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Primes in arithmetic progressions to spaced moduli

by

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1. Introduction. Let Λ be the von Mangoldt function. For $(a,q)=1$, let

$$\sum_{\substack{n\le x\\ n\equiv a\,(\mathrm{mod}\,q)}} \Lambda(n)=\frac{x}{\phi(q)}+E(x;q,a).$$

It is well known that for given $A>0$, $C>0$,

$$E(x,q):=\max_{(a,q)=1}|E(x;q,a)|\ll\frac{x}{q(\log x)^A}\tag{1.1}$$

for $x\ge 2$, $q\le(\log x)^C$. See e.g. Davenport [5].

Suppose we are given a set S with some arithmetic structure. Let

$$S(Q)=\{q\in S: Q<q\le 2Q\}.$$

Can we prove that (1.1) holds for most q in $S(Q)$, for large values of Q? That is, we seek bounds

$$\sum_{q\in S(Q)}E(x,q)\ll\frac{x|S(Q)|}{Q(\log x)^A}\tag{1.2}$$

for every $A>0$. Here $|T|$ denotes the cardinality of a finite set T. If S is the set $\mathbb{N}$ of natural numbers, then (1.2) holds for $Q\le x^{1/2}(\log x)^{-A-5}$, by the Bombieri–Vinogradov theorem; see e.g. [5].

In the present paper we study the particular case

$$S=S_f=\{f(k):k\in\mathbb{N}\}\tag{1.3}$$

where

$$f(X)=a_dX^d+\cdots+a_1X+a_0,\qquad a_j\in\mathbb{Z},\ d\ge 2,\ a_d>0.\tag{1.4}$$

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The first result for this case is due to Elliott [6]. He showed that (1.2) holds for $S = S_f$,

$$Q < x^{1/4-\varepsilon}.$$

Mikawa and Peneva [11] sharpened this, replacing the constant 1/4 by 8/19.

More is known in the special case $f(x) = x^2$. Baier and Zhao [2] used a version of the large sieve, due to Baier [1], for fractions a/q^2, $q \le Q$, $(a,q) = 1$, to obtain (1.2) for $S = \{k^2 : k \ge 1\}$ whenever

$$Q < x^{4/9-\varepsilon}.$$

In the present paper we sharpen these results.

Theorem 1. *Let f be as in* (1.4). *Let $\varepsilon > 0$. We have*

$$\sum_{q \in S_f(Q)} E(x,q) \ll \frac{x|S_f(Q)|}{Q(\log x)^A}$$

for every $A > 0$, provided that

$$Q < x^{9/20-\varepsilon}.$$

The implied constant depends at most on f, ε and A.

Theorem 2. *Let $f(x) = x^2$. The conclusion of Theorem* 1 *holds whenever*

$$Q < x^{43/90-\varepsilon}.$$

For comparison, we note that $8/19 = 0.421\ldots$, $4/9 = 0.\dot{4}$, $9/20 = 0.45$, $43/90 = 0.4\dot{7}$.

For some applications, the following theorem is more useful than Theorem 2.

Theorem 3. *We have*

$$\sum_{Q^{1/2} < p \le (2Q)^{1/2}} E(x,p^2) \ll xQ^{-1/2}(\log x)^{-A}$$

for every $A > 0$, provided that

$$Q < x^{1/2-\varepsilon}.$$

To prove Theorem 1 we start from the work of Mikawa and Peneva, and import an averaging over q in $S_f(Q)$ into the treatment of 'Type 1' sums. Theorem 2 follows the same lines, but incorporates a generalization of the large sieve inequality of Baier and Zhao [**?**] to obtain a new mean value bound for the relevant Dirichlet polynomials. For Theorem 3, we adapt the proof of Theorem 2 a little. The treatment of the bilinear forms in the remainder terms goes back to Iwaniec [9], and we need only adapt this to the present purpose.

In applications, it is sometimes useful to have a 'maximal variant' of Theorems 1, 2 or 3 in which $E(x,q)$ is replaced by $\max_{1\le y\le x} E(y,q)$. We provide this maximal variant of the theorems in Section 6.

Throughout the paper, ε denotes a positive number, which we suppose to be sufficiently small; furthermore, $\delta = \varepsilon^2$ and f is a polynomial, as in (1.4). We assume that $Q \ge 1$, and that N is a natural number.

2. The Dirichlet polynomials $\sum_{n\le N} \chi(n)n^{-s}$. Let γ be a constant, $0 < \gamma < 1$. We seek good bounds on

$$B(s,\chi) = \sum_{n\le N} \chi(n)n^{-s}$$

that are valid on the critical line for all nonprincipal $\chi \pmod q$ and all $N \ge q^\gamma$, for $q \in S_f(Q) \setminus F(Q)$. The cardinality of the exceptional set $F(Q)$ will be small compared with $|S_f(Q)|$.

Lemma 1. *Let $b > 0$ and let G be a finite subset of $\mathbb{N} \cap [b,\infty)$. Let*

$$F = \{q \in S_f(Q) : r \mid q \text{ for some } r \in G\}.$$

Then

$$|F| \ll |S_f(Q)|\,|G| b^{-1/d+\varepsilon}.$$

The implied constant depends at most on f and ε.

Remark 1. Unless otherwise stated, the dependencies of implied constants in the proof will be the same as in the statement of the lemma; similarly in subsequent proofs.

Proof. We may suppose that Q is sufficiently large, so that

$$Q^{1/d} \ll |S_f(Q)| \ll Q^{1/d}.$$

Fix $r \in G$. We need only show that

$$|\{q \in S_f(Q) : r \mid q\}| \ll |S_f(Q)| r^{-1/d+\varepsilon}.$$

We recall that for an irreducible polynomial g in $\mathbb{Z}[x]$,

$$|\{n \pmod t : g(n) \equiv 0 \pmod t\}| \ll_g t^\varepsilon$$

(see e.g. Nagell [15]). Now let

$$f = g_1 \ldots g_h$$

where $g_1,\ldots,g_h$ are irreducible, $h \le d$. If $f(n) \equiv 0 \pmod r$, then

$$r = (g_1(n)\ldots g_h(n), r) \le (g_j(n), r)^h$$

for some j. Hence for any interval $[a,b]$,

$$\begin{aligned}
&|\{n\in[a,b]:f(n)\equiv 0 \pmod r\}|\\
&\qquad \le \sum_{j=1}^{h}\sum_{\substack{t\mid r\\ t\ge r^{1/h}}}|\{n\in[a,b]:g_j(n)\equiv 0 \pmod t\}|\\
&\qquad \ll r^{\varepsilon/2}\biggl(\frac{b-a}{t}+1\biggr)|\{n \pmod t : g_j(n)\equiv 0 \pmod t\}|\\
&\qquad\qquad\qquad\qquad \text{(for some } j,\ 1\le j\le h \text{ and } t\mid r,\ t\ge r^{1/h})\\
&\qquad \ll r^{\varepsilon}\biggl(\frac{b-a}{r^{1/h}}+1\biggr).
\end{aligned}$$

We now obtain the lemma on noting that

$$\{q\in S_f(Q): r\mid q\}=\{f(n): n\in[a,b],\ f(n)\equiv 0 \pmod r\}$$

with $b-a\ll Q^{1/d}$. Since $r\ll Q$ if there is some $q\in S_f(Q)$ divisible by r,

$$|\{q\in S_f(Q): r\mid q\}|\ll r^{\varepsilon}((Q/r)^{1/d}+1)\ll |S_f(Q)|r^{-1/d+\varepsilon}. \ \blacksquare$$

For any nonprincipal character χ to modulus q, there is a divisor

$$r=\operatorname{cond}\chi$$

of q, the *conductor* of χ, and a primitive character χ' (mod r) such that

$$\chi(n)=\begin{cases}\chi'(n) & \text{if } (n,q)=1,\\ 0 & \text{if } (n,q)>1.\end{cases}$$

We say that χ is *induced* by χ' (see [5, Chapter 5]).

Lemma 2. *Let $b>0$, $4/5\le\alpha\le 1$, $T\ge 2$. Let*

$$F=F(\alpha,T,b)$$

be the set of q in $S_f(Q)$ for which

$$L(s,\chi)=0$$

for some nonprincipal χ (mod q) with $\operatorname{cond}\chi\ge b$, and some s with $\operatorname{Re}(s)\ge\alpha$, $|\operatorname{Im}(s)|\le T$. Then

$$|F|\ll |S_f(Q)|(Q^2T)^{2(1-\alpha)/\alpha}(\log QT)^{14}b^{-1/d+\varepsilon}.$$

The implied constant depends at most on f and ε.

