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Zeros of Dirichlet series with periodic coefficients

by

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1. Introduction. We first set the algebraic background related to our result. We denote by $\mathcal{F}$ the set of Dirichlet series $\sum_{n\ge 1} a_n/n^s$, where $(a_n)_{n\ge1}$ is periodic [1], and by $\mathcal{P}$ the set of Dirichlet polynomials. Then we easily see that $\mathcal{F}$ is a $\mathcal{P}$-module. For each Dirichlet character ψ we set $L_\psi(s) = \sum_{n\ge1}\psi(n)/n^s$, and we denote by $\mathcal{D}^{\mathrm{pr}}$ the set of primitive Dirichlet characters.

Theorem PDCB (Primitive Dirichlet Character Basis). *The family $(L_\psi)_{\psi\in\mathcal{D}^{\mathrm{pr}}}$ is a basis for the $\mathcal{P}$-module $\mathcal{F}$.*

Proof. The fact that every F in $\mathcal{F}$ can be written as a finite sum $F(s) = \sum_{\psi\in\mathcal{D}^{\mathrm{pr}}} P_\psi(s)L_\psi(s)$, where the $P_\psi(s)$ are Dirichlet polynomials, follows readily from the finite orthogonal basis of Lemma 1 of the paper by Codecà, Dvornicich, and Zannier [1] [2]. On the other hand, the freeness of the family $(L_\psi)_{\psi\in\mathcal{D}^{\mathrm{pr}}}$ has been established by Kaczorowski and Perelli [5, Lemma 8.1]. ∎

We now turn to the distribution of zeros. Let $a = (a_n)_{n\ge1}$ be a periodic sequence of complex numbers. We denote by $F_a(s)$ the meromorphic continuation of $\sum_{n\ge1} a_n/n^s$, and by $N_a(\sigma_1,\sigma_2,T)$ (respectively $N'_a(\sigma_1,\sigma_2,T)$) the number of zeros of $F_a(s)$ in the rectangle $\sigma_1 < \mathrm{Re}\, s < \sigma_2$, $|\mathrm{Im}\, s| \le T$, counted with their multiplicities (resp. without their multiplicities).

Theorem. *Let $a = (a_n)_{n\ge1}$ be a periodic sequence of complex numbers such that $F_a(s)$ is not of the form $P(s)L_\chi(s)$, where P is a Dirichlet polynomial and $L_\chi(s)$ is the L-function associated with a Dirichlet character χ. Then there exists a positive number η such that, for all real numbers σ_1 and σ_2 with $1/2 < \sigma_1 < \sigma_2 \le 1+\eta$, there exist positive numbers c_1, c_2, and T_0*

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[1] That is, there exists a positive integer q such that $a_{n+q} = a_n$ for all $n \ge 1$.

[2] This finite orthogonal basis also appears in Exercise 9.1.1.3 of [7].

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such that for all $T \geq T_0$ *we have*

$$c_1 T \leq N'_a(\sigma_1, \sigma_2, T) \leq N_a(\sigma_1, \sigma_2, T) \leq c_2 T.$$

Remark. When $F_a(s)$ is developed along the basis of Theorem PDCB, the condition in the above Theorem is equivalent to asking that $F_a(s)$ does not belong to one of the submodules generated by a single element $L_\psi(s)$ of the basis.

We know by their Euler product that the functions $L_\psi(s)$ do not vanish in the half-plane $\operatorname{Re} s > 1$. Thus our result allows us to specify which functions in $\mathcal{F}$ do not vanish in $\operatorname{Re} s > 1$. In the same flavor as Theorem 1 of [4] we have

Corollary. *Let* F *be a Dirichlet series with periodic coefficients. The following statements are equivalent*:

(i) $F(s)$ *does not vanish in the half-plane* $\operatorname{Re} s > 1$.
(ii) $F(s) = P(s)L_\chi(s)$, *where* χ *is a Dirichlet character and* $P(s)$ *is a Dirichlet polynomial that does not vanish in* $\operatorname{Re} s > 1$.

