Strong Pair Correlation Conjecture. From this, one has the (Weak) Pair Correlation Conjecture:

$$\sum_{\substack{0<\gamma,\gamma'\le T\\ 0<\gamma-\gamma'\le 2\pi\alpha/\log T}} 1 \sim \frac{T}{2\pi}\log T \int_0^\alpha \left[1-\left(\frac{\sin \pi u}{\pi u}\right)^2\right] du,$$

which draws connections with random matrix theory.

The author studied these further in his thesis [1] (see also [2] and [3]) and derived more precise asymptotic formulas for $F(x,T)$ when x is in various ranges under the Twin Prime Conjecture (TPC) (see Section 4). In the

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present paper, we generalize $F(x,T)$ further to

$$F_h(x,T) = \sum_{0<\gamma,\gamma'\le T} \cos((\gamma-\gamma'-h)\log x)w(\gamma-\gamma'-h).$$

Note that $F_h(x,T) = F_{-h}(x,T)$ and $F_0(x,T) = F(x,T)$. This leads to a better understanding of the distribution of larger differences between the zeros. Our main results are the following theorems. Here and throughout the paper, $\widetilde{h} = |h| + 1$.

THEOREM 1.1. *For* $1 \le x \le T/\log T$,

$$F_h(x,T) = \frac{T}{2\pi}\left[\frac{4\cos(h\log x)}{4+h^2}\log x - \frac{8h\sin(h\log x)}{(4+h^2)^2}\right] + \frac{T}{2\pi x^2}\left[\left(\log\frac{T}{2\pi}\right)^2 - 2\log\frac{T}{2\pi}\right] + O(x\log x) + O\left(\frac{\widetilde{h}T}{x^{1/2-\varepsilon}}\right).$$

THEOREM 1.2. *Assume TPC. For* $M \ge 3$ *and* $T/\log^M T \le x$,

$$\begin{aligned} F_h(x,T) &= \frac{T}{\pi}\left[\frac{2\cos(h\log x)}{4+h^2}\log x - \frac{4h\sin(h\log x)}{(4+h^2)^2}\right] \\ &+ \frac{T}{\pi}\int_1^\infty\left[-\frac{2\cos(h\log x)}{4+h^2}\frac{1}{y} - \frac{4f(y)}{y^2}\cos(h\log x) + G_1(y) + G_2(y)\right] \\ &\times \frac{\sin\frac{Ty}{x}}{\frac{Ty}{x}}\,dy \\ &- \frac{x}{\pi}\int_0^{T/x}\frac{\sin u}{u}\,du\left[\frac{3\cos(h\log x)}{9+h^2} + \frac{h\sin(h\log x)}{9+h^2}\right] \\ &- \frac{x}{\pi}\int_0^{T/x}\frac{\sin u}{u}\,du\left[\frac{\cos(h\log x)}{1+h^2} - \frac{h\sin(h\log x)}{1+h^2}\right] \\ &+ \frac{T}{\pi}\sum_{k=1}^\infty\frac{\mathfrak{S}(k)}{k^2}\int_0^1 y\cos\left(h\log\frac{kx}{y}\right)\frac{\sin\frac{Ty}{x}}{\frac{Ty}{x}}\,dy \\ &+ O\left(\widetilde{h}\,\frac{x^{1+6\varepsilon}}{T}\right) + O(\widetilde{h}x^{1/2+7\varepsilon}) + O\left(\widetilde{h}\,\frac{x^2}{T^{2-2\varepsilon}}\right) + O\left(\frac{\widetilde{h}T}{\log^{M-2}T}\right), \end{aligned}$$

where $G_1(y)$ *and* $G_2(y)$ *are defined in Lemma* 4.2.

THEOREM 1.3. *Assume TPC. For $M \ge 3$ and $T/\log^M T \le x \le T$,*

$$F_h(x,T) = \frac{T}{\pi}\left[\frac{2\cos(h\log x)}{4+h^2}\log x - \frac{4h\sin(h\log x)}{(4+h^2)^2}\right] + O(\widetilde{h}x) + O\left(\frac{\widetilde{h}T}{\log^{M-2}T}\right).$$

THEOREM 1.4. *Assume TPC. For $M \ge 3$ and $T \le x \le T^{2-29\varepsilon}$,*

$$F_h(x,T) = \frac{T}{2\pi}\log\frac{T}{2\pi e}\left[\frac{4\cos(h\log x)}{4+h^2}\right] + O\left(\widetilde{h}T\left(\frac{T}{x}\right)^{1/2-\varepsilon}\right) + O\left(\frac{\widetilde{h}T}{\log^{M-2}T}\right).$$

For real α, let $F_h(\alpha) := \left(\frac{T}{2\pi}\log T\right)^{-1} F_h(T^\alpha, T)$. Then $F_h(\alpha) = F_h(-\alpha)$. Based on the above theorems, one may make the following

CONJECTURE 1.1. *For any arbitrarily large A and $h = o(\log^{1/3} T)$, as $T \to \infty$,*

$$F_h(\alpha) = \begin{cases} (1+o(1))T^{-2\alpha}\log T + \alpha\,\dfrac{4\cos(h\log T\alpha)}{4+h^2} + o(1) & \text{if } 0 \le \alpha \le 1, \\ \dfrac{4\cos(h\log T\alpha)}{4+h^2} + o(1) & \text{if } 1 \le \alpha \le A. \end{cases}$$

By convolving $F_h(\alpha)$ with an appropriate kernel $\widehat{r}(\alpha)$,

$$(1)\qquad \left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{0<\gamma,\gamma'\le T} r\left((\gamma-\gamma'-h)\frac{\log T}{2\pi}\right) w(\gamma-\gamma'-h) = \int_{-\infty}^{\infty} F_h(\alpha)\widehat{r}(\alpha)\,d\alpha$$

where $\widehat{r}(\alpha) = \int_{-\infty}^{\infty} r(u)e^{-2\pi i\alpha u}\,du$ for even $r(u)$ only. Conjecture 1.1 and (1) lead to

CONJECTURE 1.2. *For fixed $\alpha > 0$ and $h = o(\log^{1/3} T)$,*

$$\left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{\substack{0<\gamma\ne\gamma'\le T \\ |\gamma-\gamma'-h|\le 2\pi\alpha/\log T}} 1 \sim \int_{-\alpha+(h\log T)/(2\pi)}^{\alpha+(h\log T)/(2\pi)} \left[1 - \frac{4}{4+h^2}\left(\frac{\sin\pi u}{\pi u}\right)^2\right] du.$$

CONJECTURE 1.3. *For* $0 < \alpha < \beta \ll \log T$,

$$\left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{\substack{0<\gamma\neq\gamma'\leq T \\ 2\pi\alpha/\log T\leq\gamma-\gamma'\leq 2\pi\beta/\log T}} 1 \sim \int_{\alpha}^{\beta}\left[1 - \frac{1}{1+(\pi u/\log T)^2}\left(\frac{\sin \pi u}{\pi u}\right)^2\right] du.$$

2. Some lemmas

LEMMA 2.1 ([2, Lemma 2.2]). *We have, assuming RH, for* $x \geq 1$,

$$2\sum_{\gamma}\frac{x^{i(\gamma-t)}}{1+(t-\gamma)^2} = -\frac{1}{x}\sum_{n\leq x}\frac{\Lambda(n)}{n^{-1/2+it}} - x\sum_{n>x}\frac{\Lambda(n)}{n^{3/2+it}} + \frac{x^{1/2-it}}{1/2+it} + \frac{x^{1/2-it}}{3/2-it} + \frac{\log\tau}{x} + \frac{1}{x}\left[\frac{\zeta'}{\zeta}\left(\frac{3}{2}-it\right) - \log 2\pi\right] + O\left(\frac{1}{x\tau}\right), \tag{2}$$

where the sum is over all the imaginary parts of the zeros of the Riemann zeta function, and $\tau = |t| + 2$, *and* $\Lambda(n)$ *is von Mangoldt's lambda function.*

Write (2) as $\mathcal{L}(x,t) = \mathcal{R}(x,t)$. Let

$$P(x,T) = \frac{1}{x}\sum_{n\leq x}\frac{\Lambda(n)}{n^{-1/2+it}} + x\sum_{n>x}\frac{\Lambda(n)}{n^{3/2+it}} - \frac{x^{1/2-it}}{1/2+it} - \frac{x^{1/2-it}}{3/2-it},$$

$$Q(x,T) = \frac{\log\tau}{x}, \quad R(x,T) = \frac{1}{x}\left[\frac{\zeta'}{\zeta}\left(\frac{3}{2}-it\right)-\log 2\pi\right], \quad S(x,T) = O\left(\frac{1}{x\tau}\right).$$

LEMMA 2.2. *For* $x \geq 1$,

$$\int_0^T |\mathcal{L}(x,t)+\mathcal{L}(x,t-h)|^2\,dt = 2\pi F(x,T) + 2\pi F(x,T-h) + 4\pi F_h(x,T) + O(\log^3 T) + O(h\log^2 h).$$

Proof. This follows from page 188 of Montgomery [9] and the fact that $F(x,T) \ll T\log^2 T$.

