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On the determination of the Plancherel measure for Lebedev–Whittaker transforms on GL(n)

by

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Dedicated to Andrzej Schinzel on the occasion of his 75th birthday

1. Introduction. The classical Plancherel formula states that the inner product of two functions is the same as the inner product of their (Fourier) transforms. This fact has been vastly generalized, and the measure appearing on the transform side is called the *Plancherel measure*.

The type of Plancherel measure we will consider comes from the so-called Lebedev–Whittaker transform, the earliest version of which is the Kontorovich–Lebedev transform (see [KL38, KL39]). The original transform, a type of index transform involving modified Bessel functions, was introduced to solve certain boundary-value problems. It has since found many applications in modern analytic number theory (see e.g. [IK04]), as it is a form of a Whittaker transform on GL(2). It has a natural generalization to reductive Lie groups, as has been carried out in [Wal92, §15].

The main aim of this note is to obtain a very concrete and explicit version of the Lebedev–Whittaker transform and associated Plancherel measure for the group $\mathrm{GL}(n,\mathbb{R})$, as well as sketch an elementary proof of the inverse transform for the group $\mathrm{GL}(3,\mathbb{R})$. We expect that such a realization will be useful for analytic methods in number theory on higher rank groups.

First we set some notation and definitions. For $n \geq 2$, consider an admissible irreducible cuspidal automorphic representation π for $\mathrm{GL}(n,\mathbb{A})$, where $\mathbb{A}$ is the adele group over $\mathbb{Q}$. By Flath's tensor product theorem [Fla79],

$$\pi = \bigotimes \pi_v,$$

where the tensor product goes over irreducible, admissible, unitary local

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representations of $\mathrm{GL}(n,\mathbb{Q}_p)$. We shall assume that π is unramified at infinity.

To characterize the real components of such representations in a more explicit manner, we introduce, for $n \geq 2$, the generalized upper half-plane

$$\mathfrak{h}^n := \mathrm{GL}(n,\mathbb{R})/(O(n,\mathbb{R}) \cdot \mathbb{R}^\times).$$

By the Iwasawa decomposition, every $z \in \mathfrak{h}^n$ may be uniquely written in the form $z = xy$ with $x \in U_n(\mathbb{R})$ (the group of unipotent upper triangular matrices in $\mathrm{GL}(n,\mathbb{R})$) and y a diagonal matrix of the form

$$y = \begin{pmatrix} y_1 \cdots y_{n-1} & & & \\ & \ddots & & \\ & & y_1 & \\ & & & 1 \end{pmatrix} \qquad (y_i > 0 \text{ for } i = 1, \ldots, n-1). \tag{1.1}$$

Whenever we write $z = xy \in \mathfrak{h}^n$ we assume that x, y are as described above.

Let W_∞ be a Whittaker model for π_∞. Then there exists a *spherical Whittaker function* in W_∞ which is K_∞-fixed for the maximal compact subgroup $K_\infty = O(n,\mathbb{R})$. Let $\mathfrak{D}^n$ denote the algebra of $\mathrm{GL}(n,\mathbb{R})$-invariant differential operators acting on $\mathfrak{h}^n$. Then $W : \mathfrak{h}^n \to \mathbb{C}$ is characterized up to scalars by the fact that W is an eigenfunction of $\mathfrak{D}^n$, and in addition,

$$W(uz) = \psi(u) \cdot W(z) \qquad (z \in \mathfrak{h}^n)$$

for any $u \in U_n(\mathbb{R})$ and some fixed character ψ of $U_n(\mathbb{R})$. Associated to π_∞, there exist spectral parameters $\nu = (\nu_1, \ldots, \nu_{n-1}) \in \mathbb{C}^{n-1}$ so that we may write (see [Gol06, §5.9] for the *completed* Jacquet–Whittaker function, which is used exclusively throughout this paper)

$$\begin{aligned} &W_\nu(z) \\ &= \prod_{j=1}^{n-1} \prod_{j \leq k \leq n-1} \pi^{-1/2 - v_{j,k}} \Gamma\left(\frac{1}{2} + v_{j,k}\right) \cdot \int\limits_{U_n(\mathbb{R})} I_\nu(w_n u z) \overline{\psi(u)}\, d^\times u \qquad (z \in \mathfrak{h}^n). \end{aligned} \tag{1.2}$$