We can of course replace in this statement the open half-plane $\operatorname{Re} s > 1$ by the closed half-plane $\operatorname{Re} s \geq 1$. Notice that the statement obtained by replacing $\operatorname{Re} s > 1$ by $\operatorname{Re} s > 1/2$ is equivalent to the Generalized Riemann Hypothesis (GRH).

We recall that the Dirichlet characters are exactly the arithmetic functions which are both periodic and completely multiplicative. It is natural to ask what the roles of these two properties will be in a proof of GRH. The theorem we stated, about the zeros of Dirichlet series with periodic coefficients, confirms the commonly held idea that in any proof of GRH the property of complete multiplicativity of the Dirichlet characters must play a significant role.

As a matter of fact, in our Theorem, only the result on the lower bound of the number of zeros in $\operatorname{Re} s > 1$ is really new. The upper bound

$$N_a(1/2 + u, \infty, T) \ll_{a,u} T \tag{1}$$

comes from Steuding's work. More precisely, the proof of the slightly weaker $N_a(1/2 + u, \infty, T) \ll_{a,u} T \log T$ appears in [8] (2002). In [9] (2007), the upper bound (1) is stated in Theorem 11.3, but the proof is given only in the analogous situation of the extended Selberg class. For the sake of completeness we give in the appendix the details of the proof in the case of Dirichlet series with periodic coefficients, and take the opportunity to make the dependence on u explicit.

The lower bound $N'_a(\sigma_1, \sigma_2, T) \gg T$ appears in the paper of Laurinčikas [6] with the condition $1/2 < \sigma_1 < \sigma_2 < 1$, and the restriction that the sequence a is a linear combination (with scalars in the complex plane) of at least two Dirichlet characters modulo q.

For $1/2 < \sigma_1 < \sigma_2 < 1$, the lower bound $N'_a(\sigma_1, \sigma_2, T) \gg T$ is proven by Kaczorowski and Kulas ([3]) [4, Theorem 2] for the set of Dirichlet series $F(s)$,

$$F(s) = \sum_{j=1}^{N} P_j(s) L_{\chi_j}(s), \tag{2}$$

where $N \geq 2$, $\chi_1, \ldots, \chi_n$ are pairwise inequivalent Dirichlet characters, and $P_1, \ldots, P_N$ are not identically vanishing Dirichlet polynomials. By Theorem PDCB, their result applies to the same set of Dirichlet series as our Theorem.

Finally, let us recall what seems to be the first result on the subject. In 1936, Davenport and Heilbronn [2] showed that for all rational α in $]0,1[$ different from $1/2$, the Hurwitz zeta function $\zeta(s,\alpha) = \sum_{n=0}^{\infty} (n+\alpha)^{-s}$ has infinitely many zeros in the half-plane $\operatorname{Re} s > 1$.

Let us now say a few words about our method. By previous results, we only have to prove that for a function $F_a(s)$ satisfying the hypothesis, there exists a number $\eta > 0$ such that for any $1 < \sigma_1 < \sigma_2 < 1 + \eta$ we have $N'_a(\sigma_1, \sigma_2, T) \gg T$ for T large enough.

We did not attempt to generalize the proof by Davenport and Heilbronn. Our goal was instead to extend the method of Kaczorowski and Kulas, which is to use a strong joint universal property for the Dirichlet L-functions. But this property is no longer valid in strips $1 < \sigma_1 < \sigma_2 < 1 + \eta$. In place of it we use a kind of weak joint universal property for Dirichlet L-functions, which leads us to add a new tool into the picture: the Brouwer fixed point theorem (see the proof of Lemma 2).