LEMMA 2.3. *For $x \geq 1$,*

$$\int_0^T |\mathcal{R}(x,t)+\mathcal{R}(x,t-h)|^2\,dt = \int_0^T |P(x,t)+P(x,t-h)|^2\,dt + \frac{4T}{x^2}\bigg[\Big(\log\frac{T}{2\pi}\Big)^2 - 2\log\frac{T}{2\pi} + \Big(\frac{1}{2}\sum_{n=1}^{\infty}\frac{\Lambda^2(n)(1+\cos(h\log n))}{n^3}+2\Big)\bigg] + O(\widetilde{h}\log^2 T).$$

Proof. This is similar to the proof of Theorem 3.1 in [2].

LEMMA 2.4. *For $x \geq 1$,*

$$4\pi F_h(x,T) = \int_0^T |P(x,t)+P(x,t-h)|^2\,dt - \int_0^T |P(x,t)|^2\,dt - \int_0^T |P(x,t-h)|^2\,dt + \frac{2T}{x^2}\bigg[\Big(\log\frac{T}{2\pi}\Big)^2 - 2\log\frac{T}{2\pi} + \Big(\sum_{n=1}^{\infty}\frac{\Lambda^2(n)\cos(h\log n)}{n^3}+2\Big)\bigg] + O(\widetilde{h}\log^3 T).$$

Proof. This follows from Lemmas 2.2 and 2.3 as well as their special cases when $h=0$.

LEMMA 2.5. *For any sequence of complex numbers $\{a_n\}_{n=1}^{\infty}$ satisfying $\sum_{n=1}^{\infty} n|a_n|^2 < \infty$,*

$$\int_0^T \Big|\sum_{n=1}^{\infty} a_n n^{-it}\Big|^2\,dt = \sum_{n=1}^{\infty} |a_n|^2(T+O(n)).$$

Proof. This is Parseval's identity for Dirichlet series. See [10].

LEMMA 2.6. *Assuming RH and $x \geq 1$, we have*

$$\sum_{n\leq x}\Lambda^2(n)n = \frac{1}{2}x^2\log x - \frac{1}{4}x^2 + O(x^{1/2+\varepsilon}),$$

$$\sum_{n>x}\frac{\Lambda^2(n)}{n^3} = \frac{1}{2}\,\frac{\log x}{x^2} + \frac{1}{4x^2} + O\Big(\frac{1}{x^{5/2-\varepsilon}}\Big).$$

Proof. Use partial summation and the prime number theorem.

LEMMA 2.7. *For any real a and b not both zero,*

$$\int e^{ax}\sin bx\,dx = \frac{a}{a^2+b^2}\,e^{ax}\sin bx - \frac{b}{a^2+b^2}\,e^{ax}\cos bx,$$

$$\int e^{ax}\cos bx\,dx = \frac{a}{a^2+b^2}\,e^{ax}\cos bx + \frac{b}{a^2+b^2}\,e^{ax}\sin bx,$$

$$\int xe^{ax}\sin bx\,dx = \left[\frac{ax}{a^2+b^2} - \frac{a^2-b^2}{(a^2+b^2)^2}\right]e^{ax}\sin bx - \left[\frac{bx}{a^2+b^2} - \frac{2ab}{(a^2+b^2)^2}\right]e^{ax}\cos bx,$$

$$\int xe^{ax}\cos bx\,dx = \left[\frac{ax}{a^2+b^2} - \frac{a^2-b^2}{(a^2+b^2)^2}\right]e^{ax}\cos bx + \left[\frac{bx}{a^2+b^2} - \frac{2ab}{(a^2+b^2)^2}\right]e^{ax}\sin bx.$$

Proof. One can use the integrals $\int e^{(a+ib)x}\,dx$, $\int e^{(a-ib)x}\,dx$, $\int xe^{(a+ib)x}\,dx$ and $\int xe^{(a-ib)x}\,dx$, which are simple to compute.

LEMMA 2.8. *Assuming RH and $x \ge 1$, we have*

$$\frac{1}{x^2}\sum_{n\le x}\Lambda^2(n)n\cos(h\log n) = \frac{2\cos(h\log x)}{4+h^2}\log x + \frac{h^2-4}{(4+h^2)^2}\cos(h\log x) + \frac{h\sin(h\log x)}{4+h^2}\log x - \frac{4h}{(4+h^2)^2}\sin(h\log x) + O\left(\frac{\widetilde{h}}{x^{1/2-\varepsilon}}\right),$$

$$x^2\sum_{n>x}\frac{\Lambda^2(n)}{n}\cos(h\log n) = \frac{2\cos(h\log x)}{4+h^2}\log x - \frac{h^2-4}{(4+h^2)^2}\cos(h\log x) - \frac{h\sin(h\log x)}{4+h^2}\log x - \frac{4h}{(4+h^2)^2}\sin(h\log x) + O\left(\frac{\widetilde{h}}{x^{1/2-\varepsilon}}\right).$$

Proof. We shall prove the first formula. The other one is very similar. Let $A(x) = x^{-2}\sum_{n\le x}\Lambda^2(n)n$. By partial summation and Lemma 2.6,

$$\begin{aligned}\frac{1}{x^2}\sum_{n\le x}\Lambda^2(n)n\cos(h\log n) &= \frac{A(x)}{x^2}\cos(h\log x) + \frac{h}{x^2}\int_1^x A(u)\,\frac{\sin(h\log u)}{u}\,du\\ &= \left[\frac{1}{2}\log x - \frac{1}{4}\right]\cos(h\log x) + \frac{h}{x^2}\int_1^x\left[\frac{1}{2}\log u - \frac{1}{4}\right]u\sin(h\log u)\,du\\ &\quad + O\left(\frac{\widetilde{h}}{x^{1/2-\varepsilon}}\right)\end{aligned}$$

$$= \left[\frac{1}{2}\log x - \frac{1}{4}\right]\cos(h\log x) + \frac{h}{x^2}\left[\frac{1}{2}\int_0^{\log x} ve^{2v}\sin hv\,dv - \frac{1}{4}\int_0^{\log x} e^{2v}\sin hv\,dv\right] + O\left(\frac{\widetilde{h}}{x^{1/2-\varepsilon}}\right),$$

which gives the desired result after applying Lemma 2.7 with $a = 2$ and $b = h$, and some algebra.

3. Proof of Theorem 1.1. First, note that

$$P(x,t) = \frac{1}{x^{1/2}}\left[\sum_{n\le x}\Lambda(n)\left(\frac{x}{n}\right)^{-1/2+it} + \sum_{n>x}\Lambda(n)\left(\frac{x}{n}\right)^{3/2+it}\right] + O\left(\frac{x^{1/2}}{\tau}\right).$$

Thus,

$$P(x,t)+P(x,t-h) = \frac{1}{x^{1/2}}\left[\sum_{n\le x}\Lambda(n)(1+n^{ih})\left(\frac{x}{n}\right)^{-1/2+it} + \sum_{n>x}\Lambda(n)(1+n^{ih})\left(\frac{x}{n}\right)^{3/2+it}\right] + O\left(\frac{x^{1/2}}{\tau}\right).$$

So, the first integral in Lemma 2.4 is

$$\begin{aligned}
&\frac{1}{x}\int_0^T\left|\sum_{n\le x}\Lambda(n)(1+n^{ih})\left(\frac{x}{n}\right)^{-1/2+it} + \sum_{n>x}\Lambda(n)(1+n^{ih})\left(\frac{x}{n}\right)^{3/2+it}\right|^2 dt \\
&\quad + O\left(\left[\sum_{n\le x}\Lambda(n)\left(\frac{x}{n}\right)^{-1/2} + \sum_{n>x}\Lambda(n)\left(\frac{x}{n}\right)^{3/2}\right]\int_0^T\frac{1}{\tau^2}\,dt\right) + O\left(\int_0^T\frac{x}{\tau^4}\,dt\right) \\
&= \frac{1}{x}\sum_{n\le x}\Lambda^2(n)|1+n^{ih}|^2\left(\frac{x}{n}\right)^{-1}(T+O(n)) \\
&\quad + \frac{1}{x}\sum_{n>x}\Lambda^2(n)|1+n^{ih}|^2\left(\frac{x}{n}\right)^{3}(T+O(n)) + O(x) \\
&= \frac{2T}{x^2}\sum_{n\le x}\Lambda^2(n)n\,(1+\cos(h\log n)) + 2Tx^2\sum_{n>x}\frac{\Lambda^2(n)}{n^3}\,(1+\cos(h\log n)) \\
&\quad + O(x\log x).
\end{aligned}$$