Here Γ is the Gamma function, w_n is the long element of the Weyl group, the I-function is given by

$$I_\nu(z) = \prod_{i=1}^{n-1} \prod_{j=1}^{n-1} y_i^{b_{ij} \cdot \nu_j} \qquad (z = xy \in \mathfrak{h}^n)$$

with

$$b_{ij} = \begin{cases} ij & \text{if } i + j \leq n, \\ (n-i)(n-j) & \text{if } i + j \geq n, \end{cases} \tag{1.3}$$

and

$$v_{j,k} = \sum_{i=0}^{j-1} \frac{n\nu_{n-k+i} - 1}{2}.$$

There should be no confusion between the real number $\pi = 3.14\ldots$ in (1.2) and the representation π.

In this paper we will take $z = y$, whence the Whittaker function is independent of the character ψ. Hence we lose no generality by restricting our attention from now on to the standard character

$$\psi\begin{pmatrix} 1 & x_{n-1} & * & * \\ & \ddots & x_2 & * \\ & & 1 & x_1 \\ & & & 1 \end{pmatrix} = e(x_1 + \cdots + x_{n-1}).$$

Since we assumed that the local representation π_∞ is tempered, it follows that

$$\nu_j = 1/n + it_j \tag{1.4}$$

with $t_j \in \mathbb{R}$ $(j = 1, \ldots, n-1)$. In this case we abuse notation and define $W_{it} := W_\nu$ where W_ν is given by (1.2) and $t = (t_1, \ldots, t_{n-1})$. The Haar measure on the Levy component is given by

$$d^\times y = \prod_{k=1}^{n-1} y_k^{-k(n-k)} \frac{dy_k}{y_k}.$$

Definition 1.1 (Lebedev–Whittaker transform). Let $f : \mathbb{R}_+^{n-1} \to \mathbb{C}$, let y be as in (1.1), and let $t = (t_1, \ldots, t_{n-1}) \in \mathbb{R}^{n-1}$. Then we define the *Lebedev–Whittaker transform* $f^\# : \mathbb{R}_+^{n-1} \to \mathbb{C}$ by

$$f^\#(t) := \int_{\mathbb{R}_+^{n-1}} f(y) W_{it}(y)\, d^\times y,$$

provided the above integral converges absolutely.

The inverse transform is given in the next definition. Let

$$\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$$

be linear functions of $t \in \mathbb{R}^{n-1}$ defined as follows. Recall b_{kl} defined in (1.3). For $1 \le k \le n-1$, the α_k are determined by (see (11.6.15) in [Gol06])

$$\frac{k(n-k)}{2} + \sum_{l=1}^{n-k} \frac{\alpha_l}{2} = \sum_{l=1}^{n-1} b_{kl} \cdot \nu_l, \tag{1.5}$$

and

$$\alpha_n = -\sum_{k=1}^{n-1} \alpha_k.$$

Observe that the Whittaker function W_{it} is invariant under any permutation of $\alpha_1, \ldots, \alpha_n$, and hence $f^\sharp$ inherits these symmetries. We call such a function α-*symmetric*.

Definition 1.2 (Lebedev–Whittaker inverse transform). Let $h : \mathbb{R}^{n-1} \to \mathbb{C}$ be α-symmetric. Then define the *Lebedev–Whittaker inverse transform* by

$$h^\flat(y) = \frac{1}{(4\pi)^{n-1}} \int\limits_{\mathbb{R}^{n-1}} h(t)\overline{W_{it}(y)}\, \frac{dt}{\prod_{1\le k\ne l\le n} \Gamma\big(\frac{\alpha_k-\alpha_l}{2}\big)},$$

assuming the integral converges absolutely.

In [Wal92], a very general Lebedev–Whittaker transform on reductive Lie groups is studied from which one may derive (with some work) the following very explicit result.

