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# The terms $Cx^h$ $(h \geq 3)$ in Lucas sequences: an algorithm and applications to diophantine equations

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**1. Introduction.** For each prime $p$, we denote by $v_p$ the $p$-adic valuation. Let $h \geq 2$. The positive integer $n$ is said to be *$h$-power free* when $0 \leq v_p(n) \leq h-1$ for every prime $p$. The symbol $\Diamond$ shall represent any positive integer which is an $h$th power (and this includes the integer 1). If $h = 2$, it is customary to use the symbol $\square$.

Let $P > 0$, $Q \neq 0$ be coprime integers such that $D = P^2 - 4Q \neq 0$. We define the *Lucas sequences of the first kind*, respectively *of the second kind*, with parameters $(P, Q)$, denoted by $\mathcal{U} = \mathcal{U}(P, Q) = (U_n)_{n\geq 0}$ and $\mathcal{V} = \mathcal{V}(P, Q) = (V_n)_{n\geq 0}$, as follows:

$$\begin{aligned} &U_0 = 0, \quad &&U_1 = 1, \quad &&U_n = PU_{n-1} - QU_{n-2}, \\ &V_0 = 2, \quad &&V_1 = P, \quad &&V_n = PV_{n-1} - QV_{n-2} \end{aligned}$$

(for $n \geq 2$).

$D$ is called the *discriminant* of these sequences. We shall henceforth assume that $D > 0$, hence the sequences $(U_n)_{n\geq 0}$ and $(V_n)_{n\geq 0}$ are increasing.

Noteworthy examples of Lucas sequences are the following:

1) $P = 1$, $Q = -1$. Then $D = 5$ and $\mathcal{U}$ is the sequence of Fibonacci numbers, while $\mathcal{V}$ is the sequence of Lucas numbers.

2) $P = 2$, $Q = -1$. Then $D = 8$, $\mathcal{U}$ is the sequence of Pell numbers of the first kind and $\mathcal{V}$ is the sequence of Pell numbers of the second kind.

3) Let $a > b \geq 1$ with $\gcd(a, b) = 1$, and let

$$U_n = \frac{a^n - b^n}{a - b}, \qquad V_n = a^n + b^n$$

(for all $n \geq 0$). Then $\mathcal{U}$, $\mathcal{V}$ are Lucas sequences with parameters $P = a + b$, $Q = ab$ and discriminant $D = (a - b)^2$. A special case is when $a = 2$, $b = 1$, giving $U_n = 2^n - 1$, $V_n = 2^n + 1$.

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We shall be concerned with terms in Lucas sequences of the first kind of the form $U_n = Ax^h$, where $x \geq 1$, $h \geq 2$ and $A \geq 1$ is given and $h$-power free. We first recall some known results. Cohn (and Wyler independently) proved:

*The only square Fibonacci numbers are* $U_1 = U_2 = 1$, $U_{12} = 144$.

Cohn proved also the following results:

*The only Fibonacci numbers of the form* $2\Box$ *are* $U_3 = 2$, $U_6 = 8$. *The only square Lucas number is* $V_3 = 4$. *The only Lucas numbers of the form* $2\Box$ *are* $V_0 = 2$, $V_6 = 18$.

The above results are equivalent to the determination of the solutions of the diophantine equations

$$\begin{aligned} x^2 - 5y^4 &= \pm 4, \\ x^2 - 20y^4 &= \pm 4, \\ x^4 - 5y^2 &= \pm 4, \\ 4x^4 - 5y^2 &= \pm 4, \end{aligned}$$

respectively.

These results were extended in [11].

(1.1) *Let* $PQ$ *be odd.*

1) *If* $U_n = \Box$ *then* $n \in \{1, 2, 3, 6, 12\}$; *for each pair* $(P, Q)$ *there are at most three induces* $n$ (*including* $n = 1$) *such that* $U_n = \Box$.
2) *If* $U_n = 2\Box$ *then* $n \in \{3, 6\}$.
3) *If* $V_n = \Box$ *then* $n \in \{1, 3, 5\}$.
4) *If* $V_n = 2\Box$ *then* $n \in \{0, 3, 6\}$.