Similarly (or by setting $h = 0$), each of the second and third integrals in Lemma 2.4 is

$$\frac{T}{x^2}\sum_{n\le x}\Lambda^2(n)n + Tx^2\sum_{n>x}\frac{\Lambda^2(n)}{n^3} + O(x\log x).$$

Therefore,

$$\begin{aligned}4\pi F_h(x,T) &= 2T\left[\frac{1}{x^2}\sum_{n\le x}\Lambda^2(n)n\cos(h\log n)+x^2\sum_{n>x}\frac{\Lambda^2(n)}{n^3}\cos(h\log n)\right]\\
&\quad+\frac{2T}{x^2}\left[\left(\log\frac{T}{2\pi}\right)^2-2\log\frac{T}{2\pi}\right]\\
&\quad+O\left(\frac{T}{x^2}\right)+O(\widetilde{h}\log^3 T)+O(x\log x)\\
&=2T\left[\frac{4\cos(h\log x)}{4+h^2}\log x-\frac{8h\sin(h\log x)}{(4+h^2)^2}\right]\\
&\quad+\frac{2T}{x^2}\left[\left(\log\frac{T}{2\pi}\right)^2-2\log\frac{T}{2\pi}\right]+O(x\log x)+O\left(\frac{\widetilde{h}T}{x^{1/2-\varepsilon}}\right)\end{aligned}$$

by Lemma 2.8. The theorem follows after dividing through by 4π.

4. Twin Prime Conjecture and smooth weight. We shall use a quantitative form of the Twin Prime Conjecture (TPC) as follows: For any $\varepsilon>0$,

$$\sum_{n=1}^{N}\Lambda(n)\Lambda(n+d)=\mathfrak{S}(d)N+O(N^{1/2+\varepsilon})\quad\text{uniformly in } |d|\le N.$$

Here

$$\mathfrak{S}(d)=\begin{cases}2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\prod_{p\mid d,\,p>2}\frac{p-1}{p-2} & \text{if } d \text{ is even},\\ 0 & \text{if } d \text{ is odd}.\end{cases}$$

Let K and M be some large positive integers (K may depend on ε). Set $U=\log^M T$ and $\Delta=1/(2^K U)$. We recall the smooth weight $\Psi_U(t)$ in [3] with:

1. support in $[-1/U, 1+1/U]$,
2. $0\le\Psi_U(t)\le 1$,
3. $\Psi_U(t)=1$ for $1/U\le t\le 1-1/U$,
4. $\Psi_U^{(j)}(t)\ll U^j$ for $j=1,\dots,K$.

This weight function satisfies the requirements in Goldston and Gonek [6]. One more thing to note is that

$$\operatorname{Re}\widehat{\Psi}_U(y)=\frac{\sin 2\pi y}{2\pi y}\left(\frac{\sin 2\pi\Delta y}{2\pi\Delta y}\right)^{K+1}$$

where $\widehat{f}(y)=\int_{-\infty}^{\infty}f(t)e(yt)\,dt$.

We also need to study

$$S_\alpha^h(y) := \sum_{k\le y} \mathfrak{S}(k)k^\alpha \cos\left(h\log\frac{kx}{y}\right) - \int_0^y u^\alpha \cos\left(h\log\frac{ux}{y}\right)du \quad \text{for } \alpha\ge 0,$$

$$T_\alpha^h(y) := \sum_{k>y} \frac{\mathfrak{S}(k)}{k^\alpha} \cos\left(h\log\frac{kx}{y}\right) - \int_y^\infty \frac{1}{u^\alpha} \cos\left(h\log\frac{ux}{y}\right)du \quad \text{for } \alpha> 1.$$

Then from [4],

$$S_0(y) := S_0^0(y) = -\frac{1}{2}\log y + O((\log y)^{2/3}) = -\frac{1}{2}\log y + \varepsilon(y). \tag{3}$$

By partial summation and Lemma 2.7, for $\alpha>0$,

$$\begin{aligned}S_\alpha^h(y) = \varepsilon(y)y^\alpha\cos(h\log x) - \frac{\alpha\cos(h\log x)}{2(\alpha^2+h^2)}\,y^\alpha - \frac{h\sin(h\log x)}{2(\alpha^2+h^2)}\,y^\alpha \\ -\int_0^y \varepsilon(u)u^{\alpha-1}\left[\alpha\cos\left(h\log\frac{ux}{y}\right) - h\sin\left(h\log\frac{ux}{y}\right)\right]du,\end{aligned} \tag{4}$$

and, for $\alpha>1$,

$$\begin{aligned}T_\alpha^h(y) = -\frac{\varepsilon(y)}{y^\alpha}\cos(h\log x) - \frac{\alpha\cos(h\log x)}{2(\alpha^2+h^2)}\,\frac{1}{y^\alpha} + \frac{h\sin(h\log x)}{2(\alpha^2+h^2)}\,\frac{1}{y^\alpha} \\ +\int_y^\infty \frac{\varepsilon(u)}{u^{\alpha+1}}\left[\alpha\cos\left(h\log\frac{ux}{y}\right) + h\sin\left(h\log\frac{ux}{y}\right)\right]du.\end{aligned} \tag{5}$$

Let

$$f(y) := \int_0^y \left(\varepsilon(u) - \frac{B}{2}\right)du$$

where $B=-C_0-\log 2\pi$ and C_0 is Euler's constant. Note that

$$f(y) \ll y^{1/2+\varepsilon} \tag{6}$$

(see Lemma 2.2 of [3]). From (4) and (5),

$$\begin{aligned}&S_2^h(y)\,\frac{1}{y^3} + T_2^h(y)y \\ &= -\frac{2\cos(h\log x)}{4+h^2}\,\frac{1}{y} - \frac{1}{y^3}\int_0^y u\varepsilon(u)\left[2\cos\left(h\log\frac{ux}{y}\right) - h\sin\left(h\log\frac{ux}{y}\right)\right]du \\ &\quad + y\int_y^\infty \frac{\varepsilon(u)}{u^3}\left[2\cos\left(h\log\frac{ux}{y}\right) + h\sin\left(h\log\frac{ux}{y}\right)\right]du.\end{aligned} \tag{7}$$

LEMMA 4.1. *We have*

$$I+J = -\frac{1}{y^3}\int_0^y u\varepsilon(u)\left[2\cos\left(h\log\frac{ux}{y}\right) - h\sin\left(h\log\frac{ux}{y}\right)\right]du$$

$$
\begin{aligned}
&+ y \int_y^\infty \frac{\varepsilon(u)}{u^3} \left[2\cos\left(h\log\frac{ux}{y}\right) + h\sin\left(h\log\frac{ux}{y}\right)\right] du \\
&= -\frac{4f(y)}{y}\cos(h\log x) \\
&+ \frac{1}{y^3}\int_0^y f(u)\left[(2-h^2)\cos\left(h\log\frac{ux}{y}\right) - 3h\sin\left(h\log\frac{ux}{y}\right)\right] du \\
&+ y\int_y^\infty \frac{f(u)}{u^4}\left[(6-h^2)\cos\left(h\log\frac{ux}{y}\right) + 5h\sin\left(h\log\frac{ux}{y}\right)\right] du.
\end{aligned}
$$

Proof. I can be rewritten as

$$
\begin{aligned}
&-\frac{1}{y^3}\int_0^y u\left(\varepsilon(u) - \frac{B}{2}\right)\left[2\cos\left(h\log\frac{ux}{y}\right) - h\sin\left(h\log\frac{ux}{y}\right)\right] du \\
&\quad - \frac{B}{2}\frac{1}{y^3}\int_0^y u\left[2\cos\left(h\log\frac{ux}{y}\right) - h\sin\left(h\log\frac{ux}{y}\right)\right] du = -I_1 - I_2.
\end{aligned}
$$

By a substitution $v = \log\frac{ux}{y}$ and Lemma 2.7,

$$
I_2 = \frac{B}{2}\frac{1}{y}\cos(h\log x). \tag{8}
$$

By integration by parts and (6),

$$
\begin{aligned}
I_1 = {} & \frac{f(y)}{y^2}\left[2\cos(h\log x) - h\sin(h\log x)\right] \\
& - \frac{1}{y^3}\int_0^y f(u)\left[(2-h^2)\cos\left(h\log\frac{ux}{y}\right) - 3h\sin\left(h\log\frac{ux}{y}\right)\right] du.
\end{aligned} \tag{9}
$$

Similarly, J can be rewritten as

$$
\begin{aligned}
&y\int_y^\infty \frac{\varepsilon(u) - B/2}{u^3}\left[2\cos\left(h\log\frac{ux}{y}\right) + h\sin\left(h\log\frac{ux}{y}\right)\right] du \\
&\quad + \frac{B}{2} y\int_y^\infty \frac{1}{u^3}\left[2\cos\left(h\log\frac{ux}{y}\right) + h\sin\left(h\log\frac{ux}{y}\right)\right] du = J_1 + J_2.
\end{aligned}
$$

By a substitution $v = \log\frac{ux}{y}$ and Lemma 2.7,

$$
J_2 = \frac{B}{2}\frac{1}{y}\cos(h\log x). \tag{10}
$$

By integration by parts and (6),

$$J_1 = -\frac{f(y)}{y^2}\,[2\cos(h\log x) + h\sin(h\log x)] + y\int_y^\infty \frac{f(u)}{u^4}\left[(6-h^2)\cos\left(h\log\frac{ux}{y}\right) + 5h\sin\left(h\log\frac{ux}{y}\right)\right]du. \tag{11}$$

Equations (8)–(11) together give the lemma.