Theorem 1.3 ([Wal92]). *Under suitable growth and regularity conditions on f and h as above,*

$$(f^\sharp)^\flat = f, \tag{1.6}$$

$$(h^\flat)^\sharp = h. \tag{1.7}$$

Corollary 1.4. *For f_1, f_2 as above, with transforms $h_1 = f_1^\sharp$, $h_2 = f_2^\sharp$, we have*

$$\begin{aligned} \langle f_1, f_2\rangle &= \int\limits_{\mathbb{R}_+^{n-1}} f_1(y)\overline{f_2(y)}\, d^\times y \\ &= \langle h_1, h_2\rangle = \frac{1}{(4\pi)^{n-1}} \int\limits_{\mathbb{R}^{n-1}} h_1(t)\overline{h_2(t)} \frac{dt}{\prod_{1\le k\ne l\le n} \Gamma\big(\frac{\alpha_k-\alpha_l}{2}\big)}. \end{aligned} \tag{1.8}$$

Thus the measure

$$\frac{dt}{\prod_{1\le k\ne l\le n} \Gamma\big(\frac{\alpha_k-\alpha_l}{2}\big)}$$

is the Plancherel measure for the Lebedev–Whittaker transform on $\mathrm{GL}(n,\mathbb{R})$. Notice that by taking the product of half of the Gamma functions in the denominator, i.e. by taking $\prod_{1\le k<l\le n} \Gamma\big(\frac{\alpha_k-\alpha_l}{2}\big)$, we obtain the Harish-Chandra c-function, $c(i\nu)$ (see Wallach [Wal92, §15.10.3]), so the measure can also be written, after the linear change of variables (1.4), as

$$\frac{d\nu}{c(i\nu)c(-i\nu)}.$$

Our goal is to sketch an explicit and elementary proof of (1.7) for the group GL(3). The proof uses only complex analysis (the residue theorem) and the location of poles and residues of the Gamma function. Admittedly, it relies crucially on Stade's [Sta02] formula (see §2 below), but this is again a vast generalization of Barnes' lemma. We expect our methods to have other applications in higher rank analytic number theory.

Organization. This paper is organized as follows. In §2, we recall Stade's formula, which is a key ingredient in our proof. As an afterthought, we also treat in the Appendix the case of GL(1), by giving an elementary proof of the Mellin inversion formula.

2. Stade's formula. We use the notation set up in the previous section. Recall that we are assuming that π_∞ is unramified, which implies that the eigenvalue parameters ν are tempered, i.e. $\nu_j = 1/n + it_j$ with $t_j \in \mathbb{R}$; see (1.4). Let $\mu_j = 1/n + iu_j$ with $u_j \in \mathbb{R}$, and define β_j related to u_j in the same way as α_j are related to t_j, that is, (1.5).

Stade's formula for GL(n) (see [Gol06, Prop 11.6.17]) is as follows.

Theorem 2.1 ([Sta02]). *Let $n \geq 2$. Then for $t, u \in \mathbb{R}^{n-1}$ and $s \in \mathbb{C}$ with $\Re(s) > 0$,*

$$\int_{\mathbb{R}_+^{n-1}} W_{it}(y)\overline{W_{iu}(y)}(\det y)^s \, d^\times y = \frac{1}{2\pi^{sn(n-1)/2}} \frac{\prod_{j=1}^n \prod_{k=1}^n \Gamma\left(\frac{s+\alpha_j+\overline{\beta_k}}{2}\right)}{\Gamma\left(\frac{ns}{2}\right)}. \tag{2.1}$$

3. Lebedev–Whittaker inversion for GL(3). We now specialize to $n = 3$. In this case, the Lebedev–Whittaker transform of a continuous function $f : \mathbb{R}_+^2 \to \mathbb{C}$ becomes

$$f^\sharp(t_1, t_2) := \int_{y_1=0}^{\infty} \int_{y_2=0}^{\infty} f(y_1, y_2) W_{it_1,it_2}(y_1, y_2) \, \frac{dy_1\, dy_2}{y_1^3 y_2^3}, \tag{3.1}$$

provided f has sufficient decay properties so that the above integral is absolutely convergent.