2. Lemmas. In the following two lemmas, we use the notation

$$D_n(R) := \{z = (z_j)_{1 \leq j \leq n} \in \mathbb{C}^n : |z_j| \leq R \text{ for all } 1 \leq j \leq n\}.$$

LEMMA 1. *Let q be a positive integer, and y and R be positive real numbers. Let $\chi_1, \ldots, \chi_n$ be pairwise distinct Dirichlet characters modulo q. Then there exists a real $\eta > 0$ such that for all fixed σ with $1 < \sigma \leq 1 + \eta$, and for all prime numbers $p > y$, there exists a continuous function $t_p : D_n(R) \to \mathbb{R}$ such that for all z in $D_n(R)$,*

$$z = \left(\sum_{p > y} \frac{\chi_j(p)}{p^{\sigma + i t_p(z)}} \right)_{1 \leq j \leq n}.$$

([3]) In papers [6] and [4], only $N_a(\sigma_1, \sigma_2, T) \gg T$ is stated, but it is clear that it is $N'_a(\sigma_1, \sigma_2, T) \gg T$ that is proven.

Remark. We can interpret this lemma as a linear system to be solved. There are n equations. The unknowns are the infinite family of $(p^{-it_p})_{p>y}$ that must be chosen in the unit circle, and $z \in \mathbb{C}^n$ is a parameter. The solution must be chosen continuously in the parameter z.

Proof of Lemma 1. If $n < \varphi(q)$, we extend $(\chi_j)_{1\le j\le n}$ to $(\chi_j)_{1\le j\le \varphi(q)}$, using all the Dirichlet characters modulo q. This allows us to restrict the proof to the case $n = \varphi(q)$.

We denote by C the unitary matrix of the characters modulo q, that is,

$$C := (\chi_j(a))_{\substack{1\le a\le q,\,(a,q)=1\\ 1\le j\le \varphi(q)}}.$$

We have

$$\sum_{p>y} \frac{\chi_j(p)}{p^{\sigma+it_p}} = \sum_{\substack{1\le a\le q\\ (a,q)=1}} \chi_j(a) \sum_{\substack{p>y\\ p\equiv a\,(q)}} \frac{1}{p^{\sigma+it_p}}.$$

To change variables we write

$$z = Cw, \quad \text{where} \quad z = (z_j)_{1\le j\le\varphi(q)}, \quad w = (w_a)_{\substack{1\le a\le q\\ (a,q)=1}},$$

and

$$\theta_p = -(\log p)(t_p \circ C).$$

To prove the lemma, it is sufficient to solve the system

$$\sum_{\substack{p>y\\ p\equiv a\,(q)}} \frac{e^{i\theta_p}}{p^\sigma} = w_a, \qquad 1\le a\le q,\ (a,q)=1, \tag{3}$$

in the real unknowns $(\theta_p)_{p>y}$, continuously in $w \in D_{\varphi(q)}(\|C^{-1}\|_\infty R)$. We put

$$S_a = S_a(q,y,\sigma) := \sum_{\substack{p>y\\ p\equiv a\,(q)}} \frac{1}{p^\sigma}.$$

Using the prime number theorem for arithmetic progressions, we readily find that there exists an $\eta > 0$ such that for each $1 < \sigma \le 1+\eta$ and $1\le a\le q$, $(a,q)=1$, we have

$$S_a \ge 10\|C^{-1}\|_\infty R, \tag{4}$$

and there exist prime numbers $p_{1,a}$ and $p_{2,a}$, such that

$$\frac{1}{3} \le \lambda_0 := \frac{1}{S_a} \sum_{\substack{y<p\le p_{1,a}\\ p\equiv a\,(q)}} \frac{1}{p^\sigma} \le \frac{1}{3} + \frac{1}{100}$$

and

$$\frac{1}{3} \le \lambda_1 := \frac{1}{S_a} \sum_{\substack{p_{1,a} < p \le p_{2,a} \\ p \equiv a\,(q)}} \frac{1}{p^\sigma} \le \frac{1}{3} + \frac{1}{100}.$$