LEMMA 4.2. *We have*

$$S_2^h(y)\,\frac{1}{y^3} + T_2^h(y)y = -\frac{2\cos(h\log x)}{4+h^2}\,\frac{1}{y} - \frac{4f(y)}{y^2}\cos(h\log x) + G_1(y) + G_2(y)$$

where

$$G_1(y) = \frac{1}{y^3}\int_0^y f(u)\left[(2-h^2)\cos\left(h\log\frac{ux}{y}\right) - 3h\sin\left(h\log\frac{ux}{y}\right)\right]du,$$

$$G_2(y) = y\int_y^\infty \frac{f(u)}{u^4}\left[(6-h^2)\cos\left(h\log\frac{ux}{y}\right) + 5h\sin\left(h\log\frac{ux}{y}\right)\right]du.$$

Proof. Combine (7) and Lemma 4.1.

LEMMA 4.3 ([3, Lemma 3.3]). *For any integer $n \ge 1$, we have*

$$\int_1^\infty \frac{1}{y^n}\operatorname{Re}\widehat{\Psi}_U\left(\frac{Ty}{2\pi x}\right)dy = \int_1^\infty \frac{1}{y^n}\,\frac{\sin\frac{T}{x}y}{\frac{T}{x}y}\,dy + O\left(\Delta\log\frac{1}{\Delta}\right).$$

When $n \ne 2$, the error term can be replaced by $O(\Delta)$.

LEMMA 4.4 ([3, Lemma 3.4]). *If $F(y) \ll y^{-3/2+\varepsilon}$ for $y \ge 1$, then*

$$\int_1^\infty F(y)\operatorname{Re}\widehat{\Psi}_U\left(\frac{Ty}{2\pi x}\right)dy = \int_1^\infty F(y)\,\frac{\sin\frac{T}{x}y}{\frac{T}{x}y}\,dy + O(\Delta).$$

5. Proof of Theorem 1.2. Throughout this section, we assume $\tau = T^{1-\varepsilon} \le T/\log^M T \le x \le T^{2-2\varepsilon}$, $U = \log^M T$ for $M > 2$, $H^* = \tau^{-2}x^{2/(1-\varepsilon)}$, and $\Psi_U(t)$ is defined as in the previous section. The implicit constants in the error terms may depend on ε, K and M.

Our method is that of Goldston and Gonek [6] and it is very similar to [3]. Let $s = \sigma + it$,

$$A_h(s) := \sum_{n\le x}\frac{\Lambda(n)(1+n^{ih})}{n^s}, \qquad A_h^*(s) := \sum_{n>x}\frac{\Lambda(n)(1+n^{ih})}{n^s},$$

$$A(s) := \frac{1}{2}A_0(s), \qquad A^*(s) := \frac{1}{2}A_0^*(s).$$

By Lemma 2.4, with slight modifications, one has

$$\begin{aligned}4\pi F_h(x,T) = \int_0^T \bigg| &\frac{1}{x}\bigg(A_h\Big(-\frac{1}{2}+it\Big) - \int_1^x (1+u^{ih})u^{1/2-it}\,du\bigg) \\ &+ x\bigg(A_h^*\Big(\frac{3}{2}+it\Big) - \int_x^\infty (1+u^{ih})u^{-3/2-it}\,du\bigg)\bigg|^2\,dt \\ &- 2\int_0^T \bigg|\frac{1}{x}\bigg(A\Big(-\frac{1}{2}+it\Big) - \int_1^x u^{1/2-it}\,du\bigg) \\ &+ x\bigg(A^*\Big(\frac{3}{2}+it\Big) - \int_x^\infty u^{-3/2-it}\,du\bigg)\bigg|^2\,dt + O(\widetilde{h}\log^3 T).\end{aligned}$$

Inserting $\Psi_U(t/T)$ into the integral and extending the range of integration to the whole real line, we get

$$\begin{aligned}4\pi F(x,T) = {}&\frac{1}{x^2}\,I_1(x,T) + x^2 I_2(x,T) - \frac{2}{x^2}\,I_3(x,T) - 2x^2 I_4(x,T) \\ &+ O\bigg(\frac{T(\log T)^2}{U}\bigg) + O\bigg(\frac{x^{1+6\varepsilon}}{T}\bigg)\end{aligned} \tag{12}$$

where

$$\begin{aligned}I_1(x,T) &= \int_{-\infty}^{\infty} \bigg|A_h\Big(-\frac{1}{2}+it\Big) - \int_1^x (1+u^{ih})u^{1/2-it}\,du\bigg|^2 \Psi_U\Big(\frac{t}{T}\Big)\,dt, \\ I_2(x,T) &= \int_{-\infty}^{\infty} \bigg|A_h^*\Big(\frac{3}{2}+it\Big) - \int_x^\infty (1+u^{ih})u^{-3/2-it}\,du\bigg|^2 \Psi_U\Big(\frac{t}{T}\Big)\,dt, \\ I_3(x,T) &= \int_{-\infty}^{\infty} \bigg|A\Big(-\frac{1}{2}+it\Big) - \int_1^x u^{1/2-it}\,du\bigg|^2 \Psi_U\Big(\frac{t}{T}\Big)\,dt, \\ I_4(x,T) &= \int_{-\infty}^{\infty} \bigg|A^*\Big(\frac{3}{2}+it\Big) - \int_x^\infty u^{-3/2-it}\,du\bigg|^2 \Psi_U\Big(\frac{t}{T}\Big)\,dt\end{aligned}$$

by Lemma 1 of [7] with modification $V = -T/U$ and $T - T/U$, and $W = 2T/U$. The contributions from the cross terms are estimated via Theorem 3 of [6]. Note that by partial summation with the Riemann Hypothesis and TPC,

$$\sum_{n\le x} \Lambda(n)(1+n^{ih}) = \int_1^x (1+u^{ih})\,du + O(\widetilde{h}x^{1/2+\varepsilon}),$$

$$\begin{aligned}\sum_{n\le x} &\Lambda(n)\Lambda(n+k)(1+n^{ih})(1+(n+k)^{-ih}) \\ &= \mathfrak{S}(k)\int_1^x (1+u^{ih})(1+(u+k))^{-ih}\,du + O(\widetilde{h}x^{1/2+\varepsilon}).\end{aligned}$$

By Corollary 1 of [6] (see also the calculations at the end of [6] and [7]),

$$\begin{aligned}
I_1(x,T) &= \widehat{\Psi}_U(0)T\sum_{n\le x}\Lambda^2(n)n|1+n^{ih}|^2 \\
&\quad + 4\pi\Big(\frac{T}{2\pi}\Big)^3\int_{T/2\pi x}^{\infty}\Big[\sum_{k\le 2\pi xv/T}\mathfrak{S}(k)k^2\Big(1+\Big(\frac{kT}{2\pi v}\Big)^{ih}\Big) \\
&\quad \times\Big(1+\Big(\frac{kT}{2\pi v}+k\Big)^{-ih}\Big)\Big]\operatorname{Re}\widehat{\Psi}_U(v)\,\frac{dv}{v^3} \\
&\quad - 4\pi\Big(\frac{T}{2\pi}\Big)^3\int_{T/2\pi\tau x}^{\infty}\Big[\int_0^{2\pi xv/T}u^2\Big|1+\Big(\frac{uT}{2\pi v}\Big)^{ih}\Big|^2du\Big]\operatorname{Re}\widehat{\Psi}_U(v)\,\frac{dv}{v^3} \\
&\quad + O\Big(\widetilde{h}\,\frac{x^{3+6\varepsilon}}{T}\Big)+O(\widetilde{h}x^{5/2+7\varepsilon}).
\end{aligned}$$