If $PQ$ is even, the squares and double squares are only known for special sequences. Without attempting to review all known results, we just quote the remarkable results of Ljunggren [1] about Pell numbers.

(1.2) *For Pell numbers* $(P = 2, Q = -1)$:

1) $U_n = \Box$ *if and only if* $n = 1$ *or* $7$.
2) $U_n = 2\Box$ *if and only if* $n = 2$.
3) $V_n \neq \Box$ *for all* $n \geq 0$.
4) $V_n = 2\Box$ *if and only if* $n = 0$.

Much less is known about terms of the form $x^h$ where $x > 1$, $h \geq 3$. We quote (see London and Finkelstein [2] and Pethő [4]):

(1.3) 1) $U_1 = U_2 = 1$ *and* $U_6 = 8$ *are the only cubes in the sequence of Fibonacci numbers.*

2) $V_1 = 1$ *is the only cube in the sequence of Lucas numbers.*

About the terms of the form $Ax^h$, the following important theorem will be relevant (Shorey and Stewart [12] and Pethő [3]):

(1.4) *For any given Lucas sequence* $\mathcal{U} = (U_n)_{n\geq 0}$ *and any integer* $A \geq 1$, *there exists* $C > 0$ (*depending on* $P$, $Q$ *and* $A$ *and effectively computable*) *such that if* $U_n = Ax^h$ *with* $x > 1$, $h \geq 2$, *then* $n, h, x < C$.

The bound $C$ provided by the proof of the theorem is far too large for any practical use. In the paper [7] we gave an algorithm to determine (for given parameters $P$, $Q$ and given square-free integer $A > 1$) the set $\{n \geq 1 \mid U_n = A\Box\}$ provided one already knows the squares in the sequence. A somewhat simpler version, with additional precisions, may be found in [10].

In this paper we extend the preceding algorithm. Precisely: let $\mathcal{U} = (U_n)_{n\geq 0}$ be a given Lucas sequence of the first kind, let $h \geq 2$, and let $A \geq 1$ be given. We assume that the set $N_0 = \{n \geq 1 \mid U_n = x^h \text{ with } x \geq 1\}$ is known. We describe an algorithm to determine all terms $U_n = Ax^h$ (with $x \geq 1$).

If $Q = \pm 1$, the knowledge of $N_0$ is equivalent to the solution of the diophantine equations $x^2 - Dy^{2h} = \pm 4$. The algorithm allows us to determine the solutions of each one of the equations $x^2 - DA^2y^{2h} = \pm 4$.

**2. Preliminaries.** The terms of Lucas sequences satisfy many identities and possess numerous interesting divisibility properties. We single out below a few of these properties.

Let $\alpha = (P + \sqrt{D})/2$, $\beta = (P - \sqrt{D})/2$ be the roots of $X^2 - PX + Q = 0$. Then

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad V_n = \alpha^n + \beta^n$$

for every $n \geq 0$.

For every $n \geq 0$:

$$\begin{gathered} V_n^2 - DU_n^2 = 4Q^n, \\ U_{2n} = U_n V_n, \\ U_{3n} = U_n(DU_n^2 + 3Q^n), \\ \gcd(U_n, U_m) = U_d, \end{gathered}$$

where $d = \gcd(n, m)$.

Let $1 \leq n < m$; then $U_n \mid U_m$ if and only if $n \mid m$.

The prime $p$ is said to *divide the sequence* $\mathcal{U}$ if there exists $n > 1$ such that $p \mid U_n$. Let $\mathcal{P}(\mathcal{U})$ denote the set of all primes which divide $\mathcal{U}$. If $p$ divides $\mathcal{U}$, the smallest $n$ such that $p \mid U_n$ is called the *rank of appearance* of $p$ in $\mathcal{U}$ and it is denoted by $\varrho(p)$. It follows that $p \mid U_n$ if and only if $\varrho(p) \mid n$.

For $p = 2$ we have the following facts: if $Q$ is even then 2 does not divide $\mathcal{U}$; if $P$ is even and $Q$ is odd then $\varrho(2) = 2$; if $P$ and $Q$ are odd, then $\varrho(2) = 3$.