We also write

$$\lambda_2 := \frac{1}{S_a} \sum_{\substack{p > p_{2,a} \\ p \equiv a\,(q)}} \frac{1}{p^\sigma},$$

so that $\lambda_0 + \lambda_1 + \lambda_2 = 1$. We choose

$$\theta_p = \begin{cases} 0 & \text{if } y < p \le p_{1,a}, \\ \pi + u_1 & \text{if } p_{1,a} < p \le p_{2,a}, \\ \pi - u_2 & \text{if } p_{2,a} < p \end{cases}$$

with u_1 and u_2 to be fixed later. In view of (3) it is sufficient to solve, for each a, the equation

$$\lambda_1 e^{iu_1} + \lambda_2 e^{-iu_2} = \lambda_0 - \frac{w_a}{S_a} \tag{5}$$

in the real unknowns u_1 and u_2, continuously in w_a for $|w_a| \le \|C^{-1}\|_\infty R$. We define the function F by

$$F : \,]0, \pi/2[^2 \to \mathbb{C}, \qquad (u_1, u_2) \mapsto \lambda_1 e^{iu_1} + \lambda_2 e^{-iu_2}.$$

Then F is a diffeomorphism onto its image. Moreover, as $\frac{1}{3} \le \lambda_0, \lambda_1 \le \frac{1}{3} + \frac{1}{100}$, and $\frac{1}{3} - \frac{1}{50} \le \lambda_2 \le \frac{1}{3}$, we have

$$\{s \in \mathbb{C} : |s - \lambda_0| \le 1/10\} \subset \operatorname{Im} F,$$

as illustrated in Figure 1.

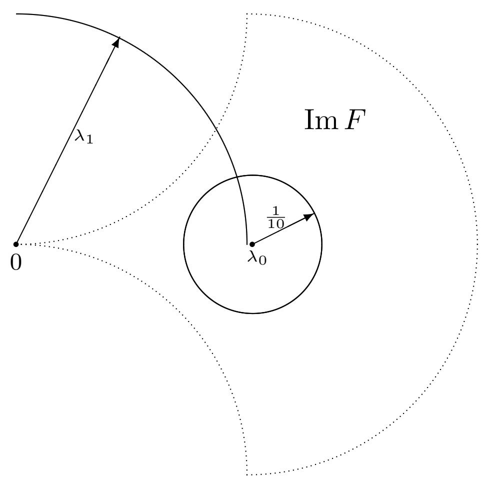

Fig. 1. The image of F (with the dotted boundary) contains the disk with center λ_0 and radius $1/10$.

Thus by (4) we can solve (5) continuously in w_a. This concludes the proof of Lemma 1. ∎

LEMMA 2. *Let q and L be positive integers, and $R \geq 1$ be real. Let $\chi_1, \ldots, \chi_n$ be pairwise distinct Dirichlet characters modulo q. For all $1 \leq j \leq n$, let h_j be a rational function in L complex variables, not identically vanishing. Then there exists a real $\eta > 0$ such that, for all σ with $1 < \sigma \leq 1 + \eta$, we have*

$$\{z \in \mathbb{C}^n : 1/R \leq |z_j| \leq R\}$$
$$\subset \left\{ \left(h_j\left(\frac{1}{p_1^{\sigma+it_{p_1}}}, \ldots, \frac{1}{p_L^{\sigma+it_{p_L}}}\right) \prod_{p>p_L} \left(1 - \frac{\chi_j(p)}{p^{\sigma+it_p}}\right)^{-1} \right)_{1\leq j\leq n} : t_p \in \mathbb{R} \right\}.$$

Proof. We first consider the particular case where all the h_j are 1. We put $y = p_L$ and $R' = \pi + \log R$. Applying Lemma 1 (and changing the letter z to w) we have continuous functions t_p such that

$$w_j = \sum_{p>y} \frac{\chi_j(p)}{p^{\sigma+it_p(w)}}, \qquad w \in D_n(1+R'),\ 1 \leq j \leq n. \tag{6}$$