Note that

$$\begin{aligned}
(13)\qquad &\Big(1+\Big(\frac{kT}{2\pi v}\Big)^{ih}\Big)\Big(1+\Big(\frac{kT}{2\pi v}+k\Big)^{-ih}\Big) \\
&= \Big|1+\Big(\frac{kT}{2\pi v}\Big)^{ih}\Big|^2+\Big(1+\Big(\frac{kT}{2\pi v}\Big)^{ih}\Big)\Big(\Big(\frac{kT}{2\pi v}+k\Big)^{-ih}-\Big(\frac{kT}{2\pi v}\Big)^{-ih}\Big) \\
&= \Big|1+\Big(\frac{kT}{2\pi v}\Big)^{ih}\Big|^2+\Big(1+\Big(\frac{kT}{2\pi v}\Big)^{ih}\Big)\Big(\frac{kT}{2\pi v}\Big)^{-ih}\Big(\Big(1+\frac{2\pi v}{T}\Big)^{-ih}-1\Big) \\
&= \Big|1+\Big(\frac{kT}{2\pi v}\Big)^{ih}\Big|^2+O\Big(\min\Big(\frac{hv}{T},1\Big)\Big).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_{T/2\pi x}^{\infty}\Big[\sum_{k\le 2\pi xv/T}\mathfrak{S}(k)k^2\Big(1+\Big(\frac{kT}{2\pi v}\Big)^{ih}\Big)\Big(1+\Big(\frac{kT}{2\pi v}+k\Big)^{-ih}\Big)\Big]\operatorname{Re}\widehat{\Psi}_U(v)\,\frac{dv}{v^3} \\
&\qquad = \int_{T/2\pi x}^{\infty}\Big[\sum_{k\le 2\pi xv/T}\mathfrak{S}(k)k^2\Big|1+\Big(\frac{kT}{2\pi v}\Big)^{ih}\Big|^2\Big]\operatorname{Re}\widehat{\Psi}_U(v)\,\frac{dv}{v^3} \\
&\qquad\quad + O\Big(\int_{T/2\pi x}^{T^{2-\varepsilon}/x}\Big(\frac{xv}{T}\Big)^3\frac{hv}{T}\,\frac{1}{v}\,\frac{dv}{v^3}+\int_{T^{2-\varepsilon}/x}^{\infty}\Big(\frac{xv}{T}\Big)^3\frac{1}{\Delta v^2}\,\frac{dv}{v^3}\Big) \\
&\qquad = \int_{T/2\pi x}^{\infty}\Big[\sum_{k\le 2\pi xv/T}\mathfrak{S}(k)k^2\Big|1+\Big(\frac{kT}{2\pi v}\Big)^{ih}\Big|^2\Big]\operatorname{Re}\widehat{\Psi}_U(v)\,\frac{dv}{v^3} \\
&\qquad\quad + O\Big(\frac{hx^2}{T^{2+\varepsilon}}+\frac{x^4}{\Delta T^{5-\varepsilon}}\Big)
\end{aligned}$$

as $\sum_{k\le x}\mathfrak{S}(k)\sim x$ and $\mathrm{Re}\,\widehat{\Psi}_U(v)\ll\min(1/v,1/\Delta v^2)$. Therefore,

$$\begin{aligned}
&I_1(x,T)\\
&\quad = T\sum_{n\le x}\Lambda^2(n)n(2+2\cos(h\log n))\\
&\qquad +4\pi\Big(\frac{T}{2\pi}\Big)^3\int\limits_{T/2\pi x}^{\infty}\Big[\sum_{k\le 2\pi xv/T}\mathfrak{S}(k)k^2\Big(2+2\cos\Big(h\log\frac{kT}{2\pi v}\Big)\Big)\\
&\qquad -\int\limits_0^{2\pi xv/T}u^2\Big(2+2\cos\Big(h\log\frac{uT}{2\pi v}\Big)\Big)\,du\Big]\,\mathrm{Re}\,\widehat{\Psi}_U(v)\,\frac{dv}{v^3}\\
&\qquad -4\pi\Big(\frac{T}{2\pi}\Big)^3\int\limits_0^{T/2\pi x}\int\limits_0^{2\pi xv/T}u^2\Big(2+2\cos\Big(h\log\frac{uT}{2\pi v}\Big)\Big)\,du\,\mathrm{Re}\,\widehat{\Psi}_U(v)\,\frac{dv}{v^3}\\
&\qquad +O\Big(\frac{\widetilde{h}x^{3+6\varepsilon}}{T}\Big)+O(\widetilde{h}x^{5/2+7\varepsilon})+O(\widetilde{h}x^2T^{1-\varepsilon})+O\Big(\frac{x^4}{\Delta T^{2-\varepsilon}}\Big).
\end{aligned}$$

Similarly, by Corollary 2 of [6],

$$\begin{aligned}
I_2(x,T)&=T\sum_{x<n}\frac{\Lambda^2(n)}{n^3}\,(2+2\cos(h\log n))\\
&\quad+\frac{8\pi^2}{T}\int\limits_0^{TH^*/2\pi x}\Big[\sum_{2\pi xv/T\le k\le H^*}\frac{\mathfrak{S}(k)}{k^2}\Big(2+2\cos\Big(h\log\frac{kT}{2\pi v}\Big)\Big)\\
&\quad-\int\limits_{2\pi xv/T}^{H^*}\frac{1}{u^2}\Big(2+2\cos\Big(h\log\frac{uT}{2\pi v}\Big)\Big)\,du\Big]\,\mathrm{Re}\,\widehat{\Psi}_U(v)v\,dv\\
&\quad+O(\widetilde{h}T^{-1}x^{-1+6\varepsilon})+O(\widetilde{h}x^{-3/2+7\varepsilon})\\
&\quad+O(\widetilde{h}T^{1-\varepsilon/2}x^{-2})+O\Big(\frac{\widetilde{h}H^*}{\Delta x^2}\Big)
\end{aligned}$$

where the last error term comes from the error term in (13). $I_3(x,T)$ and $I_4(x,T)$ are computed in [3] or one can simply set $h=0$ in $I_1(x,T)$ and $I_2(x,T)$, and divide by 4. Putting these into (12) with a substitution $y=2\pi xv/T$ and using Lemma 2.8, we get

$$\begin{aligned}
&4\pi F_h(x,T)=2T\Big[\frac{4\cos(h\log x)}{4+h^2}\log x-\frac{8h\sin(h\log x)}{(4+h^2)^2}\Big]\\
&+4T\int\limits_1^{\infty}\Big[\sum_{k\le y}\mathfrak{S}(k)k^2\cos\Big(h\log\frac{kx}{y}\Big)-\int\limits_0^{y}u^2\cos\Big(h\log\frac{ux}{y}\Big)\,du\Big]\,\mathrm{Re}\,\widehat{\Psi}_U\Big(\frac{Ty}{2\pi x}\Big)\frac{dy}{y^3}
\end{aligned}$$

$$
\begin{aligned}
&- 4T\int_0^1\int_0^y u^2\cos\Big(h\log\frac{ux}{y}\Big)\,du\,\mathrm{Re}\,\widehat{\Psi}_U\Big(\frac{Ty}{2\pi x}\Big)\frac{dy}{y^3}\\
&+ 4T\int_1^{H^*}\Big[\sum_{y\le k\le H^*}\frac{\mathfrak{S}(k)\cos\big(h\log\frac{kx}{y}\big)}{k^2} - \int_y^{H^*}\frac{\cos\big(h\log\frac{ux}{y}\big)}{u^2}\,du\Big]\\
&\times \mathrm{Re}\,\widehat{\Psi}_U\Big(\frac{Ty}{2\pi x}\Big)y\,dy\\
&+ 4T\int_0^1\Big[\sum_{k\le H^*}\frac{\mathfrak{S}(k)\cos\big(h\log\frac{kx}{y}\big)}{k^2} - \int_y^{H^*}\frac{\cos\big(h\log\frac{ux}{y}\big)}{u^2}\,du\Big]\\
&\times \mathrm{Re}\,\widehat{\Psi}_U\Big(\frac{Ty}{2\pi x}\Big)y\,dy\\
&+ O\Big(\frac{\widetilde{h}x^{1+6\varepsilon}}{T}\Big) + O(\widetilde{h}x^{1/2+7\varepsilon}) + O\Big(\frac{\widetilde{h}x^2}{T^{2-2\varepsilon}}\Big) + O\Big(\frac{\widetilde{h}T}{\log^{M-2}T}\Big).
\end{aligned}
$$

From Lemma 2.7,

$$
\int e^{ax}\cos bx\,dx = \frac{a}{a^2+b^2}\,e^{ax}\cos bx + \frac{b}{a^2+b^2}\,e^{ax}\sin bx. \tag{14}
$$

Also,

$$
\begin{aligned}
\int_0^1 \mathrm{Re}\,\widehat{\Psi}_U\Big(\frac{Ty}{2\pi x}\Big)\,dy &= \frac{x}{T}\int_0^{T/x}\frac{\sin u}{u}\,(1+O(\Delta^2u^2))\,du \\
&= \frac{x}{T}\int_0^{T/x}\frac{\sin u}{u}\,du + O\Big(\frac{\Delta^2 T}{x}\Big).
\end{aligned} \tag{15}
$$

