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Congruences for Ramanujan's ϕ function

by

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1. Introduction. Ramanujan's famous congruences for the partition function, $p(n)$,

$$(1.1)\qquad \begin{aligned} p(5n+4) &\equiv 0 \pmod 5,\\ p(7n+5) &\equiv 0 \pmod 7,\\ p(11n+6) &\equiv 0 \pmod{11}, \end{aligned}$$

have been a source of inspiration for many. There has been much interest in generating functions with congruences. A short list includes the ranks, cranks, t-core, and M_2 ranks of the partition function, [4]–[6], [15], the overpartition function [18], [19], overpartition pairs [10]. In addition, we also mention that G. E. Andrews's smallest part partition function [1], $spt(n)$, satisfies Ramanujan type congruences for the primes 5, 7, and 13. In other words, when $p = 5, 7$, or 13, there is some integer l_p $(0 \le l_p < p)$ such that $spt(pn + l_p) \equiv 0 \pmod p$ for all positive integer n.

On page 3 of his lost notebook [23], Ramanujan defines the function

$$\phi(q) := \sum_{n=0}^{\infty} \frac{(-q;q)_{2n} q^{n+1}}{(q;q^2)_{n+1}^2},$$

and then states an identity involving $\phi(q^3)$ and a sixth order mock theta function, namely,

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-q;q)_n}{(q;q^2)_{n+1}} = 2q^{-1}\phi(q^3) + \frac{(q^2;q^2)_\infty^2(-q^3;q^3)_\infty}{(q;q^2)_\infty^2(q^3;q^3)_\infty}.$$

Y.-S. Choi [13] worked out the analogous identities involving ϕ and the other sixth order mock theta functions. The function ϕ was recently studied by

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K. Hikami [17]. The notations used above and in the rest of the article are

$$(x)_0 := (x;q)_0 := 1,$$
$$(x)_n := (x;q)_n := \prod_{k=0}^{n-1}(1-xq^k),$$
$$(x_1,\dots,x_m)_n := (x_1,\dots,x_m;q)_n := (x_1;q)_n\cdots(x_m;q)_n,$$

and we require $|q|<1$ for convergence. Whenever necessary, we use more compact notations introduced by D. Hickerson,

$$J_{a,b} := (q^a, q^{b-a}, q^b; q^b)_\infty, \qquad J_a := (q^a;q^a)_\infty.$$

The main objective of this article is to present and prove congruences for the coefficients of the function ϕ.

THEOREM 1.1. *Let $\sum_{n=1}^{\infty} a(n)q^n := \phi(q)$. For any nonnegative integer n, we have the following congruences:*

$$a(9n+4) \equiv 0 \pmod 2, \tag{1.2}$$
$$a(18n+10) \equiv 0 \pmod 4, \tag{1.3}$$
$$a(25n+14) \equiv a(25n+24) \equiv 0 \pmod 4, \tag{1.4}$$
$$a(3n+2) \equiv 0 \pmod 3, \tag{1.5}$$
$$a(18n+7) \equiv a(18n+13) \equiv 0 \pmod 3, \tag{1.6}$$
$$a(10n+9) \equiv 0 \pmod 5, \tag{1.7}$$
$$a(7n+3) \equiv a(7n+4) \equiv a(7n+6) \equiv 0 \pmod 7, \tag{1.8}$$
$$a(6n+5) \equiv 0 \pmod{27}. \tag{1.9}$$

In Section 2, we prove (1.2), (1.3), and (1.5). In Sections 3 and 4, we prove (1.4) and (1.8), respectively. The proofs in Sections 2–4 depend upon the method of A. O. L. Atkin and H. P. F. Swinnerton-Dyer [5]. In Section 5, we prove (1.6) and (1.7). Identity (1.14) below serves as the key to the proofs in Section 5. Using (1.14) together with several modular equations of degree three, we prove (1.9) in Section 6. In Sections 2 and 6, respectively, we also prove stronger results,

$$\sum_{n=0}^{\infty} a(3n+2)q^{3n+2} = 3q^2\frac{J_{18}^{15}}{J_{3,18}^9 J_{6,18}^2 J_{9,18}^3},$$
$$\sum_{n=0}^{\infty} a(6n+5)q^{3n+2} = \frac{27q^2 J_{18}^{19}}{J_{3,18}^9 J_{6,18}^4 J_9^5} + \frac{324q^5 J_{18}^{34}}{J_{3,18}^{14} J_{6,18}^8 J_9^{11}},$$

from which (1.5) and (1.9) follows. These are analogs to Ramanujan's partition identities such as

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}, \tag{1.10}$$

from which (1.1) follows immediately. (See [11] for an interesting proof of (1.10).)

The congruences satisfied by the function ϕ motivated a search for similar functions satisfying simple congruences. In addition to presenting conjectures on congruences for several related functions in the last section, we also prove the following theorem.