For any odd prime $p$, if $p \mid Q$ then $P$ does not divide $\mathcal{U}$; if $p \mid P$ but $p \nmid Q$ then $\varrho(p) = 2$; if $p \mid D$ but $p \nmid PQ$ then $\varrho(p) = p$; if $p \nmid QPD$ then $\varrho(p)$ divides $p - \left(\frac{D}{p}\right)$ where $\left(\frac{D}{p}\right)$ denotes the Legendre symbol.

We observe that if $PQ$ is even, or if $PQ$ is odd and $p \neq 2$, then every prime factor $q$ of $\varrho(p)$ is at most equal to $p$.

The following result was established in [7]:

(2.1) *Let* $n = rm$ *with* $1 < r$. *Then* $U_n = U_m Z$ *where* $\gcd(U_m, Z)$ *divides* $r$.

**3. The algorithm.** Let $\mathcal{U} = (U_n)_{n\geq 0}$ be a Lucas sequence of the first kind, and let $\mathcal{P}(\mathcal{U})$ be the set of primes dividing $\mathcal{U}$. A subset $H$ of $\mathcal{P}(\mathcal{U})$ is said to be *saturated* when the following condition is satisfied: if $p \in H$ and if the prime $q$ divides $\varrho(p)$ and $q \in \mathcal{P}(\mathcal{U})$ then $q \in H$.

The empty set is saturated. If $H$ is a non-empty finite set we denote by $p[H]$ the largest prime in $H$. [The above definition of a saturated set is a variant of the one given in [7].]

(3.1) *For every finite set of primes* $H \subseteq \mathcal{P}(\mathcal{U})$ *there exists the smallest saturated set* $H^*$ *containing* $H$. $H^*$ *is finite and if* $H \neq \emptyset$ *then* $p[H^*] = p[H]$, *except when* $H = \{2\}$, $PQ$ *is odd and* $3 \nmid Q$; *in this case* $H^* = \{2, 3\}$.

*Proof.* First we note that $\emptyset^* = \emptyset$. So we assume that $H \neq \emptyset$. The proof is by induction on $p[H]$.

Let $p[H] = 2$, that is, $H = \{2\}$. Since $2 \in \mathcal{P}(\mathcal{U})$, $Q$ is odd. If $P$ is even then $\varrho(2) = 2$ so $H^* = \{2\}$. If $P$ is odd then $\varrho(2) = 3$. If $3 \mid Q$ then $3 \nmid \mathcal{P}(\mathcal{U})$ so $H^* = \{2\}$. If $3 \nmid Q$ then $3 \in \mathcal{P}(\mathcal{U})$. If $H'$ is a saturated set containing $\{2\}$ then it contains $\{2, 3\}$. We show that $\{2, 3\}$ is saturated, so $H^* = \{2, 3\}$. If $3 \mid P$ then $\varrho(3) = 2$ so $\{2, 3\}$ is saturated. If $3 \nmid P$ and $3 \mid D$ then $\varrho(3) = 3$ so $\{2, 3\}$ is saturated. If $3 \nmid PD$ then $\varrho(3)$ divides $3 - \left(\frac{D}{3}\right)$ so 2 is the only prime dividing $\varrho(3)$, hence again $\{2, 3\}$ is saturated.

Now let $p[H] = q \geq 3$, and let $H = \{q\} \cup H'$ with $q \notin H'$; so if $H' \neq \emptyset$ then $p[H'] < q$. By induction, we have: if $H' = \emptyset$ then $H'^* = \emptyset$; if $H' = \{2\}$ then $p[H'^*] \leq 3 \leq q$; if $2 < p[H']$ then $p[H'^*] = p[H'] < q$. From $q \in H \subseteq \mathcal{P}(\mathcal{U})$ the set $H_1 = \{p \text{ prime} \mid p \text{ divides } \varrho(q)\}$ is non-empty and contained in $H^*$. As already indicated, if $p \in H_1$ and $q \nmid D$ then $p < q$, so $p[H_1] < q$, so $p[H' \cup H_1] < q$. By induction $(H' \cup H_1)^*$ has been defined. If $p[H' \cup H_1] = 2$ then $p[(H' \cup H_1)^*] \leq 3 \leq q$. If $p[H' \cup H_1] > 2$ then $p[(H' \cup H_1)^*] = p[H' \cup H_1] < q$. The set $\{q\} \cup (H' \cup H_1)^*$ is saturated, it contains every saturated set containing $H$, so $H^* = \{q\} \cup (H' \cup H_1)^*$ and $p[H^*] = q = p[H]$.