Using integration by parts, (14) and (15) with an appropriate change of variables, we have

$$
\begin{aligned}
&\int_0^1\int_0^y u^2\cos\Big(h\log\frac{ux}{y}\Big)\,du\,\mathrm{Re}\,\widehat{\Psi}_U\Big(\frac{Ty}{2\pi x}\Big)\frac{dy}{y^3}\\
&\qquad = \frac{x}{T}\int_0^{T/x}\frac{\sin u}{u}\,du\Big[\frac{3}{9+h^2}\cos(h\log x) + \frac{h}{9+h^2}\sin(h\log x)\Big]\\
&\qquad\quad - \int_0^1\frac{\cos(h\log x)}{y}\int_0^y \mathrm{Re}\,\widehat{\Psi}_U\Big(\frac{Tv}{2\pi x}\Big)\,dv\,dy + O\Big(\frac{\Delta^2 T}{x}\Big)
\end{aligned}
$$

and

$$\int_0^1 \Big[\sum_{k\le H^*} \frac{\mathfrak{S}(k)\cos\big(h\log\frac{kx}{y}\big)}{k^2} - \int_y^{H^*} \frac{\cos\big(h\log\frac{ux}{y}\big)}{u^2}\,du\Big] \operatorname{Re}\widehat{\Psi}_U\Big(\frac{Ty}{2\pi x}\Big) y\,dy$$
$$= \sum_{k=1}^{\infty} \frac{\mathfrak{S}(k)}{k^2} \int_0^1 y\cos\Big(h\log\frac{kx}{y}\Big)\frac{\sin\frac{Ty}{x}}{\frac{Ty}{x}}\,dy + O\Big(\frac{1}{H^*}\Big) + O\Big(\frac{\Delta^2 T}{x}\Big)$$
$$- \frac{x}{T}\int_0^{T/x} \frac{\sin u}{u}\,du\Big[\frac{1}{1+h^2}\cos(h\log x) - \frac{h}{1+h^2}\sin(h\log x)\Big]$$
$$- \int_0^1 \frac{\cos(h\log x)}{y}\int_0^y \operatorname{Re}\widehat{\Psi}_U\Big(\frac{Tv}{2\pi x}\Big)\,dv\,dy + O\Big(\frac{\Delta^2 T}{x}\Big).$$

Therefore, with the notation $S_\alpha^h(y)$ and $T_\alpha^h(y)$,

$$4\pi F_h(x,T) = 2T\Big[\frac{4\cos(h\log x)}{4+h^2}\log x - \frac{8h\sin(h\log x)}{(4+h^2)^2}\Big]$$
$$+ 4T\int_1^{\infty} S_2^h(y)\operatorname{Re}\widehat{\Psi}_U\Big(\frac{Ty}{2\pi x}\Big)\frac{dy}{y^3}$$
$$+ 4T\int_1^{H^*} (T_2^h(y) - T_2^h(H^*))\operatorname{Re}\widehat{\Psi}_U\Big(\frac{Ty}{2\pi x}\Big) y\,dy$$
$$- 4x\int_0^{T/x} \frac{\sin u}{u}\,du\Big[\frac{3\cos(h\log x)}{9+h^2} + \frac{h\sin(h\log x)}{9+h^2}\Big]$$
$$- 4x\int_0^{T/x} \frac{\sin u}{u}\,du\Big[\frac{\cos(h\log x)}{1+h^2} - \frac{h\sin(h\log x)}{1+h^2}\Big]$$
$$+ 4T\sum_{k=1}^{\infty}\frac{\mathfrak{S}(k)}{k^2}\int_0^1 y\cos\Big(h\log\frac{kx}{y}\Big)\frac{\sin\frac{Ty}{x}}{\frac{Ty}{x}}\,dy + O\Big(\frac{\widetilde{h}x^{1+6\varepsilon}}{T}\Big)$$
$$+ O(\widetilde{h}x^{1/2+7\varepsilon}) + O\Big(\frac{\widetilde{h}x^2}{T^{2-2\varepsilon}}\Big) + O\Big(\frac{\widetilde{h}T}{\log^{M-2}T}\Big).$$

By (3) and (5), $T_2^h(H^*) \ll h(\log H^*)^{2/3}/(H^*)^2$. It follows that the contribution from $T_2^h(H^*)$ in the second integral is $O(hT^{-\varepsilon})$. Also, one can extend the upper limit of the second integral to ∞ with an error $O(hT^{-\varepsilon})$ by (3) and (5) again. Finally, we obtain the theorem by applying Lemmas 4.2–4.4, (6) and dividing by 4π.

6. Proof of Theorems 1.3 and 1.4

Proof of Theorem 1.3. This follows directly from Theorem 1.2 by observing that all the main terms except the first one are $O(x)$ because of (6).

Before proving Theorem 1.4, we need the following lemmas.

LEMMA 6.1. *We have*

$$\int_1^\infty \frac{\sin ax}{x^{2n}}\,dx$$
$$= \frac{a^{2n-1}}{(2n-1)!}\Bigg[\sum_{k=1}^{2n-1}\frac{(2n-k-1)!}{a^{2n-k}}\sin\Big(a+(k-1)\,\frac{\pi}{2}\Big)+(-1)^n\operatorname{ci}(a)\Bigg]$$

where

$$\operatorname{ci}(x) = -\int_x^\infty \frac{\cos t}{t}\,dt = C_0+\log x+\int_0^x\frac{\cos t-1}{t}\,dt$$

and C_0 *is Euler's constant.*

Proof. This is formula 3.761(3) on p. 430 of [8], which can be proved by integration by parts repeatedly.

LEMMA 6.2 ([3, Lemma 5.2]). *If* $F(y) \ll y^{-3/2+\varepsilon}$ *for* $y \geq 1$, *then for* $T \leq x$,

$$\int_1^\infty F(y)\,\frac{\sin\frac{T}{x}y}{\frac{T}{x}y}\,dy = \int_1^\infty F(y)\,dy + O\Big(\Big(\frac{T}{x}\Big)^{1/2-\varepsilon}\Big).$$

LEMMA 6.3. *We have*

$$I = \int_1^\infty \frac{1}{y^3}\int_0^y f(u)\Big[(2-h^2)\cos\Big(h\log\frac{ux}{y}\Big)-3h\sin\Big(h\log\frac{ux}{y}\Big)\Big]\,du\,dy$$
$$= \int_0^1 f(u)[\cos(h\log ux)-h\sin(h\log ux)]\,du$$
$$+\int_1^\infty \frac{f(u)}{u^2}\,du\,[\cos(h\log x)-h\sin(h\log x)].$$

Proof. Because of (6), we can change the order of integration:

$$I = \int_0^1 f(u)\int_1^\infty \frac{1}{y^3}\Big[(2-h^2)\cos\Big(h\log\frac{ux}{y}\Big)-3h\sin\Big(h\log\frac{ux}{y}\Big)\Big]\,dy\,du$$
$$+\int_1^\infty f(u)\int_u^\infty \frac{1}{y^3}\Big[(2-h^2)\cos\Big(h\log\frac{ux}{y}\Big)-3h\sin\Big(h\log\frac{ux}{y}\Big)\Big]\,dy\,du$$
$$= \int_0^1 f(u)\Big\{(2-h^2)\Big[\frac{2}{4+h^2}\cos(h\log ux)+\frac{h}{4+h^2}\sin(h\log ux)\Big]$$
$$-3h\Big[\frac{2}{4+h^2}\sin(h\log ux)-\frac{h}{4+h^2}\cos(h\log ux)\Big]\Big\}\,du$$

$$+\int_1^\infty \frac{f(u)}{u^2}\bigg\{(2-h^2)\bigg[\frac{2}{4+h^2}\cos(h\log x)+\frac{h}{4+h^2}\sin(h\log x)\bigg]$$
$$-3h\bigg[\frac{2}{4+h^2}\sin(h\log x)-\frac{h}{4+h^2}\cos(h\log x)\bigg]\bigg\}\,du,$$

by substituting $v=\log\frac{ux}{y}$ and applying Lemma 2.7. Now the result follows after some simple algebra.