Note that $f^\sharp(t_1, t_2)$ inherits the same functional equations as W_{it_1,it_2}, i.e., it is invariant under permutation of the parameters $\alpha_1, \alpha_2, \alpha_3$ defined by (cf. (1.5))

$$\alpha_1 = 2it_1 + it_2, \qquad \alpha_2 = -it_1 + it_2, \qquad \alpha_3 = -it_1 - 2it_2. \tag{3.2}$$

Recall such a function is called α-symmetric. It is convenient to also define

$$t_3 := t_1 + t_2. \tag{3.3}$$

The inverse transform is given as follows. For $h:\mathbb{R}^2\to\mathbb{C}$ with the above symmetries in (t_1,t_2), we have

$$h^\flat(y_1,y_2)=\frac{1}{(4\pi)^2}\int_{t_1=-\infty}^{\infty}\int_{t_2=-\infty}^{\infty}h(t_1,t_2)\overline{W_{it_1,it_2}(y_1,y_2)}\times\frac{dt_1\,dt_2}{\prod_{1\le l\ne l'\le 3}\Gamma\left(\frac{\alpha_l-\alpha_{l'}}{2}\right)},\tag{3.4}$$

assuming the integral converges absolutely.

For $\eta>0$ and $A\ge 10$, we introduce the class $\mathcal{H}_{\eta,A}$ of functions $h(t_1,t_2)$ which are α-symmetric, have holomorphic extension to the horizontal strip $\Im(t_1),\Im(t_2)\in(-\eta,\eta)$, and satisfy

$$h(t_1,t_2)\ll_h \exp\left(-\frac{3\pi}{4}\sum_{k=1}^{3}|t_k|\right)\prod_{k=1}^{3}(1+|t_k|)^{-A}\tag{3.5}$$

in this strip. This class is nonempty; for example, it contains $e^{-(t_1^2+t_2^2+t_3^2)}$, with $\eta=1$, say, and any $A>0$.

We first analyze convergence issues.

Remark 3.1. One of our aims will be to justify the convergence of the integral (3.1) for $f=h^\flat$. It is easy to see that the Whittaker function $W_{it}(y)$ has arbitrary polynomial decay when y is large, and decays like y_1y_2 for y_1,y_2 small. Since the Haar measure has the factor $(y_1y_2)^{-3}$, in order for (3.1) to converge absolutely, we need $h^\flat$ to have decay of the form $(y_1y_2)^{1+\varepsilon}$ (for some fixed $\varepsilon>0$). This is accomplished via η, as follows.

Lemma 3.2. *For $\eta>0$, $A\ge 10$, and $h\in\mathcal{H}_{\eta,A}$, the integral* (3.4) *defining the inverse transform $h^\flat(y)$ converges absolutely, and satisfies*

$$h^\flat(y)\ll_h (y_1y_2)^{1+\eta/2}\tag{3.6}$$

for all $0<y_1,y_2<\infty$.

Proof. To check absolute convergence, we need the double inverse Mellin transform formula [Sta01] for the Whittaker function:

$$W_{it}(y)=\frac{y_1y_2\pi^{3/2}}{(2\pi i)^2}\int_{(2)}\int_{(2)}\frac{\prod_{k=1}^{3}\Gamma\left(\frac{s_1+\alpha_k}{2}\right)\Gamma\left(\frac{s_2-\alpha_k}{2}\right)}{4\pi^{s_1+s_2}\Gamma\left(\frac{s_1+s_2}{2}\right)}y_1^{-s_1}y_2^{-s_2}\,ds_1\,ds_2,\tag{3.7}$$

where the integrals are over the vertical lines $\Re(s_j)=2$. Inserting (3.7) into (3.4), putting absolute values, writing $s_j=2+iu_j$, and using (3.5) and

Stirling's formula gives

$$(3.8)\qquad |h^\flat(y)| \ll_h \frac{1}{y_1y_2}\iint\limits_{\mathbb{R}^2}\iint\limits_{\mathbb{R}^2} \frac{\prod_{k=1}^3[(1+|iu_1+\alpha_k|)^{1/2}(1+|iu_2-\alpha_k|)^{1/2}(1+|t_k|)^{1-A}]}{(1+|u_1+u_2|)^{3/2}} \times \exp\Big[-\frac{\pi}{4}\sum_{k=1}^{3}(|iu_1+\alpha_k|+|iu_2-\alpha_k|-|u_1+u_2|-3|t_k|)\Big] \times du_1\,du_2\,dt_1\,dt_2.$$

It is straightforward to estimate the result (see [Blo11, proof of Proposition 1] where a similar calculation is carried out). The exponential factor cuts off the range of u_1, u_2, which then contributes a polynomial growth in t_1, t_2. This is offset by $|t_1t_2|^{-A}$, and the resulting integral in t converges for $A \geq 10$.