If $q \mid D$ then $\varrho(q) = q$ so $H_1 = \{q\}$. The set $\{q\} \cup H'^*$ is saturated, it contains $H$ and it is contained in any saturated set containing $H$. So $H^* = \{q\} \cup H'^*$ and $p[H^*] = p[H]$. ■

We deduce easily:

(3.2) 1) *Let $H$ be a non-empty finite saturated set*, $H \neq \{2,3\}$, *with* $p[H] = q$, *and* $H = \{q\} \cup H'$ *with* $q \notin H'$. *Then $H'$ is a saturated set.*

2) *The following conditions are equivalent*:

(a) $H = \{2,3\}$ *is saturated but* $\{2\}$ *and* $\{3\}$ *are not saturated*,

(b) $2 \nmid Q$, $3 \nmid Q$, $2 \nmid P$ *and either* $3 \mid P$ *or* $3 \nmid PD$.

*Proof.* 1) Let $H \neq \{2,3\}$, $p[H] = q$, $H = \{q\} \cup H'$ with $q \notin H'$. If $H'$ is empty then $H'$ is saturated. If $p[H'] = 2$ then $H' = \{2\}$ so $H = \{2,q\}$. If $H'$ is not saturated then by (3.1), $H'^* = \{2,3\}$. From $H'^* \subseteq H^* = H$ we have $H = \{2,3\}$, which is absurd. So $H'$ is saturated. If $p[H'] \geq 3$ then $p[H'^*] = p[H']$. If $H'$ is not saturated then $H' \subset H'^* \subseteq H^* = H$ so $H'^* = H$ so $p[H'] = p[H] = q$, which is absurd. So $H'$ is saturated.

2) (a)⇒(b). Since $\{2,3\} \subseteq \mathcal{P}(\mathcal{U})$ we have $2 \nmid Q$ and $3 \nmid Q$. If $2 \mid P$ then $\{2\}$ is saturated; so $2 \nmid P$. If $3 \nmid P$ but $3 \mid D$ then $\varrho(3) = 3$ so $\{3\}$ is saturated. Thus either $3 \mid P$ or $3 \nmid PD$.

(b)⇒(a). First $\{2,3\} \subseteq \mathcal{P}(\mathcal{U})$. From $2 \nmid P$ we have $\varrho(2) = 3$, thus $\{2,3\}$ is saturated but $\{2\}$ is not saturated. If $3 \nmid PD$ or if $3 \mid P$ then 2 is the only prime dividing $\varrho(3)$; but $2 \notin \{3\}$, so $\{3\}$ is not saturated. ■

Let $h \geq 2$, and let $H$ be a finite saturated set. Let $T(H) = \{t \geq 1 \mid t$ is $h$-power free and if $p$ is any prime dividing $t$ then $p \in H\}$. So $T(\emptyset) = \{1\}$ and $T(H)$ is a finite set.

Let $N_H = \{n \geq 1 \mid U_n = t\Diamond$ for some $t \in T(H)\}$. Thus $N_\emptyset = \{n \geq 1 \mid U_n = \Diamond\}$. We denote this set by $N_0$. We introduce the following sets. Let $f \geq 1$, and let $H_0$, $H$ be finite saturated sets such that $H_0 \subset H$. Let

$$\begin{aligned} M_{f,H_0,H} &= M_f \\ &= \Big\{ m \geq 1 \;\Big|\; m \in \Big( \prod_{p \in H \setminus H_0} \varrho(p)^{e_p} \Big) N_{H_0} \text{ and } \sum_{p \in H \setminus H_0} e_p = f \Big\}. \end{aligned}$$

In particular, $M_0 = N_{H_0}$. We shall require the lemmas below. Let

$$S = \{s \geq 1 \mid \text{if } p \notin H \text{ then } v_p(s) \equiv 0 \pmod{h}\}.$$

(3.3) Lemma. 1) *Let* $s = s_1 s_2$ *with* $\gcd(s_1, s_2) = 1$. *Then* $s \in S$ *if and only if* $s_1, s_2 \in S$.