LEMMA 6.4. *We have*

$$\begin{aligned}J &= \int_1^\infty y\int_y^\infty \frac{f(u)}{u^4}\bigg[(6-h^2)\cos\bigg(h\log\frac{ux}{y}\bigg)+5h\sin\bigg(h\log\frac{ux}{y}\bigg)\bigg]\,du\,dy\\ &= -\int_1^\infty \frac{f(u)}{u^4}\,[3\cos(h\log ux)+h\sin(h\log ux)]\,du\\ &\quad+\int_1^\infty \frac{f(u)}{u^2}\,du\,[3\cos(h\log x)+h\sin(h\log x)].\end{aligned}$$

Proof. Again, because of (6), we can change the order of integration:

$$\begin{aligned}J &= \int_1^\infty\int_1^u y\bigg[(6-h^2)\cos\bigg(h\log\frac{ux}{y}\bigg)+5h\sin\bigg(h\log\frac{ux}{y}\bigg)\bigg]\,dy\,du\\ &= \int_1^\infty \frac{f(u)}{u^4}\bigg\{(6-h^2)\bigg[\frac{-2}{4+h^2}\cos(h\log ux)+\frac{h}{4+h^2}\sin(h\log ux)\bigg]\\ &\quad+5h\bigg[\frac{-2}{4+h^2}\sin(h\log ux)-\frac{h}{4+h^2}\cos(h\log ux)\bigg]\\ &\quad-(6-h^2)\bigg[\frac{-2}{4+h^2}\cos(h\log x)+\frac{h}{4+h^2}\sin(h\log x)\bigg]\\ &\quad-5h\bigg[\frac{-2}{4+h^2}\sin(h\log x)-\frac{h}{4+h^2}\cos(h\log x)\bigg]\bigg\}\,du,\end{aligned}$$

by substituting $v=\log\frac{ux}{y}$ and applying Lemma 2.7. The result now follows after some simple algebra.

LEMMA 6.5. *We have*

$$\begin{aligned}S &= \sum_{k=1}^\infty \frac{\mathfrak{S}(k)}{k^2}\int_0^1 y\cos\bigg(h\log\frac{kx}{y}\bigg)\,dy\\ &= \bigg[\frac{1}{1+h^2}\cos(h\log x)-\frac{h}{1+h^2}\sin(h\log x)\bigg]\\ &\quad-\bigg[\frac{4-h^2}{2(4+h^2)^2}\cos(h\log x)-\frac{2h}{(4+h^2)^2}\sin(h\log x)\bigg]\end{aligned}$$

$$+\frac{B}{2}\left[\frac{2}{4+h^2}\cos(h\log x)-\frac{h}{4+h^2}\sin(h\log x)\right]+\left(1+\frac{B}{2}\right)\cos(h\log x)$$

$$+\int_1^\infty \frac{f(u)}{u^4}\,[3\cos(h\log ux)+h\sin(h\log ux)]\,du.$$

Proof. By substituting $v=\log\frac{kx}{y}$ and using Lemma 2.7,

$$S=\frac{2}{4+h^2}\sum_{k=1}^{\infty}\frac{\mathfrak{S}(k)}{k^2}\cos(h\log kx)-\frac{h}{4+h^2}\sum_{k=1}^{\infty}\frac{\mathfrak{S}(k)}{k^2}\sin(h\log kx).$$

Recall the definition of $S_0(u)$ from (3) and use partial summation to obtain

$$\begin{aligned}
S&=\frac{2}{4+h^2}\int_1^\infty \frac{S_0(u)+u}{u^3}\,[2\cos(h\log ux)+h\sin(h\log ux)]\,du\\
&\quad-\frac{h}{4+h^2}\int_1^\infty \frac{S_0(u)+u}{u^3}\,[-h\cos(h\log ux)+2\sin(h\log ux)]\,du\\
&=\int_1^\infty \frac{S_0(u)+u}{u^3}\cos(h\log ux)\,du\\
&=\int_1^\infty \frac{u-\frac{1}{2}\log u+\varepsilon(u)}{u^3}\cos(h\log ux)\,du\\
&=\int_1^\infty \frac{1}{u^2}\cos(h\log ux)\,du-\frac{1}{2}\int_1^\infty\frac{\log u}{u^3}\cos(h\log ux)\,du\\
&\quad+\frac{B}{2}\int_1^\infty\frac{1}{u^3}\cos(h\log ux)\,du+\int_1^\infty\frac{\varepsilon(u)-B/2}{u^3}\cos(h\log ux)\,du\\
&=I_1-\frac{1}{2}I_2+\frac{B}{2}I_3+I_4.
\end{aligned}$$

By an appropriate substitution and Lemma 2.7,

$$\begin{aligned}
I_1&=\frac{1}{1+h^2}\cos(h\log x)-\frac{h}{1+h^2}\sin(h\log x),\\
I_2&=\frac{4-h^2}{(4+h^2)^2}\cos(h\log x)-\frac{2h}{(4+h^2)^2}\sin(h\log x),\\
I_3&=\frac{2}{4+h^2}\cos(h\log x)-\frac{h}{4+h^2}\sin(h\log x).
\end{aligned}$$

Finally, by integration by parts,

$$\begin{aligned}
I_4&=\int_1^\infty\frac{\cos(h\log ux)}{u^3}\,df(u)\\
&=\left(1+\frac{B}{2}\right)\cos(h\log x)+\int_1^\infty\frac{f(u)}{u^4}\,[3\cos(h\log ux)+h\sin(h\log ux)]\,du
\end{aligned}$$

because $f(1) = -1 - B/2$. Combining these results for I_1, I_2, I_3, I_4, we get the result.

LEMMA 6.6. *We have*

$$\int_0^1 f(u)[\cos(h\log ux) - h\sin(h\log ux)]\,du$$
$$= -\frac{1}{2}\left[\frac{4+3h^2}{(4+h^2)^2}\cos(h\log x) + \frac{h^3}{(4+h^2)^2}\sin(h\log x)\right]$$
$$-\left(\frac{1}{2}+\frac{B}{2}\right)\left[\frac{2+h^2}{4+h^2}\cos(h\log x) - \frac{h}{4+h^2}\sin(h\log x)\right]$$
$$-\frac{1}{2}\left[\frac{3+h^2}{9+h^2}\cos(h\log x) - \frac{2h}{9+h^2}\sin(h\log x)\right].$$

Proof. The key is $\varepsilon(u) = \frac{1}{2}\log u - u$ when $0 \le u \le 1$ (see (3)). So,

$$f(u) = \int_0^u \left[\varepsilon(v) - \frac{B}{2}\right] dv = \frac{1}{2}u\log u - \left(\frac{1}{2}+\frac{B}{2}\right)u - \frac{1}{2}u^2.$$

Putting this into the integral and evaluating the integral piece by piece with suitable substitution and Lemma 2.7, one gets the result.

Proof of Theorem 1.4. First observe that when $T \le x \le T^{2-29\varepsilon}$, the error term in Theorem 1.2 is $O(hT/\log^{M-2}T)$. Rewrite Theorem 1.2 as

$$F_h(x,T) = T_1 + T_2 + T_3 + T_4 + T_5 + O\left(\frac{\widetilde{h}T}{\log^{M-2}T}\right).$$

Since $\frac{\sin u}{u} = 1 + O(u^2)$,

$$T_3 = -\frac{T}{\pi}\left[\frac{3\cos(h\log x)}{9+h^2} + \frac{h\sin(h\log x)}{9+h^2}\right] + O\left(T\left(\frac{T}{x}\right)^2\right),$$
$$T_4 = -\frac{T}{\pi}\left[\frac{\cos(h\log x)}{1+h^2} - \frac{h\sin(h\log x)}{1+h^2}\right] + O\left(T\left(\frac{T}{x}\right)^2\right).$$

By Lemma 6.5,

$$T_5 = \frac{T}{\pi}\sum_{k=1}^{\infty}\frac{\mathfrak{S}(k)}{k^2}\int_0^1 y\cos\left(h\log\frac{kx}{y}\right)dy + O\left(T\left(\frac{T}{x}\right)^2\right)$$
$$= \frac{T}{\pi}\left[\frac{\cos(h\log x)}{1+h^2} - \frac{h\sin(h\log x)}{1+h^2}\right]$$
$$-\frac{T}{\pi}\left[\frac{4-h^2}{2(4+h^2)^2}\cos(h\log x) - \frac{2h}{(4+h^2)^2}\sin(h\log x)\right]$$

$$+\frac{T}{\pi}\frac{B}{2}\left[\frac{2\cos(h\log x)}{4+h^2}-\frac{h\sin(h\log x)}{4+h^2}\right]+\frac{T}{\pi}\left(1+\frac{B}{2}\right)\cos(h\log x)$$
$$+\frac{T}{\pi}\int_1^\infty\frac{f(u)}{u^4}\,[3\cos(h\log ux)+h\sin(h\log ux)]\,du+O\left(T\left(\frac{T}{x}\right)^2\right).$$