Proof. Let $q\in F$. Suppose that $L(s,\chi)=0$, where χ and s are as in the statement of the lemma, χ being induced by the primitive character χ' to modulus $r\ge b$. Then

$$L(s,\chi')=0$$

[5, Section 5]. Let us write $N(\sigma, T, \chi')$ for the number of zeros of $L(s, \chi')$ with $\mathrm{Re}(s) \ge \sigma$, $|\mathrm{Im}(s)| \le T$. Let

$$G = \{r : b \le r \le 2Q,\ L(s,\chi') = 0 \text{ for some primitive character } \chi' \pmod r \text{ and some } s,\ \mathrm{Re}(s) \ge \alpha,\ |\mathrm{Im}(s)| \le T\}.$$

Obviously

$$|G| \le \sum_{r \le 2Q} \sideset{}{^*}\sum_{\lambda \,(\mathrm{mod}\, r)} N(\alpha, T, \lambda)$$

where the asterisk denotes a restriction to primitive characters. The above discussion yields

$$F \subseteq \{q \in S_f(Q) : r \mid q \text{ for some } r \in G\}. \tag{2.1}$$

Combining Lemma 1 with (2.1), we obtain

$$|F| \ll |S_f(Q)| b^{-1/d+\varepsilon} \sum_{r \le 2Q} \sideset{}{^*}\sum_{\lambda \,(\mathrm{mod}\, r)} N(\alpha, T, \lambda).$$

We now complete the proof by appealing to the bound

$$\sum_{r \le 2Q} \sideset{}{^*}\sum_{\lambda \,(\mathrm{mod}\, r)} N(\alpha, T, \lambda) \ll (Q^2T)^{2(1-\alpha)/\alpha} (\log QT)^{14}$$

given by Montgomery [12, Theorem 12.2]. ∎

Lemma 3. *Let $1/2 < \alpha < 1$. Let $T \ge T_0(\alpha, \varepsilon)$. Suppose that χ is a nonprincipal character modulo q, and*

$$L(s, \chi) \ne 0 \quad (\mathrm{Re}(s) \ge \alpha,\ |\mathrm{Im}(s)| \le T).$$

Then for $\sigma \ge \alpha$, $|t| \le T/2$,

$$\log L(\sigma + it, \chi) \ll (\log qT)^{(1-\sigma)/(1-\alpha)+\varepsilon}. \tag{2.2}$$

The implied constant depends at most on α and ε.

Proof. We argue as in Titchmarsh [16, proof of Theorem 14.2]. Let $\eta = \eta(\alpha, \varepsilon) > 0$ be sufficiently small, and $\sigma_1 = \sigma_1(\alpha, \varepsilon) > 0$ sufficiently large. Apply the Borel–Carathéodory theorem to the function $\log L(s, \chi)$ and the circles with center $2 + it$ and radii r, $2 - \alpha$, where $|t| \le T$ and

$$0 < r \le 2 - \alpha - \eta.$$

On the larger circle,

$$\mathrm{Re}(\log L(s, \chi)) = \log |L(s, \chi)| < \log 4qT$$

([5, (14) of Chapter 12]). Hence, on the smaller circle,

$$|\log L(s, \chi)| \le \frac{4 - 2\alpha}{\eta} \log 4qT + \frac{4 - 2\alpha - \eta}{\eta} |\log L(2 + it, \chi)|.$$

Thus for $\mathrm{Re}(s) \geq \alpha + \eta$, $|\mathrm{Im}(s)| \leq T$ it is clear that

$$|\log L(s,\chi)| \ll \log qT.$$

In proving (2.2) we may suppose that

$$\alpha + \eta \leq \sigma \leq 1 + \eta, \qquad |t| \leq T/2.$$

We apply Hadamard's three circles theorem to the circles with center $\sigma_1 + it$ passing through the points $1 + \eta + it$, $\sigma + it$ and $\alpha + \eta + it$. The radii are

$$r_1 = \sigma_1 - (1+\eta), \qquad r_2 = \sigma_1 - \sigma, \qquad r_3 = \sigma_1 - (\alpha + \eta).$$

If the maxima of $|\log L(s,\chi)|$ on the circles are M_1, M_2, M_3, then

$$M_2 \leq M_1^{1-a} M_3^a, \quad \text{where} \quad a = \frac{\log(r_2/r_1)}{\log(r_3/r_1)}.$$

Hence

$$\log L(\sigma + it, \chi) \ll M_3^a \ll (\log qT)^a.$$

It remains to bound a. We have

$$\log\left(\frac{r_2}{r_1}\right) = \log\left(1 + \frac{1+\eta-\sigma}{\sigma_1 - 1 - \eta}\right) = \frac{1+\eta-\sigma}{\sigma_1 - 1 - \eta}(1 + O(\sigma_1^{-1})),$$
$$\log\left(\frac{r_3}{r_1}\right) = \log\left(1 + \frac{1-\alpha}{\sigma_1 - 1 - \eta}\right) = \frac{1-\alpha}{\sigma_1 - 1 - \eta}(1 + O(\sigma_1^{-1})),$$

where the implied constants are absolute. Hence

$$a = \frac{1+\eta-\sigma}{1-\alpha}(1 + O(\sigma_1^{-1})) < \frac{1-\sigma}{1-\alpha} + \varepsilon$$

as required, if η and σ_1 are chosen suitably. ∎

The following version of Perron's formula is a slight variant of [3, Lemma 13].

LEMMA 4. *Let $b \geq 0$, $c > 0$ and let $\lambda \in \mathbb{R}$, $\lambda + c > 1 + b$. For $K > 0$ and complex numbers a_l $(l \geq 1)$ with $|a_l| \leq K l^b$, write*

$$h(s) = \sum_{l=1}^{\infty} \frac{a_l}{l^s} \qquad (\mathrm{Re}(s) > 1 + b).$$

Then for $T > 1$,

$$\sum_{l \leq N} \frac{a_l}{l^\lambda} = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} h(s+\lambda) \frac{(N+1/2)^s}{s}\, ds + O\left(\frac{KN^c}{T}\right).$$

The implied constant depends at most on c, $\lambda + c - 1 - b$.

Let χ be a nonprincipal character modulo q. We apply the lemma with $a_l = \chi(l)$, $K = 1$, $b = \lambda = 0$, $c = 1 + \varepsilon$. Thus

$$\sum_{n\le N} \chi(n) = \frac{1}{2\pi i}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT} L(s,\chi)\frac{(N+1/2)^s}{s}\,ds + O\Big(\frac{N^{1+\varepsilon}}{T}\Big). \tag{2.3}$$

This leads to the following result.

Lemma 5. *Let $\gamma > 0$, $1/2 < \alpha < 1$ and suppose that the nonprincipal character χ (mod q) satisfies*

$$L(s,\chi) \ne 0 \qquad (\mathrm{Re}(s) \ge \alpha,\ |\mathrm{Im}(s)| \le 2q).$$

Then

$$\sum_{n\le N} \chi(n) \ll N^{\alpha+\varepsilon} \qquad (N \ge q^\gamma).$$

The implied constant depends at most on α, γ and ε.

Proof. We may suppose that $N > T_0(\alpha,\varepsilon)$. In view of the Pólya–Vinogradov inequality, we may further suppose that $N < q$. By (2.3),

$$\sum_{n\le N} \chi(n) = \frac{1}{2\pi i}\int_{1+\varepsilon-iN}^{1+\varepsilon+iN} L(s,\chi)\frac{(N+1/2)^s}{s}\,ds + O(N^\varepsilon).$$

We replace the integral by

$$\int_{\alpha+\varepsilon/2-iN}^{\alpha+\varepsilon/2+iN} L(s,\chi)\frac{(N+1/2)^s}{s}\,ds, \tag{2.4}$$

incurring an error that is the sum of the integrals over horizontal segments. On these segments the integrand is

$$O\Big(N^\varepsilon \max_{\substack{\mathrm{Re}(s)\ge\alpha+\varepsilon/2\\ |\mathrm{Im}(s)|\le q}} |L(s,\chi)|\Big) = O(N^\varepsilon q^{\gamma\varepsilon}) = O(N^{2\varepsilon})$$

by an application of Lemma 3. Likewise the integral in (2.4) is

$$O\bigg(N^{\alpha+2\varepsilon/3}\int_{-N}^{N}\frac{dt}{|\alpha+it|}\bigg) = O(N^{\alpha+\varepsilon}).$$

The lemma follows on combining these estimates. ∎

Lemma 6. *Let $0 < \gamma < 1$. There is a subset $F(Q)$ of $S_f(Q)$, with*

$$|F(Q)| \ll |S_f(Q)|Q^{-\beta},$$

such that for $q \in S_f(Q) \setminus F(Q)$, χ nonprincipal modulo q and $\mathrm{Re}(s) = 1/2$ *we have*

$$\sum_{n\le N} \chi(n)n^{-1/2+it} \ll |s|N^{1/2-\beta} \qquad (N \ge q^\gamma). \tag{2.5}$$

Here $\beta = \beta(\gamma,d) > 0$. The implied constants depend only on f and γ.

Proof. Let $s = 1/2 + it$ and

$$T(\chi, u) = \sum_{n \le u} \chi(n).$$

Suppose for a moment that

$$T(\chi, u) \ll u^{1-\beta} \qquad (u \ge q^{\gamma/2}).$$

Then for $N \ge q^{\gamma}$,

$$\begin{aligned}
\sum_{n \le N} \chi(n) n^{-1/2+it} &= \int_{1-}^{N} u^{-1/2+it}\, dT(\chi, u) \\
&= T(\chi, u) u^{-1/2+it}\Big|_{1-}^{N} - \left(-\frac{1}{2} + it\right) \int_{1}^{N} u^{-3/2+it} T(\chi, u)\, du \\
&\ll |s| N^{1/2-\beta} + |s| \int_{1}^{q^{\gamma/2}} u^{-1/2}\, du \ll |s| N^{1/2-\beta}
\end{aligned}$$

provided that $\beta \le 1/4$.