It remains to control the decay in y. Using the above argument and pulling the contours of the s_1- and s_2-integrals to $\Re(s_j) = \varepsilon$ gives

$$h^\flat(y) \ll_\varepsilon (y_1y_2)^{1-\varepsilon},$$

which is just shy of our goal. So we must pull the contour in (3.7) past the poles on the lines $\Re(s_j) = 0$, say to the lines $\Re(s_j) = -1/2$, and pick up the resulting polar contributions (1) (the α_k are distinct except for a set of measure zero):

$$(3.9)\qquad W_{it}(y) = \sum_{\{\delta_1,\delta_2,\delta_3\}} \frac{y_1^{1+\delta_1}y_2^{1-\delta_2}\Gamma\big(\frac{-\delta_1+\delta_2}{2}\big)\Gamma\big(\frac{-\delta_1+\delta_3}{2}\big)\Gamma\big(\frac{\delta_2-\delta_3}{2}\big)}{\pi^{-\delta_1+\delta_2-3/2}} + \frac{y_1y_2\pi^{3/2}}{(2\pi i)^2}\int\limits_{(-1/2)}\int\limits_{(-1/2)}(\cdot)y_1^{-s_1}y_2^{-s_2}\,ds_1\,ds_2,$$

where the sum on $\{\delta_1,\delta_2,\delta_3\}$ runs over all permutations of $\{\alpha_1,\alpha_2,\alpha_3\}$.

The analysis on the remainder is the same as above, with y-dependence of the form $(y_1y_2)^{3/2}$, which is more than adequate for our purpose.

We now focus on the first of the six polar contributions; the others are handled similarly. Putting the contribution into (3.4) gives

$$\mathcal{P}_1 := \frac{1}{(4\pi)^2}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} h(t_1,t_2)\frac{y_1^{1-\alpha_1}y_2^{1+\alpha_2}\Gamma\big(\frac{\alpha_1-\alpha_2}{2}\big)\Gamma\big(\frac{\alpha_1-\alpha_3}{2}\big)\Gamma\big(\frac{-\alpha_2+\alpha_3}{2}\big)}{\pi^{\alpha_1-\alpha_2-3/2}\prod_{k=1}^3\Gamma\big(\frac{3it_k}{2}\big)\Gamma\big(\frac{-3it_k}{2}\big)}\,dt_1\,dt_2.$$

(1) Note that if we were to pull all the way left, we would recover the Taylor expansion in [Mez11, §3.2].

The three Gamma factors in the numerator cancel half of the factors in the denominator. Recalling from (3.2) that $\alpha_1 = 2it_1 + it_2$ and $\alpha_2 = -it_1 + it_2$, we see that if we pull the contour of the t_1-integral from $\Im(t_1) = 0$ to $\Im(t_1) = \eta/2$, say, then putting absolute values gives a y-dependence of the form $y_1^{1+\eta} y_2^{1+\eta/2}$. Then by Stirling's formula and (3.5), the rest of the integral converges absolutely, for $A \gg 1$. ∎

We now present our main result in the following theorem, which is just a restatement of (1.7). The added caveat is that we give very precise growth and regularity conditions under which the theorem holds.

THEOREM 3.3. *For $\eta > 0$, $A \geq 10$ and $h \in \mathcal{H}_{\eta,A}$, we have*

$$\boxed{(h^\flat)^\sharp = h.}$$

Proof. Assume that $h(t_1, t_2) \in \mathcal{H}_{\eta,A}$. For any $0 \leq \varepsilon < \eta/10$, define the function [2]

$$\mathcal{H}(t_1, t_2, \varepsilon) := \int_{y_1=0}^{\infty} \int_{y_2=0}^{\infty} h^\flat(y_1, y_2) W_{it_1,it_2}(y_1, y_2)(y_1^2 y_2)^\varepsilon \frac{dy_1}{y_1^3} \frac{dy_2}{y_2^3}.$$

The integral above converges absolutely by (3.6), (3.8), and (3.9), and its value at $\varepsilon = 0$ is exactly $(h^\flat)^\sharp$. Hence we must show that

$$\mathcal{H}(t_1, t_2, \varepsilon) \to h(t_1, t_2)$$

as $\varepsilon \to 0$. For simplicity, we assume that the α_j are all distinct. The case when the α_j are not distinct can be handled in a similar manner.