2) *Let* $d \geq 1$ *be such that if* $p \mid d$ *then* $p \notin H$ *and let $s$ be a multiple of $d$. Then* $s \in S$ *if and only if* $s/d \in S$.

*Proof.* 1) We have $v_p(s) = v_p(s_1) + v_p(s_2)$ and $\min\{v_p(s_1), v_p(s_2)\} = 0$ for every prime $p$. The assertion follows at once from the definition of $S$.

2) For every prime $p \notin H$, $v_p(s) = v_p(s/d)$ so the assertion follows at once. ∎

(3.4) 1) *There exists the smallest* $l \geq 0$ *such that* $(N_H \setminus N_{H_0}) \cap M_{l+1,H_0,H} = \emptyset$.

2) $N_H \setminus N_{H_0} \subseteq \bigcup_{f=1}^{l} M_{f,H_0,H}$.

*Proof.* 1) For every $t \in T(H)$ the set $\{n \geq 1 \mid U_n = t\Diamond\}$ is finite, by (1.4). Since $T(H)$ is finite, the sets $N_H$ and also $N_H \setminus N_{H_0}$ are finite.

If $N_H \setminus N_{H_0} = \emptyset$ the statement is true with $l = 0$. Now let $N_H \setminus N_{H_0} \neq \emptyset$. For each $n \in N_H \setminus N_{H_0}$ the set $F_n = \{f \geq 1 \mid n \in M_f\}$ is finite. Indeed, assume that there exist $f_1 < f_2 < \ldots$ in $F_n$. So for every $i \geq 1$ there exist $(e_{pi})_{p \in H \setminus H_0}$ with $\sum_{p \in H \setminus H_0} e_{pi} = f_i$ and $\prod_{p \in H \setminus H_0} \varrho(p)^{e_{pi}}$ divides $n$. Then there exists $p_0 \in H \setminus H_0$ such that the sequence $(e_{p_0 i})_{i \geq 1}$ has an infinite subsequence $e_{p_0 i_1} < e_{p_0 i_2} < \ldots$ so $n$ is divisible by an arbitrarily large power of $\varrho(p_0)$, which is absurd. Therefore $F_n$ is finite for every $n \in N_H \setminus N_{H_0}$. Let $l = \max_{n \in N_H \setminus N_{H_0}} \{\max F_n\}$. It follows that $(N_H \setminus N_{H_0}) \cap M_{l+1,H_0,H} = \emptyset$ and that $l$ is the smallest such index.

2) The statement is trivial when $N_H \setminus N_{H_0} = \emptyset$. Now assume that $N_H \setminus N_{H_0} \neq \emptyset$ and that there exists $n$ such that $n \in N_H \setminus N_{H_0}$ but $n \notin \bigcup_{f=1}^{l} M_f$. We choose $n$ minimal. Since $U_n = t\Diamond$ with $t \in T(H) \setminus T(H_0)$ there exists $p \in H \setminus H_0$ such that $p$ divides $U_n$. Therefore $\varrho(p) \mid n$ and we write $n = m\varrho(p)$. Then by (2.1), $U_n = U_m Z$ with $d = \gcd(U_m, Z) \mid \varrho(p)$. We have

$$\frac{U_n}{d^2} = \frac{U_m}{d} \cdot \frac{Z}{d}.$$

Let $q$ be any prime dividing $d$, so $q \mid U_m$, hence $q \in \mathcal{P}(\mathcal{U})$. Thus $q \in H$ because $H$ is saturated. By hypothesis $U_n = t\Diamond \in T(H)\Diamond \subseteq S$. By (3.3), $U_n/d^2 \in S$, hence $U_m/d \in S$ and so $U_m \in S$, that is, $U_m = t_1\Diamond \in T(H)\Diamond$, so $m \in N_H$.

If $m \in N_{H_0}$ then $n = \varrho(p)m \in M_1$, which is contrary to the assumption. If $m \notin N_{H_0}$, since $n$ is minimal and $m < n$, we have $m \in \bigcup_{f=1}^{l} M_f$, so $n = \varrho(p)m \in \bigcup_{f=2}^{l+1} M_f$. But $(N_H \setminus N_{H_0}) \cap M_{l+1} = \emptyset$, so this is also absurd. ∎

The above results are the basis for the algorithm.