Now let α be a positive constant, $4/5 \le \alpha < 1$, to be determined below. We take $F(Q) = F(\alpha, 4Q, (2Q)^{\gamma/2})$ in the notation of Lemma 2. We first show that for $q \in S_f(Q) \setminus F(Q)$ and a nonprincipal character $\chi \pmod q$,

$$T(\chi, u) \ll u^{1-\beta} \qquad (u \ge q^{\gamma/2}). \tag{2.6}$$

Suppose first that $\operatorname{cond} \chi \ge (2Q)^{\gamma/2}$. Since $q \notin F(Q)$,

$$L(s, \chi) \ne 0 \qquad (\operatorname{Re}(s) \ge \alpha,\ |\operatorname{Im}(s)| \le 4Q).$$

By Lemma 5, with $\gamma/2$ in place of γ,

$$T(\chi, u) \ll u^{\alpha+\varepsilon} \qquad (u \ge q^{\gamma/2}).$$

This gives the bound (2.6), provided that we choose $\beta \le 1 - \alpha - \varepsilon$, and (2.5) follows.

Now suppose that χ has conductor $r < (2Q)^{\gamma/2}$ and is induced by the primitive character χ'. Let $u \ge q^{\gamma/2}$. Then

$$\begin{aligned}
T(\chi, u) &= \sum_{n \le u} \Big(\sum_{\substack{d \mid n \\ d \mid q}} \mu(d)\Big) \chi'(n) = \sum_{d \mid q} \mu(d) \chi'(d) \sum_{m \le u/d} \chi'(m) \qquad (2.7) \\
&\ll \tau(q) r^{1/2} \log r \qquad \text{(by the Pólya–Vinogradov inequality)} \\
&\ll q^{\gamma/4+\varepsilon} \ll u^{1-\beta} \qquad (u \ge q^{\gamma/2}).
\end{aligned}$$

This establishes that (2.5) holds for all $\chi \pmod q$.

It remains to bound $|F(Q)|$. According to Lemma 2,

$$|F(Q)| \ll |S_f(Q)| Q^{6(1-\alpha)/\alpha} (\log Q)^{14} Q^{-\gamma/3d}.$$

We choose α so that $6(1-\alpha)/\alpha = \gamma/(6d)$. This gives the desired bound provided that we take $\beta < \gamma/(6d)$. ∎

3. First stage of proof of Theorems 1, 2 and 3. By the Brun–Titchmarsh theorem [13],

$$E(x,q) \ll \frac{x}{\phi(q)} \ll \frac{x}{Q}\log\log x \qquad (q \in S_f(Q)).$$

With $F(Q)$ as in Lemma 6,

$$\sum_{q\in F(Q)} E(x,q) \ll \frac{x|F(Q)|}{Q}\log\log x \ll \frac{x|S_f(Q)|}{Q(\log x)^A}.$$

Thus we need only show that

$$\sum_{q\in S_f(Q)\setminus F(Q)} E(x,q) \ll \frac{x|S_f(Q)|}{Q(\log x)^A}.$$

We use a particular case of Vaughan's identity (see e.g. [5, Chapter 24]). Let $Z = Qx^{\varepsilon/4}$. Then

$$\Lambda(n) = a_1(n) + a_2(n) + a_3(n) + a_4(n)$$

with

$$a_1(n) = \begin{cases} \Lambda(n) & \text{if } n \le Z, \\ 0 & \text{if } n > Z, \end{cases} \qquad a_3(n) = \sum_{\substack{hd=n\\ d\le Z}} \mu(d)\log h,$$

$$a_2(n) = -\sum_{\substack{mdr=n\\ m\le Z,\, d\le Z}} \Lambda(m)\mu(d), \qquad a_4(n) = -\sum_{\substack{mk=n\\ m>Z,\, k>Z}} \Lambda(m)\Big(\sum_{\substack{d\mid k\\ d\le Z}} \mu(d)\Big).$$

Let

$$E_i(x;q,a) = \sum_{\substack{n\le x\\ n\equiv a \,(\mathrm{mod}\, q)}} a_i(n) - \frac{1}{\phi(q)} \sum_{\substack{n\le x\\ (n,q)=1}} a_i(n).$$

For $q \in S_f(Q)$,

$$\sum_{i=1}^{4} E_i(x;q,a) = \psi(x;q,a) - \frac{1}{\phi(q)} \sum_{\substack{n\le x\\ (n,q)=1}} \Lambda(n)$$
$$= \psi(x;q,a) - \frac{x}{\phi(q)} + O\Big(\frac{x(\log x)^{-A}}{Q}\Big)$$

by the prime number theorem. Thus to prove Theorem 1 or 2 it suffices to show for $1 \le i \le 4$ that

$$H_i(Q) := \sum_{q\in S_f(Q)\setminus F(Q)} \max_{(a,q)=1} |E_i(x;q,a)| \ll \frac{x|S_f(Q)|}{Q(\log x)^A}. \tag{3.1}$$

The case $i = 1$ is obvious from the Brun–Titchmarsh theorem. A partial summation, together with an elementary argument, gives

$$E_3(x; q, a) \ll Zx^{\varepsilon} \ll \frac{x}{Q(\log x)^A},$$

and yields (3.1) for $i = 3$.

For $i = 4$, we appeal to the work of Mikawa and Peneva [11, Section 3.1]. Their bound $Q < x^{8/19-\varepsilon}$ is not used in this part of the argument, which gives

$$\sum_{q \in S_f(Q)} \max_{(a,q)=1} |E_4(x; q, a)| \ll \frac{x|S_f(Q)|}{Q(\log x)^A}.$$

Turning to $H_2(Q)$, let $q \in S_f(Q)$, $(a, q) = 1$. Then

$$E_2(x; q, a) = -\sum_{\substack{m,n \le Z\\ (mn,q)=1}} \Lambda(m)\mu(n)\bigg\{\sum_{\substack{l \le x/mn\\ lmn \equiv a \,(\mathrm{mod}\, q)}} 1 - \frac{1}{\phi(q)} \sum_{\substack{l \le x/mn\\ (lmn,q)=1}} 1\bigg\}.$$

We can change the inner summation condition $(lmn, q) = 1$ to $(l, q) = 1$ because $(mn, q) = 1$. An easy computation yields

$$\frac{1}{\phi(q)} \sum_{\substack{l \le x/mn\\ (l,q)=1}} 1 - \frac{x}{qmn} = O(\tau(q)/\phi(q)),$$

$$E_2(x; q, a) = -I(x; q, a) + O\bigg(\frac{Z^2\tau(q)\log x}{\phi(q)}\bigg),$$

where

$$I(x; q, a) = \sum_{\substack{m,n \le Z\\ (mn,q)=1}} \Lambda(m)\mu(n)\bigg\{\sum_{\substack{l \le x/mn\\ lmn \equiv a \,(\mathrm{mod}\, q)}} 1 - \frac{x}{qmn}\bigg\}.$$

Thus it suffices for the proof of Theorem 1 or 2 to show for Q in the appropriate interval that

$$\sum_{q \in S_f(Q) \setminus F(Q)} \max_{(a,q)=1} |I(x; q, a)| \ll \frac{x|S_f(Q)|}{Q(\log x)^A}. \tag{3.2}$$

Likewise for Theorem 3 it suffices to show that

$$\sum_{p^2 \in (Q,2Q] \setminus F(Q)} \max_{p \nmid a} |I(x, p^2, a)| \ll xQ^{-1/2}(\log x)^{-A}.$$

4. Sums over characters of absolute values of Dirichlet polynomials. Our strategy resembles that of Iwaniec [9, Section 2] in dealing with sieve remainder terms. We begin with some material about sums over sets of characters $\chi \pmod q$, $q \in S_f(Q) \setminus F(Q)$, of the absolute values of certain Dirichlet polynomials.

PROPOSITION 1. *Let $M_1, \ldots, M_{15}$ be numbers with $M_1 \ge \cdots \ge M_{15} \ge 1$, and suppose $\{1, \ldots, 15\}$ partitions into subsets A and B such that*

$$\prod_{i \in A} M_i \ll x^{9/20 - 3\varepsilon/4}, \qquad \prod_{i \in B} M_i \ll x^{9/20 - 3\varepsilon/4}. \tag{4.1}$$

Let $a_i(m)$ $(M_i/2 < m \le M_i)$ be a complex sequence with

$$|a_i(m)| \le \log m \qquad (1 \le i \le 15,\ M_i/2 < m \le M_i).$$

Suppose that whenever $M_i > x^{1/8}$ then either

$$a_i(m) = 1 \qquad (M_i/2 < m \le M_i)$$

or

$$a_i(m) = \log m \qquad (M_i/2 < m \le M_i).$$

Let $M_i(s, \chi) = \sum_{M_i/2 < m \le M_i} a_i(m) \chi(m) m^{-s}$ and

$$L = \frac{x}{M_1 \ldots M_{15}}, \qquad B(s, \chi) = \sum_{n \le L} \chi(n) n^{-s}.$$

Then for $\mathrm{Re}(s) = 1/2$ *and*

$$Q \ll x^{9/20 - \varepsilon}, \tag{4.2}$$

we have

$$\sum_{q \in S_f(Q) \setminus F(Q)} \sum_{\substack{\chi \pmod q \\ \chi \ne \chi_0}} |B(s, \chi)| \prod_{i=1}^{15} |M_i(s, \chi)| \ll |s|^3 \, |S_f(Q)| x^{1/2 - 3\delta}. \tag{4.3}$$

PROPOSITION 2. *For $f(X) = X^2$, the assertion of Proposition* 1 *remains true if we replace* $9/20$ *by* $43/90$ *in* (4.1), *and replace* (4.2) *by*

$$Q \ll x^{43/90 - \varepsilon}. \tag{4.4}$$

PROPOSITION 3. *Suppose that $f(X) = X^2$ and*

$$Q \ll x^{1/2 - \varepsilon}. \tag{4.5}$$

The assertion of Proposition 1 *remains true if we replace* $9/20$ *by* $1/2$ *in* (4.1), *and replace q in* (4.3) *by p^2, with p prime.*

The following basic lemmas are needed.