We define the error term E by

$$\left(\sum_{p>y} \log\left(1 - \frac{\chi_j(p)}{p^{\sigma+it_p}}\right)\right)_{1\leq j\leq n} = \left(-\sum_{p>y} \frac{\chi_j(p)}{p^{\sigma+it_p}}\right)_{1\leq j\leq n} + E((t_p)_{p>y}). \tag{7}$$

The real number $\sigma > 1$ being fixed, the function E is continuous for the product topology. Moreover, for all j and all $(t_p)_{p>y}$, we have

$$|E_j((t_p)_{p>y})| \leq \sum_p \frac{1}{p^2} < 1. \tag{8}$$

Let $z \in D_n(R')$ be fixed. From (8) we see that, for all j and all $(t_p)_{p>y}$,

$$|z_j + E_j((t_p)_{p>y})| \leq 1 + R'.$$

Thus we have the following continuous function:

$$F : D_n(1+R') \to D_n(1+R'), \qquad w \mapsto z + E((t_p(w))_{p>y}).$$

The Brouwer fixed point theorem shows that there exists a $w \in D_n(1+R')$ such that $F(w) = w$. Together with (6) and (7) this yields

$$\left(-\sum_{p>y} \log\left(1 - \frac{\chi_j(p)}{p^{\sigma+it_p(w)}}\right)\right)_{1\leq j\leq n} = z.$$

Taking exponentials allows us to conclude the case when $h_j \equiv 1$.

We now consider the case with a general h. Let us choose $(t_{p_1}, \ldots, t_{p_L})$ such that for all j, $h_j(1/p_1^{\sigma+it_{p_1}}, \ldots, 1/p_L^{\sigma+it_{p_L}})$ has neither zeros nor poles for $1 \le \sigma \le 2$. We put

$$c := \min_{1\le j\le n} \min_{1\le\sigma\le 2} \left| h_j\left(\frac{1}{p_1^{\sigma+it_{p_1}}}, \ldots, \frac{1}{p_L^{\sigma+it_{p_L}}}\right)\right|,$$
$$C := \max_{1\le j\le n} \max_{1\le\sigma\le 2} \left| h_j\left(\frac{1}{p_1^{\sigma+it_{p_1}}}, \ldots, \frac{1}{p_L^{\sigma+it_{p_L}}}\right)\right|.$$

Applying the particular case where $h_j \equiv 1$ with $\widetilde{R} = \max(C/R, R/c)$ allows us to conclude the general case. ∎

3. Proof of the Theorem. Let $F(s)$ in $\mathcal{F}$ be such that it is not of the form $P(s)L_\chi(s)$, with P in $\mathcal{P}$. By previous results (see the introduction), we only need to show that there exists an $\eta > 0$ such that, for $1 < \sigma_1 < \sigma_2 < 1 + \eta$ and T large enough, we have $N'(\sigma_1, \sigma_2, T) \gg T$.

By Theorem PDCB, we can write

$$F(s) = \sum_{j=1}^{n} F_j(s),$$

where $n \ge 2$,

$$\frac{F_j(s)}{L_{\psi_j}(s)} = \sum_{k\ge 1} \frac{c_{j,k}}{k^s}$$

are Dirichlet polynomials for $1 \le j \le n$, and $\psi_1, \ldots, \psi_n$ are distinct primitive Dirichlet characters. Let q be the least common multiple of the conductors of $\psi_1, \ldots, \psi_n$. Choose $y = p_L$ such that if p divides a k for which there is a j such that $c_{j,k} \ne 0$, then $p \le y$. Denoting by χ_j the Dirichlet character modulo q that is induced by ψ_j we can thus write

$$F_j(s) = h_j\left(\frac{1}{p_1^s}, \ldots, \frac{1}{p_L^s}\right) \prod_{p>p_L} \left(1 - \frac{\chi_j(p)}{p^s}\right)^{-1}$$

where h_j is a rational function, not identically vanishing, such that

$$h_j \text{ has no poles in } \{(z_1, \ldots, z_L) \in \mathbb{C}^L : |z_l| < 1\}. \tag{9}$$