*Description of the algorithm.* Let $h \geq 2$, $C \geq 1$. We wish to determine the set $N^* = \{n \geq 1 \mid U_n = C\Diamond\}$. Writing $C = A\Diamond$ where $A \geq 1$ and $A$ is $h$-power free, we have $N^* = \{n \geq 1 \mid U_n = A\Diamond\}$. We assume that the set $N_0 = \{n \geq 1 \mid U_n = \Diamond\}$ is known. If there exists a prime $p$ such that $p \mid A$ and $p \notin \mathcal{P}(\mathcal{U})$ then $N^* = \emptyset$. Now we assume that if $p \mid A$ then $p \in \mathcal{P}(\mathcal{U})$. Let $H$ be the smallest saturated set containing all the prime factors of $A$.

Let $H_0 = \emptyset$, and let $H_1 = \{2,3\}$ if $\{2,3\} \subseteq H$, $\{2,3\}$ is saturated but $\{2\}$ and $\{3\}$ are not saturated. Otherwise let $H_1 = \{q_1\}$ where $q_1$ is the smallest prime in $H$. Let $q_2$ be the smallest prime in $H \setminus H_1$ and let $H_2 = H_1 \cup \{q_2\}$. Define $H_3, \ldots$ in a similar way. Let $r$ be such that $H_r = H$. So $3 < q_2$ or $q_1 < q_2$ and $q_2 < q_3 < \ldots < q_r$. By the preceding results if $H_1 = \{2, 3\}$ then

$$N_{H_1} \setminus N_0 \subseteq \bigcup_{f=1}^{l} M_{f,\emptyset,\{2,3\}}$$

where

$$M_{f,\emptyset,\{2,3\}} = \{m = \varrho(2)^{e_2}\varrho(3)^{e_3}N_0 \mid e_2 + e_3 = f\}.$$

By direct calculation we may determine the elements of

$$(N_{\{2,3\}} \setminus N_0) \cap M_{1,\emptyset,\{2,3\}}, \qquad (N_{\{2,3\}} \setminus N_0) \cap M_{2,\emptyset,\{2,3\}},$$

etc. until we determine $l$ such that

$$(N_{\{2,3\}} \setminus N_0) \cap M_{l+1,\emptyset,\{2,3\}} = \emptyset.$$

We note that $l$ depends on $\{2, 3\}$. This determines $N_{\{2,3\}} \setminus N_0$.

Let $i \geq 1$ and $H_i = H_{i-1} \cup \{q_i\}$, where we may take $i = 1$ when $H_1 = \{q_1\}$, so $H_0 = \emptyset$. Similarly, by direct calculation we may determine the sets

$$(N_{H_i} \setminus N_{H_{i-1}}) \cap M_{1,H_{i-1},H_i}, \qquad (N_{H_i} \setminus N_{H_{i-1}}) \cap M_{2,H_{i-1},H_i},$$

etc. until we reach the smallest $l$ such that

$$(N_{H_i} \setminus N_{H_{i-1}}) \cap M_{l+1,H_{i-1},H_i} = \emptyset.$$

This determines the sets $N_{H_i} \setminus N_{H_{i-1}}$ for $i = 1, \ldots, r$. Finally, we identify by calculation the subset $N^*$ of $N_{H_r}$.

The implementation of the algorithm is usually very simple, as will be illustrated in the next section.

**4. Numerical examples.** In this section we shall give numerical examples to show how the algorithm indicated in the preceding section may be applied. We modify somewhat the notation used before to make it more adapted to handle the examples. The following lemma will be useful to eliminate superfluous calculations.