LEMMA 7. *We have, for $q \ge 2$, $L \ge 1$,*

$$\sum_{\substack{\chi \pmod q \\ \chi \ne \chi_0}} \Big| \sum_{l \le L} \chi(l) l^{-1/2 - it} \Big|^4 \ll q(|t| + 1) \log^6 qL(|t| + 1).$$

Proof. This is [9, Lemma 3]. ∎

LEMMA 8. *For any complex numbers a_n $(N < n \ll N)$,*

$$\sum_{\chi\,(\mathrm{mod}\,q)} \Big| \sum_{N<n\ll N} a_n \chi(n) \Big|^2 \ll (N+q) \sum_{N<n\ll 2N} |a_n|^2.$$

Proof. See [12, Theorem 6.2]. ■

LEMMA 9. *For any complex numbers a_n $(N < n \ll N)$ and $V > 0$, and $G = \sum_{N<n\ll N} |a_n|^2$,*

$$\Big| \Big\{ \chi \ (\mathrm{mod}\ q) : \Big| \sum_{N<n\ll N} a_n \chi(n) \Big| > V \Big\} \Big| \ll GNV^{-2} + q^{1+\varepsilon} G^3 N V^{-6}.$$

Proof. See e.g. Jutila [10]. ■

Proof of Proposition 1. We prove (4.3) simply by showing for a fixed q in $S_f(Q) \setminus F(Q)$ that, writing

$$M = \prod_{i\in A} M_i, \qquad N = \prod_{i \in B} M_i \tag{4.6}$$

and

$$M(s,\chi) = \prod_{i\in A} M_i(s,\chi), \qquad N(s,\chi) = \prod_{i\in B} M_i(s,\chi), \tag{4.7}$$

we have

$$\sum_{\substack{\chi\,(\mathrm{mod}\,q)\\ \chi\neq\chi_0}} |B(s,\chi)M(s,\chi)N(s,\chi)| \ll |s|^3 x^{1/2-3\delta}. \tag{4.8}$$

We have trivially

$$B(s,\chi) \ll L^{1/2}, \qquad M(s,\chi) \ll M^{1/2+\delta}, \qquad N(s,\chi) \ll N^{1/2+\delta}.$$

Thus the characters $\chi \neq \chi_0$ for which one of these three Dirichlet polynomials has absolute value less than $(\phi(q)x^{5\delta})^{-1}$ can be neglected. We partition the remaining characters into $O((\log x)^3)$ subsets $A_q(U,V,W)$ of characters satisfying

$$U < |B(s,\chi)| \le 2U, \qquad V < |M(s,\chi)| \le 2V, \qquad W < |N(s,\chi)| \le 2W,$$

where $U \ll L^{1/2}$, $V \ll M^{1/2+\delta}$, $W \ll N^{1/2+\delta}$. To prove (4.8), it suffices to show for each triple U, V, W that

$$UVW|A_q(U,V,W)| \ll |s|^3 x^{1/2-4\delta}.$$

From the above lemmas applied to $B(s,\chi), M(s,\chi), N(s,\chi), B(s,\chi)^2$ we obtain

$$|A_q(U,V,W)| \ll x^\delta |s|^{1+\delta} P,$$

where

$$P = \min\Big(\frac{M+Q}{V^2}, \frac{N+Q}{W^2}, \frac{Q}{U^4}, \frac{M}{V^2} + \frac{QM}{V^6}, \frac{N}{W^2} + \frac{QN}{W^6}, \frac{L^2}{U^4} + \frac{QL^2}{U^{12}} \Big).$$

Thus it suffices to show

$$UVWP \ll x^{1/2-5\delta}.$$

We consider four cases.

Case 1: $P \le 2V^{-2}M$, $P \le 2W^{-2}N$. In this case we apply Lemma 6 with $\gamma = 1/10$; we have $MN \le x^{9/10}$ and $L \ge x^{1/10}$. Since $q \in S_f(Q)\setminus F(Q)$, we obtain

$$U \ll |s| L^{1/2} x^{-5\delta},$$

and

$$UVWP \le 2UVW \min(V^{-2}M,\, W^{-2}N) \ll U(MN)^{1/2} \ll |s|\, x^{1/2-5\delta}.$$

Case 2: $P > 2V^{-2}M$, $P > 2W^{-2}N$. In this case,

$$\begin{aligned} P &\le 2\min\{QV^{-2}, QW^{-2}, QMV^{-6}, QNW^{-6}, QU^{-4}, L^2U^{-4}\}\\ &\quad + 2\min\{QV^{-2}, QW^{-2}, QMV^{-6}, QNW^{-6}, QU^{-4}, QL^2U^{-12}\}\\ &\le 2(QV^{-2})^{5/16}(QW^{-2})^{5/16}(QMV^{-6})^{1/16}(QNW^{-6})^{1/16}\\ &\quad \times(\min(QU^{-4}, L^2U^{-4}))^{1/4}\\ &\quad +2\min\{(QV^{-2})^{5/16}(QW^{-2})^{5/16}(QMV^{-6})^{1/16}(QNW^{-6})^{1/16}(QU^{-4})^{1/4},\\ &\qquad (QV^{-2})^{7/16}(QW^{-2})^{7/16}(QMV^{-6})^{1/48}(QNW^{-6})^{1/48}(QL^2U^{-12})^{1/12}\}\\ &= 2(UVW)^{-1}Q(MN)^{1/16}\{\min(1, Q^{-1/4}L^{1/2}) + \min(1, L^{1/6}(MN)^{-1/24})\}\\ &\ll (UVW)^{-1}(x^{1/16}Q^{31/32} + x^{1/20}Q) \ll (UVW)^{-1}x^{1/2-\varepsilon} \end{aligned}$$

since $Q \ll x^{9/20-\varepsilon}$.

Case 3: $P > 2V^{-2}M$, $P \le 2W^{-2}N$. In this case,

$$\begin{aligned} P &\le 2\min\{QV^{-2}, NW^{-2}, QMV^{-6}, QU^{-4}, L^2U^{-4}\}\\ &\quad + 2\min\{QV^{-2}, NW^{-2}, QMV^{-6}, QU^{-4}, QL^2U^{-12}\}\\ &\le 2(QV^{-2})^{1/8}(NW^{-2})^{1/2}(QMV^{-6})^{1/8}(\min(QU^{-4}, L^2U^{-4}))^{1/4}\\ &\quad + 2\min\{(QV^{-2})^{1/8}(NW^{-2})^{1/2}(QMV^{-6})^{1/8}(QU^{-4})^{1/4},\\ &\qquad (QV^{-2})^{3/8}(NW^{-2})^{1/2}(QMV^{-6})^{1/24}(QL^2U^{-12})^{1/12}\}\\ &= 2(UVW)^{-1}(QN)^{1/2}M^{1/8}\{\min(1, Q^{-1/4}L^{1/2}) + \min(1, L^{1/6}M^{-1/12})\}\\ &\ll (UVW)^{-1}(x^{1/8}Q^{7/16}N^{3/8} + x^{1/12}Q^{1/2}N^{5/12}) \ll (UVW)^{-1}x^{1/2-\varepsilon} \end{aligned}$$

since $Q \ll x^{9/20-\varepsilon}$ and $N < Qx^{\varepsilon/2}$. (There is a little to spare in Case 3.)

Case 4: $P > 2W^{-2}N$, $P \le 2V^{-2}M$. We proceed as in Case 3, interchanging the roles of M and N.

This completes the proof of Proposition 1. ∎

We break the argument for Proposition 2 into a number of lemmas. We maintain the definitions (4.6), (4.7) and let $M = x^{\alpha_1}$, $N = x^{\alpha_2}$, $Q = x^{\theta}$. We may suppose that $\theta > 9/20 - \varepsilon$ and $\alpha_2 \le \alpha_1$.