Choosing $R = 1$ we get by Lemma 2 a real $\eta > 0$, which will be the one we use here. Let σ_1 and σ_2 be real numbers such that $1 \le \sigma_1 < \sigma_2 \le 1 + \eta$. We choose

$$\sigma = \frac{\sigma_1 + \sigma_2}{2}.$$

By Lemma 2, there is a sequence $(t_p)_p$ of real numbers such that for all j with $1 \le j \le n$,

$$h_j\left(\frac{1}{p_1^{\sigma+it_{p_1}}}, \ldots, \frac{1}{p_L^{\sigma+it_{p_L}}}\right) \prod_{p>p_L} \left(1 - \frac{\chi_j(p)}{p^{\sigma+it_p}}\right)^{-1} = e^{2i\pi j/n}.$$

We write

$$G_j(s) := h_j\left(\frac{1}{p_1^{s+it_{p_1}}}, \dots, \frac{1}{p_L^{s+it_{p_L}}}\right) \prod_{p>p_L} \left(1 - \frac{\chi_j(p)}{p^{s+it_p}}\right)^{-1}.$$

As $n \geq 2$, we have

$$\sum_{j=1}^{n} G_j(\sigma) = 0. \tag{10}$$

We now choose a circle $C = C(\sigma, r)$ centered at $\sigma = (\sigma_1 + \sigma_2)/2$ and with a radius r with $0 < r < (\sigma_2 - \sigma_1)/2$, such that $\sum_{j=1}^{n} G_j(s)$ does not vanish on C. We write

$$\gamma := \min_{s\in C} \left|\sum_{j=1}^{n} G_j(s)\right| > 0.$$

Because of (9) and the uniform convergence of the infinite products, we can choose a prime number $p_M \geq p_L$ such that for all j with $1 \leq j \leq n$,

$$\left|F_j(z) - h_j\left(\frac{1}{p_1^z}, \dots, \frac{1}{p_L^z}\right) \prod_{p_L<p\leq p_M} \left(1 - \frac{\chi_j(p)}{p^z}\right)^{-1}\right| < \frac{\gamma}{3n}, \qquad \operatorname{Re} z \geq \sigma - r,$$

and

$$\left|G_j(s) - h_j\left(\frac{1}{p_1^{s+it_{p_1}}}, \dots, \frac{1}{p_L^{s+it_{p_L}}}\right) \prod_{p_L<p\leq p_M} \left(1 - \frac{\chi_j(p)}{p^{s+it_p}}\right)^{-1}\right| < \frac{\gamma}{3n}, \qquad \operatorname{Re} s \geq \sigma - r.$$

By Weyl's criterion, we know that the set $\{p_1^{it}, \dots, p_M^{it}\}$ is uniformly distributed in $\{z : |z| = 1\}^M$. Using (9) once more it follows that the set of $t \in \mathbb{R}$ such that for all s with $|s - \sigma| \leq r$ and all j with $1 \leq j \leq n$,

$$\begin{aligned}\Bigg|h_j\left(\frac{1}{p_1^{s+it}}, \dots, \frac{1}{p_L^{s+it}}\right) &\prod_{p_L<p\leq p_M} \left(1 - \frac{\chi_j(p)}{p^{s+it}}\right)^{-1} \\ &- h_j\left(\frac{1}{p_1^{s+it_{p_1}}}, \dots, \frac{1}{p_L^{s+it_{p_L}}}\right) \prod_{p_L<p\leq p_M} \left(1 - \frac{\chi_j(p)}{p^{s+it_p}}\right)^{-1}\Bigg| < \frac{\gamma}{3n},\end{aligned}$$

has positive lower density. For these real t, we thus have

$$\max_{s\in C} \left|\sum_{j=1}^{n} F_j(s+it) - G_j(s)\right| < \gamma = \min_{s\in C} \left|\sum_{j=1}^{n} G_j(s)\right|.$$

As $\sum_{j=1}^{n} G_j(\sigma) = 0$ (formula (10)), it follows by Rouché's theorem that $F(s+it) = \sum_{j=1}^{n} F_j(s+it)$ has at least one zero in $|s - \sigma| < r$. By the positive lower density of these t, we conclude that $N'_F(\sigma_1, \sigma_2, T) \gg_{F,\sigma_1,\sigma_2} T$ for sufficiently large T.