(4.1) LEMMA. *Let $l > 1$ be such that every prime factor of $l$ belongs to $H$. If $m \notin N_H$ then $lm \notin N_H$.*

*Proof.* We have $U_{ml} = U_m Z$ with $d = \gcd(U_m, Z)$ dividing $l$. From $m \notin N_H$ there exists a prime $p \notin H$ such that $v_p(m) \not\equiv 0 \pmod{h}$. If $ml \in N_H$ then $v_p(ml) \equiv 0 \pmod{h}$, therefore $v_p(l) \not\equiv 0 \pmod{h}$, so $p \mid l$, which is a contradiction. ■

(4.2) *Determination of the Fibonacci numbers of the form* $25 \cdot 7x^3$. We have $P = 1$, $Q = -1$, $D = 5$. The ranks of appearance of 5, 7 are $\varrho(5) = 5$, $\varrho(7) = 8$. So the smallest saturated set containing $\{5, 7\}$ is $H = \{2, 3, 5, 7\}$.

It was recalled in (1.3) that $N_0 = \{n \mid U_n \text{ is a cube}\} = \{1, 2, 6\}$. Let

$$N_1 = N_{\{2,3\}} = \{n \geq 1 \mid U_n = 2\Diamond, 3\Diamond, 4\Diamond, 6\Diamond, 9\Diamond, 12\Diamond, 18\Diamond, 36\Diamond\}.$$

For $f \geq 1$ let

$$M_{1,f} = \{2^{e_2}3^{e_3}N_0 \mid e_2 + e_3 = f\}.$$

So $N_1 \subseteq M_{1,1} \cup M_{1,2} \cup M_{1,3} \cup \ldots$ Now

$$M_{1,1} = 2N_0 \cup 3N_0 = \{2, 4, 12, 3, 6, 18\}.$$

By direct calculation, or looking at tables, $N_1 \cap M_{1,1} = \{3, 4, 12\}$. Next

$$M_{1,2} = 4N_0 \cup 6N_0 \cup 9N_0 = \{4, 8, 24, 6, 12, 36, 9, 18, 54\}.$$

We observe that $6, 8, 9 \notin N_1$, hence by (4.1), $12, 36, 24, 18, 54 \notin N_1$, so $N_1 \cap M_{1,2} = \{4\}$.

Next $M_{1,3} = 8N_0 \cup 12N_0 \cup 18N_0 \cup 27N_0$. As already said $8, 12, 18, 9 \notin N_1$, hence $N_1 \cap M_{1,3} = \emptyset$, by Lemma (4.1).

In conclusion, $N_1 = \{3, 4, 12\}$.

Let

$$N_2 = \{n \geq 1 \mid U_n = 5\Diamond, 10\Diamond, 15\Diamond, 20\Diamond, 30\Diamond, 45\Diamond, 60\Diamond, 90\Diamond\}.$$

[In the preceding notation, $N_2 = N_{\{2,3,5\}} \setminus N_{\{2,3\}}$.]

Hence $N_2 \subseteq M_{2,1} \cup M_{2,2} \cup \ldots$ where $M_{2,1} = 5\overline{N}_1, M_{2,2} = 25\overline{N}_1, \ldots$ with $\overline{N}_1 = N_0 \cup N_1 = \{1, 2, 3, 4, 6, 12\}$. Now we observe that $3, 10 \notin N_2$, so $25, 45, 20, 30, 60, 90 \notin N_2$ by (4.1). We have $M_{2,1} = \{5, 10, 15, 20, 30, 60\}$, so $N_2 \cap M_{2,1} = \{5\}$. Next $M_{2,2} = \{25, 50, 75, 100, 150, 300\}$ and again by (4.1), $N_2 \cap M_{2,2} = \emptyset$. Thus $N_2 = \{5\}$.

Let $\overline{N}_2 = \overline{N}_1 \cup N_2 = \{1, 2, 3, 4, 5, 6, 12\}$. Let

$$N_3 = \{n \geq 1 \mid U_n = 2^i 3^j 5^k 7^l \Diamond,\ 0 \leq i, j, k \leq 2,\ l = 1, 2\}.$$

Since $\varrho(7) = 8$ we have $N_3 \subseteq M_{3,1} \cup M_{3,2} \cup \ldots$ where $M_{3,1} = 8\overline{N}_2$, $M_{3,2} = 64\overline{N}_2, \ldots$

Explicitly $M_{3,1} = \{8, 16, 24, 32, 40, 48, 96\}$ and by calculation and (4.1), $N_3 \cap M_{3,1} = \{8\}$. By similar considerations $N_3 \cap 64\overline{N}_2 = \emptyset$. Thus $N_3 = \{8\}$.