It suffices to show for $0 \le \lambda \le \theta$ that

$$\sum_{q \in S_f(Q) \setminus F(Q)} \sum_{\substack{\chi \,(\mathrm{mod}\, q),\, \chi \ne \chi_0 \\ x^{\lambda} < \mathrm{cond}\, \chi \le 2x^{\lambda}}} |B(s,\chi) M_1(s,\chi) \ldots M_{15}(s,\chi)| \ll |s|^3 x^{1/2 - 4\delta} Q^{1/2}. \tag{4.9}$$

A strategy which works for some triples $\lambda, \alpha_1, \alpha_2$ is to show that, for $q \in S_f(Q) \setminus F(Q)$,

$$\sum_{\substack{\chi \,(\mathrm{mod}\, q),\, \chi \ne \chi_0 \\ x^{\lambda} < \mathrm{cond}\, \chi \le 2x^{\lambda}}} |B(s,\chi) M(s,\chi) N(s,\chi)| \ll |s|^3 x^{1/2 - 4\delta}. \tag{4.10}$$

Lemma 10. *Let $q \in S_f(Q) \setminus F(Q)$. Suppose that*

$$\alpha_1 + \alpha_2 < 8 - 16\lambda - 200\delta, \tag{4.11}$$

$$\alpha_1 < 1 - 6\lambda/5 - 20\delta. \tag{4.12}$$

Then (4.10) *holds. In particular, it holds if $\lambda \le (5\theta + \varepsilon)/6$.*

Proof. When χ is counted in the sum in (4.10),

$$M(s,\chi) = \sum_{(n,q)=1} a(n) \chi'(n) n^{-s}$$

with $a(n) \ll x^{\delta}$ and some primitive character χ' (mod r), $r \le 2x^{\lambda}$; similarly for $N(s,\chi)$. We may improve our bounds for mean and large values of these Dirichlet polynomials, replacing q by x^{λ} in each case. Thus

$$|A_q(U,V,W)| \ll \min(MV^{-2} + x^{\lambda+\delta} V^{-2}, NW^{-2} + x^{\lambda+\delta} W^{-2}, \\ MV^{-2} + x^{\lambda+\delta} MV^{-6}, NW^{-2} + x^{\lambda+\delta} NW^{-6}).$$

To get variants of the other quantities in the definition of P, we observe that

$$B(s,\chi) = \sum_{n \le L} \Big(\sum_{\substack{d|q \\ d|n}} \mu(d) \Big) \chi'(n) n^{-s} = \sum_{d|q} \frac{\mu(d)\chi'(d)}{d^s} \sum_{k \le L/d} \chi'(k) k^{-s}.$$

If $|B(s,\chi)| \ge U$, then

$$\Big| \sum_{k \le L/d} \chi'(k) k^{-s} \Big| \ge U x^{-\delta/12}$$

for some d with $d \,|\, q$, and consequently

$$|A_q(U,V,W)| \ll \min(x^{\lambda+\delta} U^{-4} |s|^{1+\delta}, x^{\delta} L^2 U^{-4} + x^{\lambda+\delta} L^2 U^{-12}).$$

Let

$$P' = \min\Big(\frac{M+x^\lambda}{V^2}, \frac{N+x^\lambda}{W^2}, \frac{x^\lambda}{U^4}, \frac{M}{V^2} + \frac{x^\lambda M}{V^6}, \frac{N}{W^2} + \frac{x^\lambda N}{W^6}, \frac{L^2}{U^4} + \frac{x^\lambda L^2}{U^{12}}\Big).$$

The bound (4.10) will follow if we show that

$$UVWP' \ll |s|^{1+\delta} x^{1/2-7\delta}.$$

As in the preceding proof, we break the argument into Cases 1–4, defined exactly as before with P replaced by P'. Case 1 proceeds as before. In Case 2,

$$\begin{aligned} P' &\le 2\min\{x^\lambda V^{-2}, x^\lambda W^{-2}, x^\lambda MV^{-6}, x^\lambda NW^{-6}, x^\lambda U^{-4}\} \\ &\le 2(x^\lambda V^{-2})^{5/16}(x^\lambda W^{-2})^{5/16}(x^\lambda MV^{-6})^{1/16}(x^\lambda NW^{-6})^{1/16}(x^\lambda U^{-4})^{1/4} \\ &= 2(UVW)^{-1}x^\lambda (MN)^{1/16} \ll (UVW)^{-1}x^{1/2-7\delta} \end{aligned}$$

from (4.11). In Case 3, the argument used in proving (4.8) yields

$$P' \ll (UVW)^{-1}(x^{1/8+7\lambda/16}N^{3/8} + x^{1/12+\lambda/2}N^{5/12}) \ll (UVW)^{-1}x^{1/2-7\delta}.$$

To see this, note that

$$\frac{1}{8} + \frac{7\lambda}{16} + \frac{3\alpha_2}{8} < \frac{1}{2} - 7\delta$$

since $\alpha_2 < 1 - 7\lambda/6 - 20\delta$, and

$$\frac{1}{12} + \frac{\lambda}{2} + \frac{5\alpha_2}{12} < \frac{1}{2} - 7\delta$$

since $\alpha_2 < 1 - 6\lambda/5 - 20\delta$. In Case 4, proceed as in Case 3, with M and N interchanged.

This proves the first assertion of the lemma. For the second assertion, we observe that if $\lambda \le (5\theta + \varepsilon)/6$, then

$$\begin{aligned} \alpha_1 &\le \theta + \varepsilon/4 < 1 - \theta - \varepsilon < 1 - 6\lambda/5 - 20\delta, \\ \alpha_1 + \alpha_2 &\le 2\theta + \varepsilon/2 < 8 - 80\theta/6 - 20\varepsilon < 8 - 16\lambda - 200\delta. \end{aligned}$$

We obtain (4.10) in view of the first assertion of the lemma. ∎

In view of Lemma 10, we suppose for the remainder of the proof of Proposition 2 that

$$\lambda > (5\theta + \varepsilon)/6. \tag{4.13}$$

We now bring the work of Baier and Zhao into play.

LEMMA 11. *Let $a_1, \ldots, a_N$ be complex numbers and*

$$T(\alpha) = \sum_{n=1}^{N} a_n e(n\alpha), \qquad G = \sum_{n=1}^{N} |a_n|^2.$$

Let $g \in \mathbb{N}$, $g \le Q$. Then

$$\sum_{q \le Q} \sum_{\substack{a=1 \\ (a,gq^2)=1}}^{gq^2} \left| T\left(\frac{a}{gq^2}\right)\right|^2 \ll (QN)^{\varepsilon}(g^2Q^3 + gQ^{1/2}N)G.$$

Proof. We deduce this from the work of Baier and Zhao [**?**], where the case $g = 1$ is treated. By [12, Theorem 2.1],

$$\sum_{q \le Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{gq^2} \left| T\left(\frac{a}{gq^2}\right)\right|^2 \ll K(\varDelta)(N + \varDelta^{-1})G. \tag{4.14}$$

Here

$$K(\varDelta) = \max_{\alpha \in \mathbb{R}} \sum_{q \le Q} \sum_{\substack{a=1 \\ (a,gq^2)=1 \\ \|a/(gq^2) - \alpha\| \le \varDelta}}^{gq^2} 1. \tag{4.15}$$

We observe that the conditions of summation in (4.15) imply

$$\left\| \frac{a}{q^2} - g\alpha \right\| \le g\varDelta. \tag{4.16}$$

If there are $\mathcal{N}(\alpha)$ solutions of (4.16) with $(a,q) = 1$, $1 \le a \le q^2$, $q \le Q$, then there are $g\mathcal{N}(\alpha)$ solutions with $(a,q) = 1$, $1 \le a \le gq^2$, $q \le Q$. Now according to [**?**, Section 11], with $g\varDelta$ in place of $\varDelta$,

$$\mathcal{N}(\alpha) \ll (Q\varDelta^{-1})^{\varepsilon}(Q^3(g\varDelta) + Q^{7/4}(g\varDelta)^{1/2} + Q(g\varDelta)^{1/4} + Q^{1/2}).$$

Take $\varDelta = N^{-1}$ to obtain

$$(N + \varDelta^{-1})K(N^{-1}) \ll (QN)^{\varepsilon}(g^2Q^3 + g^{3/2}Q^{7/4}N^{1/2} + g^{5/4}QN^{3/4} + gQ^{1/2}N).$$

The lemma follows on combining this with (4.14), since

$$g^{3/2}N^{1/2}Q^{7/4} = (g^2Q^3)^{1/2}(gQ^{1/2}N)^{1/2},$$
$$g^{5/4}QN^{3/4} \le (g^2Q^3)^{1/4}(gQ^{1/2}N)^{3/4}. \ \blacksquare$$

LEMMA 12. *Let $c_1, \ldots, c_J$ be complex numbers. Let*

$$T(J,\lambda) = \sum_{Q^{1/2} < q \le 2Q^{1/2}} \sum_{\substack{\chi \,(\mathrm{mod}\, q^2),\, \chi \ne \chi_0 \\ x^{\lambda} < \mathrm{cond}\, \chi \le 2x^{\lambda}}} \Big| \sum_{m=1}^{J} c_m \chi(m) \Big|^2.$$

Then

$$T(J,\lambda) \ll (QJ)^{2\delta}(Q^{3/2} + Q^{7/4}x^{-3\lambda/2}J) \sum_{m=1}^{J} |c_m|^2.$$

Proof. The conductor of a character χ counted in $T(J,\lambda)$ may be written as gk^2 where g is square-free, $gk^2 \in (x^\lambda, 2x^\lambda]$. These χ counted by $T(J,\chi)$ arising from a given primitive character χ' to modulus gk^2 may be written as

$$\chi'_v(m) = \begin{cases} \chi'(m) & \text{if } (m,v)=1, \\ 0 & \text{if } (m,v)>1, \end{cases}$$

where v takes integer values such that

$$vgk^2 = q^2 \in (Q,2Q]. \tag{4.17}$$

Clearly all such v have

$$g \mid v, \qquad v \in (Qx^{-\lambda}/2, 2Qx^{-\lambda}). \tag{4.18}$$

Let

$$a_{v,m} = \begin{cases} c_m & \text{if } (m,v)=1, \\ 0 & \text{if } (m,v)>1. \end{cases}$$