Insert the definition of $h^\flat$ in the above integral to obtain

$$\mathcal{H}(t_1, t_2, \varepsilon) = \int_{y_1=0}^{\infty} \int_{y_2=0}^{\infty} \Bigg(\frac{1}{(4\pi)^2} \int_{t_1'=-\infty}^{\infty} \int_{t_2'=-\infty}^{\infty} h(t_1', t_2') \overline{W_{it_1',it_2'}(y_1, y_2)} \times \frac{dt_1'\, dt_2'}{\Gamma\big(\frac{3it_1'}{2}\big)\Gamma\big(\frac{-3it_1'}{2}\big)\Gamma\big(\frac{3it_2'}{2}\big)\Gamma\big(\frac{-3it_2'}{2}\big)\Gamma\big(\frac{3it_1'+3it_2'}{2}\big)\Gamma\big(\frac{-3it_1'-3it_2'}{2}\big)} \Bigg) \times W_{it_1,it_2}(y)(y_1^2 y_2)^\varepsilon \frac{dy_1}{y_1^3} \frac{dy_2}{y_2^3}.$$

Next, interchange the orders of integration and insert Stade's formula (2.1) with $s = \varepsilon$. After simplifying, we obtain the following:

(2) As kindly pointed out to us by the referee, a similar trick is used in the GL(2) case in [Yak96, (2.30)].

$$\begin{aligned}
&\mathcal{H}(t_1,t_2,\varepsilon)\\
&= \frac{1}{(4\pi)^2} \int_{t_1'=-\infty}^{\infty} \int_{t_2'=-\infty}^{\infty} h(t_1',t_2') \Bigg[\frac{1}{\pi^{3\varepsilon}\Gamma(3\varepsilon/2)2} \Gamma\Big(\frac{\varepsilon+2it_1+it_2-2it_1'-it_2'}{2}\Big)\\
&\qquad\times \Gamma\Big(\frac{\varepsilon+2it_1+it_2+it_1'-it_2'}{2}\Big)\Gamma\Big(\frac{\varepsilon+2it_1+it_2+it_1'+2it_2'}{2}\Big)\\
&\qquad\times \Gamma\Big(\frac{\varepsilon-it_1+it_2-2it_1'-it_2'}{2}\Big)\Gamma\Big(\frac{\varepsilon-it_1+it_2+it_1'-it_2'}{2}\Big)\\
&\qquad\times \Gamma\Big(\frac{\varepsilon-it_1+it_2+it_1'+2it_2'}{2}\Big)\Gamma\Big(\frac{\varepsilon-it_1-2it_2-2it_1'-it_2'}{2}\Big)\\
&\qquad\times \Gamma\Big(\frac{\varepsilon-it_1-2it_2+it_1'-it_2'}{2}\Big)\Gamma\Big(\frac{\varepsilon-it_1-2it_2+it_1'+2it_2'}{2}\Big)\Bigg]\\
&\qquad\times \frac{dt_1'\,dt_2'}{\Gamma(\frac{3it_1'}{2})\Gamma(\frac{-3it_1'}{2})\Gamma(\frac{3it_2'}{2})\Gamma(\frac{-3it_2'}{2})\Gamma(\frac{3it_1'+3it_2'}{2})\Gamma(\frac{-3it_1'-3it_2'}{2})}.
\end{aligned}$$