By Lemma 4.3, (6) and Lemmas 6.2–6.4 and 6.6,

$$\begin{aligned}
T_2 = &-\frac{2x\cos(h\log x)}{\pi(4+h^2)}\int_1^\infty\frac{\sin\frac{Ty}{x}}{y^2}\,dy-\frac{4T}{\pi}\cos(h\log x)\int_1^\infty\frac{f(y)}{y^2}\,dy\\
&+\frac{T}{\pi}\int_1^\infty(G_1(y)+G_2(y))\,dy+O\left(T\left(\frac{T}{x}\right)^{1/2-\varepsilon}\right)+O\left(\frac{\widetilde{h}T}{\log^M T}\right)\\
= &-\frac{2x\cos(h\log x)}{\pi(4+h^2)}\int_1^\infty\frac{\sin\frac{Ty}{x}}{y^2}\,dy+O\left(\frac{\widetilde{h}T}{\log^M T}\right)\\
&+\frac{T}{\pi}\int_0^1 f(u)[\cos(h\log ux)-h\sin(h\log ux)]\,du\\
&-\frac{T}{\pi}\int_1^\infty\frac{f(u)}{u^4}\,[3\cos(h\log ux)+h\sin(h\log ux)]\,du+O\left(T\left(\frac{T}{x}\right)^{1/2-\varepsilon}\right)\\
= &-\frac{2x\cos(h\log x)}{\pi(4+h^2)}\int_1^\infty\frac{\sin\frac{Ty}{x}}{y^2}\,dy+O\left(\frac{\widetilde{h}T}{\log^M T}\right)\\
&-\frac{T}{\pi}\frac{1}{2}\left[\frac{4+3h^2}{(4+h^2)^2}\cos(h\log x)+\frac{h^3}{(4+h^2)^2}\sin(h\log x)\right]\\
&-\frac{T}{\pi}\left(\frac{1}{2}+\frac{B}{2}\right)\left[\frac{2+h^2}{4+h^2}\cos(h\log x)-\frac{h}{4+h^2}\sin(h\log x)\right]\\
&-\frac{T}{\pi}\frac{1}{2}\left[\frac{3+h^2}{9+h^2}\cos(h\log x)-\frac{2h}{9+h^2}\sin(h\log x)\right]\\
&-\frac{T}{\pi}\int_1^\infty\frac{f(u)}{u^4}\,[3\cos(h\log ux)+h\sin(h\log ux)]\,du+O\left(T\left(\frac{T}{x}\right)^{1/2-\varepsilon}\right).
\end{aligned}$$

Therefore, with miraculous cancellations,

$$\begin{aligned}
T_2+T_3+T_4+T_5 = &-\frac{2x\cos(h\log x)}{\pi(4+h^2)}\int_1^\infty\frac{\sin\frac{Ty}{x}}{y^2}\,dy+\frac{T}{\pi}\,\frac{2B\cos(h\log x)}{4+h^2}\\
&+\frac{T}{\pi}\,\frac{4h\sin(h\log x)}{(4+h^2)^2}+O\left(T\left(\frac{T}{x}\right)^{1/2-\varepsilon}\right)+O\left(\frac{\widetilde{h}T}{\log^M T}\right).
\end{aligned}$$

By Lemma 6.1 and $B = -C_0 - \log 2\pi$,

$$\begin{aligned} F_h(x,T) &= \frac{T}{\pi}\left[\frac{2\cos(h\log x)}{4+h^2}\log x\right] - \frac{2T\cos(h\log x)}{\pi(4+h^2)}\left[\frac{\sin(T/x)}{T/x} - \operatorname{ci}\left(\frac{T}{x}\right)\right] \\ &\quad + \frac{T}{\pi}\,\frac{2B\cos(h\log x)}{4+h^2} + O\left(T\left(\frac{T}{x}\right)^{1/2-\varepsilon}\right) + O\left(\frac{\widetilde{h}T}{\log^{M-2}T}\right) \\ &= \frac{T}{\pi}\left[\frac{2\cos(h\log x)}{4+h^2}\log x\right] - \frac{2T\cos(h\log x)}{\pi(4+h^2)}\left[1 - C_0 - \log\frac{T}{x}\right. \\ &\quad \left. + C_0 + \log 2\pi\right] + O\left(T\left(\frac{T}{x}\right)^{1/2-\varepsilon}\right) + O\left(\frac{\widetilde{h}T}{\log^{M-2}T}\right) \\ &= \frac{T}{2\pi}\log\frac{T}{2\pi e}\left[\frac{4\cos(h\log x)}{4+h^2}\right] + O\left(T\left(\frac{T}{x}\right)^{1/2-\varepsilon}\right) + O\left(\frac{\widetilde{h}T}{\log^{M-2}T}\right). \end{aligned}$$

7. Sketch for Conjecture 1.2. Fix $\alpha > 0$. Let $r(u)$ be an even function which is almost the characteristic function of the interval $[-\alpha,\alpha]$ with $\widehat{r}(\alpha) \ll 1/\alpha^2$ (see page 87 of [1] for a detailed construction). We use Conjecture 1.1 to compute the right hand side of (1):

$$\begin{aligned} I &= \int_{-\infty}^{\infty} F_h(\alpha)\widehat{r}(\alpha)\,d\alpha = 2\int_0^{\infty} F_h(\alpha)\widehat{r}(\alpha)\,d\alpha \\ &= 2(1+o(1))\log T\int_0^1 T^{-2\alpha}\,\widehat{r}(\alpha)\,d\alpha + 2\,\frac{4}{4+h^2}\int_0^1 \alpha\cos(h\log T\alpha)\widehat{r}(\alpha)\,d\alpha \\ &\quad + 2\,\frac{4}{4+h^2}\int_1^{\infty}\cos(h\log T\alpha)\widehat{r}(\alpha)\,d\alpha + O\left(\frac{1}{A}\right) + o(1) \\ &= \frac{4}{4+h^2}\int_{-\infty}^{\infty}\cos(h\log T\alpha)\widehat{r}(\alpha)\,d\alpha \\ &\quad - \frac{4}{4+h^2}\int_{-1}^{1}(1-|\alpha|)\cos(h\log T\alpha)\widehat{r}(\alpha)\,d\alpha \\ &\quad + (1+o(1))\log T\int_{-\infty}^{\infty} T^{-2|\alpha|}\widehat{r}(\alpha)\,d\alpha + O\left(\frac{1}{A}\right) + o(1) \\ &= \frac{4}{4+h^2}\int_{-\infty}^{\infty}\widehat{r}_1(\alpha)\,d\alpha - \frac{4}{4+h^2}\int_{-1}^{1}(1-|\alpha|)\widehat{r}_1(\alpha)\,d\alpha \\ &\quad + (1+o(1))\log T\int_{-\infty}^{\infty} T^{-2|\alpha|}\widehat{r}(\alpha)\,d\alpha + O\left(\frac{1}{A}\right) + o(1) \\ &= \frac{4}{4+h^2}\,I_1 - \frac{4}{4+h^2}\,I_2 + (1+o(1))I_3 + O\left(\frac{1}{A}\right) + o(1) \end{aligned}$$

where $r_1(u) = r(u + (h \log T)/(2\pi))$. As $\int_{-\infty}^{\infty} \widehat{r}_1(\alpha)\, d\alpha = r_1(0)$,

$$I_1 = r_1(0) = r\left(\frac{h \log T}{2\pi}\right).$$

By $\int f\widehat{g} = \int \widehat{f} g$, the transform pair and the definition of $r(u)$,

$$f(t) = \max(1 - |t|, 0), \qquad \widehat{f}(u) = \left(\frac{\sin \pi u}{\pi u}\right)^2,$$

$$I_2 = \int_{-\infty}^{\infty} r_1(u)\left(\frac{\sin \pi u}{\pi u}\right)^2 du = \int_{-\alpha+(h\log T)/(2\pi)}^{\alpha+(h\log T)/(2\pi)} \left(\frac{\sin \pi u}{\pi u}\right)^2 du + o(1).$$

Similarly, by the transform pair,

$$f(t) = e^{-2a|t|}, \qquad \widehat{f}(u) = \frac{4a}{4a^2 + (2\pi u)^2},$$

$$I_3 = \int_{-\alpha}^{\alpha} \frac{4 \log^2 T}{4 \log^2 T + (2\pi u)^2}\, du + o(1) = \int_{-\alpha+(h\log T)/(2\pi)}^{\alpha+(h\log T)/(2\pi)} 1\, du + o(1).$$

Therefore,

$$I = \frac{4}{4+h^2}\, r\left(\frac{h \log T}{2\pi}\right) + \int_{-\alpha+(h\log T)/(2\pi)}^{\alpha+(h\log T)/(2\pi)} \left[1 - \frac{4}{4+h^2}\left(\frac{\sin \pi u}{\pi u}\right)^2\right] du + O\left(\frac{1}{A}\right) + o(1). \tag{16}$$

Now, the left hand side of (1) is

$$\frac{4}{4+h^2}\, r\left(\frac{h \log T}{2\pi}\right) + \left(\frac{T}{2\pi} \log T\right)^{-1} \sum_{\substack{0<\gamma\neq\gamma'\leq T \\ |\gamma-\gamma'-h|\leq 2\pi\alpha/\log T}} (1 + o(1)). \tag{17}$$

Combining (16) and (17), we get Conjecture 1.2 by making A arbitrarily large. The only shaky point in the above argument is the error analysis. All of these become rigorous following pages 87–90 of [1].

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