For a given triple k, g, v satisfying (4.17), (4.18), we have

$$\sideset{}{^*}\sum_{\chi' \,(\mathrm{mod}\, gk^2)} \Big| \sum_{m=1}^{J} c_m \chi'_v(m) \Big|^2 = \sideset{}{^*}\sum_{\chi' \,(\mathrm{mod}\, gk^2)} \Big| \sum_{m=1}^{J} a_{v,m} \chi'(m) \Big|^2$$
$$\leq \frac{\phi(gk^2)}{gk^2} \sum_{\substack{a=1 \\ (a,gk^2)=1}}^{gk^2} \Big| T_v\Big(\frac{a}{gk^2}\Big)\Big|^2,$$

where

$$T_v(\alpha) = \sum_{m=1}^{J} a_{v,m} e(m\alpha).$$

Here we appeal to (10) in [5, Section 27]. Combining this with Lemma 11 we find that, for a given pair g, v satisfying (4.18),

$$\sum_{Q^{1/2}/(vg)^{1/2} < k \leq (2Q)^{1/2}/(vg)^{1/2}} \ \sideset{}{^*}\sum_{\chi' \,(\mathrm{mod}\, gk^2)} \Big| \sum_{m=1}^{J} c_m \chi'_v(m) \Big|^2$$
$$\ll (QJ)^\delta \Big(\frac{g^2 Q^{3/2}}{(vg)^{3/2}} + \frac{gQ^{1/4}}{(vg)^{1/4}} J \Big) \sum_{m=1}^{J} |c_m|^2$$
$$\ll (QJ)^\delta (Q^{3/2} v^{-1} + Q^{1/4} J v^{1/2}) \sum_{m=1}^{J} |c_m|^2.$$

Summing over all pairs v, g satisfying (4.18), we obtain

$$T(J,\lambda) \ll (QJ)^{2\delta} (Q^{3/2} + Q^{1/4} J (Qx^{-\lambda})^{3/2}) \sum_{m=1}^{J} |c_m|^2,$$

as claimed. ∎

LEMMA 13. *Let*

$$H(s,\chi)=\sum_{n\le H} a_n\chi(n)n^{-s},\qquad K(s,\chi)=\sum_{n\le K} b_n\chi(n)n^{-s},$$

with $|a_n|\le \tau(n)^B$, $|b_n|\le\tau(n)^B$ *for an absolute constant* B. *If*

$$HK\ll x,\qquad K\le H\ll x^{1+3\lambda/2-9\theta/4-16\delta},$$

then

$$\sum_{q\in S_f(Q)}\ \sum_{\substack{\chi\,(\mathrm{mod}\,q),\,\chi\ne\chi_0\\ x^\lambda<\mathrm{cond}\,\chi\le 2x^\lambda}} |H(s,\chi)K(s,\chi)|\ll x^{1/2-6\delta}Q^{1/2}. \tag{4.19}$$

Proof. By Lemma 12 and the Cauchy–Schwarz inequality, the left-hand side of (4.19) is

$$\begin{aligned}&\ll x^{2\delta}(Q^{3/4}+Q^{7/8}x^{-3\lambda/4}H^{1/2})(Q^{3/4}+Q^{7/8}x^{-3\lambda/4}K^{1/2})\\ &\ll x^{2\delta}(Q^{3/2}+Q^{7/4}x^{-3\lambda/2+1/2}+Q^{13/8}x^{-3\lambda/4}H^{1/2}).\end{aligned}$$

Now

$$x^{2\delta}Q^{3/2}\ll Q^{1/2}x^{1/2-6\delta}$$

since $\theta<1/2-\varepsilon$. Also

$$x^{2\delta}Q^{7/4}x^{-3\lambda/2+1/2}\ll Q^{1/2}x^{1/2-6\delta}$$

from (4.13). Finally,

$$x^{2\delta}Q^{13/8}x^{-3\lambda/4}H^{1/2}\ll Q^{1/2}x^{1/2-6\delta}$$

since $H\ll x^{1-9\theta/4+3\lambda/2-16\delta}$. ∎

LEMMA 14. *Let* $\beta_1\ge\dots\ge\beta_R\ge 0$, $\beta_1+\dots+\beta_R\ge 1/2$, $R\ge 2$. *Suppose that* $\beta_1+\beta_2\le 3/5$. *Then there is a sum*

$$\sigma=\sum_{j=1}^{r}\beta_j,\qquad 2\le r\le R,$$

such that $\sigma\in[2/5,3/5]$.

Proof. Suppose the contrary; then $\beta_1+\beta_2<2/5$,

$$\beta_1+\beta_2+\beta_3\le\frac{3}{2}(\beta_1+\beta_2)<\frac{3}{5},\quad\text{hence}\quad \beta_1+\beta_2+\beta_3<\frac{2}{5}.$$

Arguing in this way we prove for $j=4,\dots,R$ that

$$\beta_1+\dots+\beta_j\le\frac{j}{j-1}(\beta_1+\dots+\beta_{j-1})<\frac{3}{5},\quad\text{hence}\quad \beta_1+\dots+\beta_j<\frac{2}{5}.$$

When $j=R$, we have a contradiction. ∎

LEMMA 15. *Suppose that*

$$\lambda \geq -\frac{4}{15} + \frac{3\theta}{2} + 12\delta.$$

Then (4.9) *holds.*

Proof. We decompose $B(s,\chi)$ into $O(\log x)$ Dirichlet polynomials of the form

$$M_{16}(s,\chi) = \sum_{M_{16}/2<m\leq M_{16}} \chi(m)m^{-s}.$$

It suffices to prove the analog of (4.9) with M_{16} in place of B and 6δ in place of 4δ. Fix M_{16} and rearrange $M_1,\ldots,M_{16}$ as $N_1 \geq \cdots \geq N_{16}$; write $N_i(s)$ for the corresponding Dirichlet polynomials and

$$N_i = x^{\beta_i}. \tag{4.20}$$

Thus $\beta_1 \geq \cdots \geq \beta_{16} \geq 0$, $\beta_1 + \cdots + \beta_{16} \leq 1$.

We treat the rather trivial case

$$\beta_1 + \cdots + \beta_{16} < 1/2$$

by applying Lemma 13 with $K(s,\chi) = 1$,

$$H(s,\chi) = N_1(s,\chi)\ldots N_{16}(s,\chi), \qquad H = x^{\beta_1+\cdots+\beta_{16}} < x^{1/2} < x^{1+3\lambda/2-9\theta/4-\varepsilon}$$

since $3\lambda/2 > 5\theta/4$ and $\theta < 1/2 - \varepsilon$.

Now suppose that $\beta_1 + \cdots + \beta_{16} \geq 1/2$, so that Lemma 14 is applicable.

Suppose first that $\beta_1 + \beta_2 > 3/5$. We write $N_0(s) = N_3(s)\ldots N_{16}(s)$,

$$A(U_0,U_1,U_2) = \{\chi \pmod q : q \in S_f(Q), \chi \neq \chi_0,\, x^\lambda < \operatorname{cond}\chi \leq 2x^\lambda,$$
$$U_j < |N_j(s)| \leq 2U_j \ (j = 0,1,2)\}.$$

Arguing as in the proof of Proposition 1, it suffices to show that

$$U_0U_1U_2|A(U_0,U_1,U_2)| \ll Q^{1/2}|s|^3x^{1/2-6\delta}. \tag{4.21}$$

Since $N_1 \geq x^{3/10}$, we have

$$|A(U_0,U_1,U_2)| \ll Q^{1/2}|s|^{1+\delta}x^{\theta+\delta}U_1^{-4}$$

from Lemma 7 (and, if needed, a partial summation). Next

$$|A(U_0,U_1,U_2)| \ll Q^{1/2}|s|^{1+\delta}x^{\theta+\delta}U_2^{-4}$$

from Lemma 7 (if $N_2^2 > x^\theta$) and Lemma 8 (if $N_2^2 \leq x^\theta$). We have

$$|A(U_0,U_1,U_2)| \ll Q^{1/2}x^{\theta+\delta}U_0^{-2}$$

from Lemma 8, since $N_0 \ll x^{2/5} \ll x^\theta$. Hence

$$|A(U_0,U_1,U_2)| \ll Q^{1/2}|s|^{1+\delta}x^{\theta+\delta}(U_1^{-4})^{1/4}(U_2^{-4})^{1/4}(U_0^{-2})^{1/2},$$

and (4.21) follows at once.