Now make the change of variables $(t_1',t_2') \mapsto (\alpha_1',\alpha_2')$, where (see (3.2))

$$\alpha_1' = 2it_1' + it_2', \qquad \alpha_2' = -it_1' + it_2'.$$

The Jacobian is $|\det(\partial\alpha'/\partial t')| = -3$. Similarly, we use the notation (3.2) to simplify the appearance of the above expression, which is now

$$\begin{aligned}
&\mathcal{H}(t_1,t_2,\varepsilon) = \frac{1}{(4\pi)^2} \int_{\alpha_1'=-i\infty}^{i\infty} \int_{\alpha_2'=-i\infty}^{i\infty} h\Big(\frac{\alpha_1'-\alpha_2'}{3i}, \frac{\alpha_1'+2\alpha_2'}{3i}\Big)\\
&\times \Bigg[\frac{1}{\pi^{3\varepsilon}\Gamma(3\varepsilon/2)2}\Gamma\Big(\frac{\varepsilon+\alpha_1-\alpha_1'}{2}\Big)\Gamma\Big(\frac{\varepsilon+\alpha_1-\alpha_2'}{2}\Big)\Gamma\Big(\frac{\varepsilon+\alpha_1+\alpha_1'+\alpha_2'}{2}\Big)\\
&\times \Gamma\Big(\frac{\varepsilon+\alpha_2-\alpha_1'}{2}\Big)\Gamma\Big(\frac{\varepsilon+\alpha_2-\alpha_2'}{2}\Big)\Gamma\Big(\frac{\varepsilon+\alpha_2+\alpha_1'+\alpha_2'}{2}\Big)\\
&\times \Gamma\Big(\frac{\varepsilon+\alpha_3-\alpha_1'}{2}\Big)\Gamma\Big(\frac{\varepsilon+\alpha_3-\alpha_2'}{2}\Big)\Gamma\Big(\frac{\varepsilon+\alpha_3+\alpha_1'+\alpha_2'}{2}\Big)\Bigg]\\
&\times \frac{\frac{-1}{3}\,d\alpha_1'\,d\alpha_2'}{\Gamma(\frac{2\alpha_1'+\alpha_2'}{2})\Gamma(\frac{-2\alpha_1'-\alpha_2'}{2})\Gamma(\frac{\alpha_1'-\alpha_2'}{2})\Gamma(\frac{-\alpha_1'+\alpha_2'}{2})\Gamma(\frac{\alpha_1'+2\alpha_2'}{2})\Gamma(\frac{-\alpha_1'-2\alpha_2'}{2})}.
\end{aligned}$$

Shift the lines of integration from $\alpha_1' \in \{i\mathbb{R}\}$ to $\alpha_1' \in \{\varepsilon_0 + i\mathbb{R}\}$, with $\varepsilon < \varepsilon_0 < \eta/2$. We pass through poles at

$$\begin{aligned}
\alpha_1' &= \varepsilon + \alpha_1, &&\text{with residue } \mathcal{R}_1,\\
\alpha_1' &= \varepsilon + \alpha_2, &&\text{with residue } \mathcal{R}_2,\\
\alpha_1' &= \varepsilon + \alpha_3, &&\text{with residue } \mathcal{R}_3.
\end{aligned}$$

Consider $\mathcal{R}_1$. After a computation, we have

$$\mathcal{R}_1 = \frac{i}{4\pi} \int\limits_{\alpha_2'=-i\infty}^{i\infty} \frac{h\big(\frac{\varepsilon+\alpha_1-\alpha_2'}{3i}, \frac{\varepsilon+\alpha_1+2\alpha_2'}{3i}\big)}{\pi^{3\varepsilon}\Gamma(3\varepsilon/2)2} \bigg[\Gamma\Big(\frac{\alpha_2-\alpha_1}{2}\Big)\Gamma\Big(\frac{\alpha_3-\alpha_1}{2}\Big)$$
$$\times \Gamma\Big(\frac{\varepsilon+\alpha_2-\alpha_2'}{2}\Big)\Gamma\Big(\frac{\varepsilon+\alpha_3-\alpha_2'}{2}\Big)\Gamma\Big(\frac{2\varepsilon-\alpha_3+\alpha_2'}{2}\Big)\Gamma\Big(\frac{2\varepsilon-\alpha_2+\alpha_2'}{2}\Big)\bigg]$$
$$\times \frac{\frac{-1}{3}\,d\alpha_2'}{\Gamma\big(\frac{-2\varepsilon-2\alpha_1-\alpha_2'}{2}\big)\Gamma\big(\frac{-\varepsilon-\alpha_1+\alpha_2'}{2}\big)\Gamma\big(\frac{\varepsilon+\alpha_1+2\alpha_2'}{2}\big)\Gamma\big(\frac{-\varepsilon-\alpha_1-2\alpha_2'}{2}\big)}.$$