Appendix

Proposition. *Let $a = (a_n)_{n\ge 1}$ be a periodic sequence not identically equal to zero. Then*

$$N_a(1/2+u,\infty,T) \ll_a T\,\frac{\log(1/u)}{u}$$

for $0 < u \le 1/2$ and $T \ge 1$.

Proof. We establish the upper bound for the number $N_a^+(1/2+u,\infty,T)$ of zeros in $1/2+u < \operatorname{Re} s < \infty$, $0 \le \operatorname{Im} s \le T$. The proof is similar for zeros with negative imaginary part.

Let $\zeta(s,r)$ denote the Hurwitz zeta function. From Theorem 1 of [3] we have, for $1/2 < \sigma < 1$,

$$\begin{aligned}(11)\qquad \int_0^T |F_a(\sigma+it)|^2\,dt &= \frac{T}{q^{2\sigma}}\sum_{j=1}^{q}|a_j|^2\zeta(2\sigma,j/q) + O\biggl(\frac{q^{2-2\sigma}T^{2-2\sigma}\sum_{j=1}^{q}|a_j|^2}{(2\sigma-1)(1-\sigma)}\biggr)\\ &= O_a\biggl(\frac{T}{(2\sigma-1)(1-\sigma)}\biggr),\end{aligned}$$

since $\zeta(2\sigma,r) = O_r((2\sigma-1)^{-1})$. By Jensen's inequality,

$$(12)\qquad \int_0^T \log|F_a(\sigma+it)|\,dt \le \frac{T}{2}\log\biggl(\frac{1}{T}\int_0^T|F_a(\sigma+it)|^2\,dt\biggr) = O_a(T\log(1/u))$$

for $\sigma = (1+u)/2$, according to (11).

Let a_m be the first nonzero term of the sequence $(a_n)_{n\ge1}$, and let $c \ge 2$ be large enough so that, for $\operatorname{Re} s \ge c$, we have $F_a(s) = (a_m/m^s)(1+\theta(s))$ with $|\theta(s)| \le 1/2$. We apply Littlewood's lemma (see [10, Section 3.8]) to the rectangle R with vertices $c+i$, $c+iT$, $(1+u)/2+iT$, $(1+u)/2+i$, to get

$$\begin{aligned}2\pi\sum_{\substack{\beta>(1+u)/2\\ 1<\gamma\le T}}(\beta-(1+u)/2) &= \int_1^T \log|F_a((1+u)/2+it)|\,dt - \int_1^T \log|F_a(c+it)|\,dt\\ &\quad + \int_{(u+1)/2}^{c} \arg F_a(\sigma+iT)\,d\sigma - \int_{(u+1)/2}^{c}\arg F_a(\sigma+i)\,d\sigma.\end{aligned}$$

The second integral is clearly $O_a(T)$ since $\log|F_a(c+it)| \ll 1$. Steuding shows on page 302 of [8] that $|\arg(1+\theta(\sigma+iT))| \ll \log T$ if σ is from a bounded interval. Thus the third integral is $O_a(T)$. The last integral is bounded. Together with (12) this shows that

$$\sum_{\substack{\beta>(1+u)/2\\ 1<\gamma\le T}} (\beta-(1+u)/2) = O_a(T\log(1/u)).$$

The desired bound now follows from

$$\frac{u}{2}N_a^+(1/2+u,\infty,T) = \sum_{\substack{\beta>1/2+u\\ 0\le\gamma\le T}} \frac{u}{2} \le \sum_{\substack{\beta>(1+u)/2\\ 0\le\gamma\le T}} (\beta-(1+u)/2) \ll_a T\log(1/u). \quad \blacksquare$$

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