Now suppose that $\beta_1 + \beta_2 \le 3/5$. By Lemma 14, there is a subset W of $\{1, \dots, 16\}$ such that

$$x^{2/5} \ll \prod_{j \in W} M_j \ll x^{3/5}.$$

We now apply Lemma 13 with $\{H, K\} = \{\prod_{j\in W}(2M_j), \prod_{j\le 16,\, j\notin W}(2M_j)\}$, $H \ge K$. We have

$$x^{1/2} \ll H \ll x^{3/5} \ll x^{1+3\lambda/2-9\theta/4-16\delta}$$

by hypothesis. This gives the analog of (4.9) with M_{16} in place of B and 6δ in place of 4δ, and the lemma follows at once. ∎

Lemma 16. *Suppose that*

$$\alpha_1 \ge \frac{9\theta}{4} - \frac{3\lambda}{2} + 16\delta.$$

Then (4.9) *holds.*

Proof. Since $\alpha_1 < 1/2$, this is a straightforward consequence of Lemma 13 with $K(x, \chi) = M(s, \chi)$, $H(s, \chi) = N(s, \chi)B(s, \chi)$. ∎

Lemma 17. *Suppose that*

$$\alpha_1 < 4 - 8\lambda - 100\delta.$$

Then (4.9) *holds.*

Proof. We have (4.11) since $\alpha_2 \le \alpha_1$. In view of Lemma 10, we need only show that

$$\alpha_1 < 1 - \frac{6\lambda}{5} - 20\delta.$$

By Lemma 16, we may suppose that

$$\alpha_1 < \frac{9\theta}{4} - \frac{3\lambda}{2} + 16\delta.$$

Hence we can establish

$$\alpha_1 < 1 - \frac{6\lambda}{5} - 20\delta$$

by using $\lambda > (5\theta + \varepsilon)/6$, $\theta < 1/2$ to obtain

$$1 + \frac{3\lambda}{10} > \frac{9\theta}{4} + 40\delta. \ ∎$$

Proof of Proposition 2. We recall that it suffices to prove (4.9). By Lemma 15, we may suppose that

$$\lambda < -\frac{4}{15} + \frac{3\theta}{2} + 12\delta. \tag{4.22}$$

In view of Lemmas 16 and 17, it remains to show that the intervals $[9\theta/4-3\lambda/2+16\delta,\theta+\varepsilon/4]$ and $[0,4-8\lambda-100\delta)$ overlap. That is, we need to show

$$4-8\lambda-100\delta>\frac{9\theta}{4}-\frac{3\lambda}{2}+16\delta,$$

or

$$\frac{13\lambda}{2}<4-\frac{9\theta}{4}-116\delta.$$

Indeed, from (4.22),

$$\frac{13\lambda}{2}<-\frac{26}{15}+\frac{39\theta}{4}+78\delta<4-\frac{9\theta}{4}-116\delta$$

since $\theta<43/90-\varepsilon$. ■

Proof of Proposition 3. As in the preceding proof it suffices to show that for each tuple $M_1,\ldots,M_{15}$,

$$\sum_{p^2\in(Q,2Q]\setminus F(Q)}\ \sum_{\substack{\chi\,(\mathrm{mod}\,p^2),\,\chi\neq\chi_0\\ x^\lambda<\mathrm{cond}\,\chi\le 2x^\lambda}}|B(s,\chi)M_1(s,\chi)\ldots M_{15}(s,\chi)| \ll |s|^3x^{1/2-4\delta}Q^{1/2}. \tag{4.23}$$

The conductor of each character counted in (4.23) is either p or p^2, so that

$$\mathrm{cond}\,\chi\in(Q^{1/2},(2Q)^{1/2}]\cup(Q,2Q].$$

Thus the sum in (4.23) is empty unless

$$\lambda=\theta/2\quad\text{or}\quad\lambda=\theta.$$

For $\lambda=\theta/2$, we obtain (4.23) as a consequence of Lemma 10. (Note that no inequality stronger than $\theta<1/2-\varepsilon$ was used in the proofs of Lemmas 10–17.) For $\lambda=\theta$, we have

$$\lambda>-\frac{4}{15}+\frac{3\theta}{2}+12\delta,$$

with something to spare. Now (4.23) is a consequence of Lemma 15. ■

5. Proofs of Theorems 1, 2 and 3. We work with the Riesz means

$$A_k(x,q,a,d)=\frac{1}{k!}\sum_{\substack{l\le x\\ l\equiv a\,(\mathrm{mod}\,q)\\ l\equiv 0\,(\mathrm{mod}\,d)}}\left(\log\frac{x}{l}\right)^k.$$

Ultimately we are interested in A_0; the presence of the factor s^{-5} in (5.5) below is the reason for working initially with A_4.

Let us write the associated remainder term as

$$r_k(x,q,a,d)=A_k(x,q,a,d)-\frac{x}{qd}.$$

We borrow from Iwaniec [9, (2.5)] the inequalities

$$r_{k-1}(x,q,a,d) \le \left(\frac{e^\lambda - 1}{\lambda} - 1\right)\frac{x}{qd} + \frac{1}{\lambda}[r_k(e^\lambda x,q,a,d) - r_k(x,q,a,d)],$$

$$\begin{aligned} r_{k-1}&(x,q,a,d) \\ &\ge \left(\frac{1-e^{-\lambda}}{\lambda} - 1\right)\frac{x}{qd} + \frac{1}{\lambda}\,[r_k(x,q,a,d) - r_k(e^{-\lambda}x,q,a,d)]. \end{aligned} \tag{5.1}$$

If $u_d \ge 0$ $(D_1 < d \le D)$, it follows that

$$\begin{aligned} \sum_{D_1<d\le D} u_d r_{k-1}(x,q,a,d) &\le \left(\frac{e^\lambda - 1}{\lambda} - 1\right)\frac{x}{q}\sum_{D_1<d\le D}\frac{u_d}{d} \\ &\quad + \frac{1}{\lambda}\Big[\sum_{D_1<d\le D} u_d r_k(e^\lambda x,q,a,d) - \sum_{D_1<d\le D} u_d r_k(x,q,a,d)\Big]. \end{aligned} \tag{5.2}$$

There is a similar lower bound for the left-hand side of (5.2), which follows from (5.1). We see that for $0 < \lambda < 1$,

$$\begin{aligned} \sum_{q\in S_f(Q)\setminus F(Q)} &\Big|\sum_{D_1<d\le D} u_d r_{k-1}(x,q,a,d)\Big| \\ &\ll \frac{\lambda x|S_f(Q)|}{Q}\sum_{D_1<d\le D}\frac{u_d}{d} \\ &\quad + \frac{1}{\lambda}\sum_{q\in S_f(Q)\setminus F(Q)}\Big\{\Big|\sum_{D_1<d\le D} u_d r_k(e^\lambda x,q,a,d)\Big| \\ &\quad + \Big|\sum_{D_1<d\le D} u_d r_k(x,q,a,d)\Big| + \Big|\sum_{D_1<d\le D} u_d r_k(e^{-\lambda}x,q,a,d)\Big|\Big\}. \end{aligned} \tag{5.3}$$

For $q \in S_f(Q)\setminus F(Q)$, let $a^{(q)}$ be an integer coprime to q. Suppose that

$$\sum_{D_1<d\le D}\frac{u_d}{d} \ll x^{\eta/6}$$

and

$$\sum_{q\in S_f(Q)\setminus F(Q)}\Big|\sum_{D_1<d\le D} u_d r_k(x,q,a^{(q)},d)\Big| \ll \frac{|S_f(Q)|}{Q}\,x^{1-\eta}$$

for some $\eta > 0$, whenever $Q \ll x^\alpha$. Taking $\lambda = x^{-\eta/2}$, we deduce from (5.3) that

$$\sum_{q\in S_f(Q)\setminus F(Q)}\Big|\sum_{D_1<d\le D} u_d r_{k-1}(x,q,a^{(q)},d)\Big| \ll \frac{|S_f(Q)|}{Q}\,x^{1-\eta/3}$$

for $Q \ll x^\alpha$.

We are now ready to make a suitable inference from the work of Section 4 about remainders $r_0(x,q,a^{(q)},d)$.

LEMMA 18. *Let $a_i(m)$ $(M_i/2 < m \le M_i)$ be nonnegative sequences satisfying the hypotheses of Proposition* 1. *Let*

$$u_d = \sum_{\substack{d=m_1\dots m_{15}\\ M_i/2<m_i\le M_i\ (i=1,\dots,15)}} a_1(m_1)\dots a_{15}(m_{15})$$

for $D_1 < d \le D$, with $D = M_1 \dots M_{15}$, $D_1 = 2^{-15}D$. Suppose that (4.2) *holds. Then for every $A > 0$,*

$$\sum_{q\in S_f(Q)\setminus F(Q)} \Big| \sum_{D_1<d\le D} u_d r_0(x,q,a^{(q)},d)\Big| \ll \frac{x|S_f(Q)|}{Q(\log x)^A}.$$

Proof. In view of the above discussion, it suffices to prove that

$$\sum_{q\in S_f(Q)\setminus F(Q)} \Big| \sum_{D_1<d\le D} u_d r_4(x,q,a^{(q)},d)\Big| \ll \frac{x^{1-\delta}|S_f(Q)|}{Q}$$

for $Q \ll x^{9/20-\varepsilon}$. We represent $r_4(x,q,a^{(q)},d)$ in the form

$$\begin{aligned} r_4(x,q,a^{(q)},d) &= \frac{1}{24\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \bar\chi(a^{(q)})\chi(d) \sum_{b\le x/d} \chi(b)\Big(\log\frac{x}{bd}\Big)^4 - \frac{x}{qd}\\ &= \frac{1}{24\phi(q)} \sum_{\substack{\chi \,(\mathrm{mod}\, q)\\ \chi\ne\chi_0}} \bar\chi(a^{(q)})\chi(d) \sum_{b\le x/d} \chi(b)\Big(\log\frac{x}{bd}\Big)^4 + O\Big(\frac{x^\delta}{q}\Big) \end{aligned}$$

for $(d,q)=1$. Since $D < x^{1-\varepsilon}$, it suffices to show that

$$\sum_{q\in S_f(Q)\setminus F(Q)} \sum_{\chi \,(\mathrm{mod}\, q),\, \chi\ne\chi_0} \Big| \sum_{D_1<d\le D} u_d\chi(d) \sum_{b\le x/d} \chi(b)\Big(\log\frac{x}{bd}\Big)^4\Big| \ll |S_f(Q)|x^{1-\delta}. \tag{5.4}$$