The set $N^*$ is contained in $N_3$. But $U_8 = 21 \neq 25 \cdot 7\Diamond$. So $N^* = \emptyset$. ∎

We deduce that the diophantine equations

$$X^2 - 5^5 \cdot 7^2 Y^6 = \pm 4$$

do not have solutions in non-zero integers.

(4.3) *Determination of the Pell numbers of the form* $20x^3$. We have $P = 2$, $Q = -1$, $D = 8$. Pethő showed in [5] that $N_0 = \{n \geq 1 \mid U_n \text{ is a cube}\} = \{1\}$. Since $\varrho(2) = 2$, $\varrho(5) = 3$, $\varrho(3) = 4$ the saturated set containing $\{2, 3\}$ is $H = \{2, 3, 5\}$. We note that $\{2\}$ is saturated. Let $N_1 = \{n \geq 1 \mid U_n = 2\Diamond, 4\Diamond\}$. Then $N_1 \subseteq 2N_0 \cup 4N_0 \cup \ldots$ But $N_1 \cap 2N_0 = \{2\}$, $N_1 \cap 4N_0 = \emptyset$. Thus $N_1 = \{2\}$.

Let $\overline{N}_1 = N_0 \cup N_1 = \{1, 2\}$, and

$$N_2 = \{n \geq 1 \mid U_n = 3\Diamond, 6\Diamond, 9\Diamond, 12\Diamond, 18\Diamond, 36\Diamond\}.$$

Then $N_2 \subseteq 4\overline{N}_1 \cup 16\overline{N}_1 \cup \dots$ We have $N_2 \cap 4\overline{N}_1 = \{4\}$. Since $8 \notin N_2$, by (4.1), $16, 32 \notin N_2$. Therefore $N_2 \cap 16\overline{N}_2 = \emptyset$. Thus $N_2 = \{4\}$.

Let $\overline{N}_2 = N_2 \cup \overline{N}_1 = \{1, 2, 4\}$. Let

$$N_3 = \{n \geq 1 \mid U_n = 2^i 3^j 5^k \Diamond,\ 0 \leq i, j \leq 2,\ 1 \leq k \leq 2\}.$$

Since $\varrho(5) = 3$ we have $N_3 \subseteq 3\overline{N}_2 \cup 9\overline{N}_2 \cup \dots$ We note that $6 \notin N_3$ and $9 \notin N_3$, so by (4.1), $12, 18, 36 \notin N_3$. Hence $N_3 \cap 3\overline{N}_2 = \{3\}$ and $N_3 \cap 9\overline{N}_2 = \emptyset$. Therefore $N_3 = \{3\}$. The set $N^* \subseteq N_3$, but $U_3 = 5 \neq 20x^3$, so $N^* = \emptyset$. ∎

We deduce that the diophantine equations

$$X^2 - 2^7 \cdot 5^2 Y^6 = \pm 4,$$

or equivalently

$$X^2 - 2^5 \cdot 5^2 Y^6 = \pm 1$$

do not have solutions in non-zero integers.

(4.4) *Determination of all $n \geq 1$ such that $7^n - 1 = 3x^3$.* Let $n \geq 1$ and $x \geq 1$ be such that $7^n - 1 = 3x^3$. Since $6 \mid 7^n - 1$ we have

$$\frac{7^n - 1}{7 - 1} = 4y^3.$$

Let $P = 8$, $Q = 7$, so $D = 36$ and let $\mathcal{U}$ be the sequence with terms $U_n = (7^n - 1)/(7 - 1)$. Let $N^* = \{n \geq 1 \mid U_n = 4\Diamond\}$ where $\Diamond$ denotes any non-zero cube.

We have $N_0 = \{n \geq 1 \mid U_n = \Diamond\} = \{1, 2\}$ (see [6]). Since $\varrho(2) = 2$, $\{2\}$ is a saturated set. Let $N_1 = \{n \geq 1 \mid U_n = 2\Diamond, 4\Diamond\}$. Then $N_1 \subseteq 2N_0 \cup 4N_0 \cup \dots$ But $N_1 \cap 2N_0 = \emptyset$ so $N_1 = \emptyset$. Therefore $7^n - 1$ is never equal to $3x^3$. ∎

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