Next, in the $\mathcal{R}_1$-integral, we shift the line of integration to the left, from $\alpha_2' \in \{i\mathbb{R}\}$ to $\alpha_2' \in \{\varepsilon_0 + i\mathbb{R}\}$. Now there are poles at

$$\alpha_2' = \varepsilon + \alpha_2, \qquad \text{with residue } \mathcal{R}_{1,1},$$
$$\alpha_2' = \varepsilon + \alpha_3, \qquad \text{with residue } \mathcal{R}_{1,2}.$$

In total there are six such residues $\mathcal{R}_{j,k}$, $1 \le j \le 3$, $1 \le k \le 2$. In fact, by the invariance of h under permutations of $\alpha_1, \alpha_2, \alpha_3$, these residues all have the same contribution. We now evaluate $\mathcal{R}_{1,1}$. After some simplification, we obtain

$$\mathcal{R}_{1,1} = \frac{h\big(\frac{\alpha_1-\alpha_2}{3i}, \frac{3\varepsilon+\alpha_1+2\alpha_2}{3i}\big)\Gamma\big(\frac{-2\alpha_1-\alpha_2}{2}\big)\Gamma\big(\frac{-\alpha_1-2\alpha_2}{2}\big)}{6\cdot\pi^{3\varepsilon}\Gamma\big(\frac{-3\varepsilon-2\alpha_1-\alpha_2}{2}\big)\Gamma\big(\frac{-3\varepsilon-\alpha_1-2\alpha_2}{2}\big)} \to \frac{1}{6}h(t_1,t_2)$$

as $\varepsilon \to 0$. Hence the contribution from the six residues adds up to exactly $h(t_1, t_2)$. The remaining integrals all contain the factor $\Gamma\big(\frac{3\varepsilon}{2}\big)$ in the denominator, making the integrals vanish as $\varepsilon \to 0$. This completes the proof, under the assumption that the α_j are all distinct.

Had the α_j not been distinct, we would have had poles of order two in the contour shifting argument; the rest of the analysis is similar. ■

REMARK 3.4. It is clear that the above algorithm should extend to $\mathrm{GL}(n)$ although the combinatorics will be much more complex. We plan to return to this question at a later date.

Appendix. An elementary proof of Mellin inversion. Fix some Schwartz class function $f : \mathbb{R}^+ \to \mathbb{C}$. Define the *Mellin transform*

$$\tilde{f}(s) := \int\limits_0^\infty f(y)y^s \, \frac{dy}{y} \tag{A.1}$$

and the *Mellin inverse transform*

$$h(x) := \frac{1}{2\pi i} \int\limits_{(2)} \tilde{f}(s)x^{-s}\, ds. \tag{A.2}$$

THEOREM 3.5 (Mellin inversion). $f(x) = h(x)$.

Proof. We require the well-known formula

$$\text{(A.3)} \qquad \frac{1}{2\pi i}\int_{(2)} x^s \frac{ds}{s(s+1)} = \begin{cases} 1-1/x & \text{if } x>1, \\ 0 & \text{if } x<1. \end{cases}$$

Starting with (A.1), integrate by parts twice:

$$\tilde{f}(s) = -\int_0^\infty f'(y)\frac{y^s}{s}\,dy = \int_0^\infty f''(y)\frac{y^{s+1}}{s(s+1)}\,dy.$$

Insert this into (A.2), reverse the order of integration and apply (A.3):

$$\begin{aligned} h(x) &= \frac{1}{2\pi i}\int_{(2)}\Big(\int_0^\infty f''(y)\frac{y^{s+1}}{s(s+1)}\,dy\Big)x^{-s}\,ds \\ &= \int_0^\infty f''(y)\Big(\frac{1}{2\pi i}\int_{(2)}\Big(\frac{y}{x}\Big)^s\frac{ds}{s(s+1)}\Big)y\,dy = \int_x^\infty f''(y)\Big(1-\frac{x}{y}\Big)y\,dy. \end{aligned}$$

And now integrate by parts twice (in the reverse direction):

$$h(x) = \int_x^\infty f''(y)(y-x)\,dy = -\int_x^\infty f'(y)(1)\,dy = f(x),$$

as claimed. ∎

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