We now use the integral representation

$$\int_{(1/2)} \frac{y^s}{s^5}\,ds = \begin{cases} (\log y)^4 & \text{if } y>1,\\ 0 & \text{if } y\le 1\end{cases} \tag{5.5}$$

(e.g. Montgomery and Vaughan [14, p. 143]). This gives

$$\sum_{D_1<d\le D} u_d\chi(d) \sum_{b\le x/d} \chi(b)\Big(\log\frac{x}{bd}\Big)^4 = \int_{(1/2)} x^s \sum_{D_1<d\le D} u_d\chi(d)d^{-s} \sum_{b\le x/D_1} \chi(b)b^{-s}\,\frac{ds}{s^5}$$

and

$$\sum_{q\in S_f(Q)\setminus F(Q)}\ \sum_{\substack{\chi \,(\mathrm{mod}\, q)\\ \chi\ne\chi_0}}\Big|\sum_{D_1<d\le D} u_d\chi(d)\sum_{b\le x/d}\chi(b)\Big(\log\frac{x}{bd}\Big)^4\Big|$$
$$\ll x^{1/2}\int_{(1/2)}\ \sum_{q\in S_f(Q)\setminus F(Q)}\ \sum_{\substack{\chi \,(\mathrm{mod}\, q)\\ \chi\ne\chi_0}}\Big|\sum_{D_1<d\le D} u_d\chi(d)d^{-s}\Big|\,|B(s,\chi)|\,\frac{|ds|}{|s|^5}.$$

Now (5.4) follows from Proposition 1. ∎

Proof of Theorem 1. Let $a^{(q)}$ be an integer coprime to q for which $I(s;q,a)$ is maximal. The left-hand side of (3.2) is

$$\sum_{q\in S_f(Q)\setminus F(Q)}\Big|\sum_{m,n\le Z}\Lambda(m)\mu(n)r(x,q,a^{(q)},mn)\Big|.$$

We recall Heath-Brown's decomposition [8] of $\Lambda(m)$ and the slight variant, used e.g. in [4], for the arithmetic function $\mu(n)$. Taking $k=4$ in both cases, we see that

$$\Lambda(m)=\sum_{(I_1,\dots,I_8)}\ \sum_{\substack{m_i\in I_i\\ m_1\dots m_8=m}}(\log m_1)\mu(m_5)\mu(m_6)\mu(m_7)\mu(m_8)\qquad (1\le m\le Z),$$
$$\mu(n)=\sum_{(J_1,\dots,J_7)}\ \sum_{\substack{n_i\in I_i\\ n_1\dots n_7=n}}\mu(n_4)\dots\mu(n_7)\qquad (1\le n\le Z).$$

Here $I_i=(a_i,2a_i]$, $J_j=(b_j,2b_j]$, $\prod_i a_i<Z$, $\prod_j b_j<Z$, $2a_i\le Z^{1/4}$ if $i>4$, $2b_j\le Z^{1/4}$ if $j>3$. Some of the intervals I_i, J_j may contain only the integer 1. There are $O((\log x)^8)$ tuples $(I_1,\dots,I_8)$ and $O((\log x)^7)$ tuples $(J_1,\dots,J_7)$ in these expressions. Now write $\mu(m)=a(m)+b(m)$ where $a(m)=\max(\mu(m),0)$. Then

$$\sum_{m\le Z,\,n\le Z}\Lambda(m)\mu(n)r_0(x,q,a^{(q)},mn)$$
$$=\sum_{(I_1,\dots,I_8)}\ \sum_{(J_1,\dots,J_7)}\ \sum_{m_i\in I_i,\,n_j\in J_j}(\log m_1)(a(m_5)+b(m_5))$$
$$\times\dots(a(n_7)+b(n_7))r_0(x,q,a^{(q)},m_1\dots m_8\,n_1\dots n_7).$$

This splits in an obvious way into $O((\log x)^{15})$ sums with an attached $\pm$ sign, in each of which the coefficients are nonnegative. Now (3.2) follows on applying Lemma 18 to each of the sums. This completes the proof of Theorem 1. ∎

In just the same way, Theorem 2 follows from Proposition 2 and Theorem 3 follows from Proposition 3.

6. A maximal variant of Theorems 1, 2 and 3

THEOREM 4. *The results of Theorems* 1 *and* 2 *remain valid when* $E(x,q)$ *is replaced by*

$$\max_{1\le y\le x} E(y,q).$$

The result of Theorem 3 *remains valid when* $E(x,p^2)$ *is replaced by*

$$\max_{1\le y\le x} E(y,p^2).$$

Proof. As above, we write $\theta = 9/20-\varepsilon$ (Theorem 1), $\theta = 43/90-\varepsilon$ (Theorem 2).

We write

$$v = x/(\log x)^A.$$

For $q < x^{1/2}$, $1\le t\le x$, we have

$$\max_{(a,q)=1} |\{p : p\equiv a \pmod q,\ t<p\le t+v\}| \ll \frac{v}{\phi(q)\log x}. \tag{6.1}$$

This can easily be deduced from [7, Theorem 2.2], for example.

Let $v = x_0, x_1, \dots, x_N$ be a sequence of equally spaced positive numbers,

$$x_j - x_{j-1} = v \quad (j=1,\dots,N), \qquad x\le x_N < x+v. \tag{6.2}$$

By Theorem 1 or 2, for $Q < x^\theta$,

$$\sum_{q\in S_f(Q)} E(x_j,q) \ll \frac{x_j|S_f(Q)|}{Q(\log x)^{3A+1}} \qquad (0\le j\le N). \tag{6.3}$$

Let

$$G_j = \left\{q\in S_f(Q) : E(x_j,q) > \frac{x_j}{Q(\log x)^{A+1}}\right\}.$$

From (6.3),

$$|G_j| \ll \frac{|S_f(Q)|}{(\log x)^{2A}}.$$

The union $G = \bigcup_{j=1}^N G_j$ thus satisfies

$$|G| \ll \frac{N|S_f(Q)|}{(\log x)^{2A}} \ll \frac{x}{v}\,\frac{|S_f(Q)|}{(\log x)^{2A}} \ll \frac{|S_f(Q)|}{(\log x)^{A}} \tag{6.4}$$

from (6.2).

Now suppose that $q\in S_f(Q)\setminus G$ and let $1\le y\le x$. If $y<v$, then (6.1) yields

$$E(y,q) \ll \frac{v}{\phi(q)\log x}.$$

If $v<y\le x$, then $y\in(x_{j-1},x_j]$ for some j, $1\le j\le N$. Thus, for some λ

in $(0,1]$,

$$\begin{aligned}
&|\{p : p \equiv a \ (\mathrm{mod}\ q),\ p \le y\}| \\
&\qquad = |\{p : p \equiv a \ (\mathrm{mod}\ q),\ p \le x_{j-1}\}| \\
&\qquad\quad + \lambda|\{p : p \equiv a \ (\mathrm{mod}\ q),\ x_{j-1} < p \le x_j\}| \\
&\qquad = \frac{x_{j-1}}{\phi(q)\log x_{j-1}} + O\biggl(\frac{x}{Q(\log x)^{A+1}}\biggr) + O\biggl(\frac{v}{\phi(q)\log x}\biggr)
\end{aligned}$$

by (6.1) and the condition $q \in S_f(Q) \setminus G_j$. After an application of the mean value theorem, we obtain

$$|\{p : p \equiv a \ (\mathrm{mod}\ q), p \le y\}| = \frac{y}{\phi(q)\log x} + O\biggl(\frac{v}{\phi(q)\log x}\biggr).$$

We have established that, for $q \in S_f(Q) \setminus G$,

$$\max_{1 \le y \le x} E(y,q) \ll \frac{v}{\phi(q)\log x},$$

and so

$$\sum_{q \in S_f(Q)\setminus G} \max_{1\le y\le x} E(y,q) \ll \frac{v}{\log x} \sum_{q\in S_f(Q)} \frac{1}{\phi(q)} \ll \frac{v|S_f(Q)|\log\log x}{Q\log x} \ll \frac{x|S_f(Q)|}{Q(\log x)^A}. \tag{6.5}$$

On the other hand, for $q \in G$,

$$\max_{1\le y\le x} E(y,q) \ll \frac{x}{\phi(q)\log x} \ll \frac{x\log\log x}{Q\log x}$$

from (5.1). Recalling (6.4), we get

$$\sum_{q\in G} \max_{1\le y\le x} E(y,q) \ll \frac{|G|x\log\log x}{Q\log x} \ll \frac{x|S_f(Q)|}{Q(\log x)^A}. \tag{6.6}$$

The maximal variant of Theorems 1 and 2 follows on combining (6.5), (6.6). The maximal variant of Theorem 3 is proved in similar fashion. ∎

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