THEOREM 1.2. *For any integer $p \geq 2$ and $1 \leq j \leq p-1$ with p and j coprime, let*

$$\sum_{n=0}^{\infty} a_{j,p}(n)q^n = \frac{1}{(q^j, q^{p-j}, q^p; q^p)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{pn(n+1)/2+jn+j}}{1-q^{pn+j}}.$$

Then

$$\sum_{n=0}^{\infty} a_{j,p}(pn+(p-j)j)q^n = p\frac{(q^p;q^p)_\infty^4}{(q;q)^3(q^j,q^{p-j};q^p)_\infty^2}. \tag{1.11}$$

In particular,

$$\sum_{n=0}^{\infty} a_{j,p}(pn+(p-j)j)q^n \equiv 0 \pmod{p}. \tag{1.12}$$

Besides the appearance of the function ϕ on page 3 in Ramanujan's lost notebook [23], a rank type generating function,

$$\sum_{n=0}^{\infty} \frac{(-q;q)_{2n}q^{n+1}}{(xq;q^2)_{n+1}(q/x;q^2)_{n+1}}, \tag{1.13}$$

from which we recover $\phi(q)$ by setting $x=1$, also appears in several identities in Ramanujan's lost notebook. Detailed proofs of these identities can be found in [2, Chap. 12]. One such identity, [2, Entry 12.4.3],

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(-1)^n(q^2;q^4)_n q^{2n^2}}{(-xq^4;q^4)_n(-q^4/x;q^4)_n} \\ &\qquad + (1+x)(1+1/x)\sum_{n=0}^{\infty} \frac{(-q;q)_{2n}q^{n+1}}{(xq;q^2)_{n+1}(q/x;q^2)_{n+1}} \\ &= \frac{(-xq^2;q^4)_\infty(-q^2/x;q^4)_\infty(q^2;q^2)_\infty}{(q;q^2)_\infty^2(-xq^4;q^4)_\infty(-q^4/x;q^4)_\infty(xq;q^2)_\infty(q/x;q^2)_\infty}, \end{aligned} \tag{1.14}$$

gives a connection between ϕ and the generating function for the ranks of partitions without repeated odd parts. (See [21] for an interesting article on the M_2 rank differences for partitions without repeated odd parts.) Identity (1.14) is equivalent to [2, (12.4.7)],

$$\frac{(q^2;q^4)_\infty}{(q^4;q^4)_\infty}\sum_{n=-\infty}^{\infty}\frac{q^{4n^2+2n}}{1+xq^{4n}}-\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2}}{1-xq^{2n-1}}$$
$$=\frac{(-xq^2;q^4)_\infty(-q^2/x;q^4)_\infty(q^2;q^2)_\infty}{(q;q^2)_\infty^2(-x;q^4)_\infty(-q^4/x;q^4)_\infty(xq;q^2)_\infty(q/x;q^2)_\infty}.$$

Another identity, [2, (12.3.20)], gives a transformation formula for ϕ, relating it to a generalized Lambert series:

$$\left(1+\frac{1}{x}\right)\sum_{n=0}^{\infty}\frac{(-q;q)_{2n}q^{n+1}}{(xq;q^2)_{n+1}(q/x;q^2)_{n+1}} \tag{1.15}$$
$$=\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n\left(\frac{q^{(n+1)^2}}{1-xq^{2n+1}}+\frac{q^{(n+1)^2}}{x-q^{2n+1}}\right).$$

Substituting $x=1$ and dividing by 2 in (1.15), we obtain

$$\phi(q)=\frac{1}{2}\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(n+1)^2}}{1-q^{2n+1}}. \tag{1.16}$$

In Theorems 2.5 and 3.2 below, we present a 3-dissection and a 5-dissection of (1.13) with x replaced by a third root of unity and a fifth root of unity, respectively. Theorem 2.5 could then be used to give a combinatorial interpretation of congruence (1.5), but we do not discuss this in the article.

In our proofs, we often require the Jacobi triple product identity [7, pp. 33–36],

$$\sum_{n=-\infty}^{\infty}a^{n(n+1)/2}b^{n(n-1)/2}=(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{1.17}$$

Also, we require, from [7, p. 49],

$$\frac{(q;q)_\infty}{(-q;q)_\infty}=\frac{(q^9;q^9)_\infty}{(-q^9;q^9)_\infty}-2q(q^3,q^{15},q^{18};q^{18})_\infty \tag{1.18}$$

$$=\frac{(q^{25};q^{25})_\infty}{(-q^{25};q^{25})_\infty}-2q(q^{15},q^{35},q^{50};q^{50})_\infty+2q^4(q^5,q^{45},q^{50};q^{50})_\infty, \tag{1.19}$$

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}=(-q^3,-q^6,q^9;q^9)_\infty+q\frac{(q^{18};q^{18})_\infty}{(q^9;q^{18})_\infty}. \tag{1.20}$$

There is an equally easy version for the 7-dissection, namely,

$$\frac{(q;q)_\infty}{(-q;q)_\infty}=\frac{(q^{49};q^{49})_\infty}{(-q^{49};q^{49})_\infty}-2q(q^{35},q^{63},q^{98};q^{98})_\infty \\ +2q^4(q^{21},q^{77},q^{98};q^{98})_\infty-2q^9(q^7,q^{91},q^{98};q^{98})_\infty. \tag{1.21}$$

2. Proofs of (1.2), (1.3), and (1.5). First, we prove three lemmas required in our proofs.

LEMMA 2.1. *We have*

$$\frac{(-q;q)_\infty}{(q;q)_\infty}=\frac{J_{18}^{12}}{J_{3,18}^8J_{6,18}^4J_{9,18}}+\frac{2qJ_{18}^{12}}{J_{3,18}^7J_{6,18}^4J_{9,18}^2}+\frac{4q^2J_{18}^{12}}{J_{3,18}^6J_{6,18}^4J_{9,18}^3}. \tag{2.1}$$

Proof. It suffices to show that

$$\frac{(q;q)_\infty}{(-q;q)_\infty}\left(\frac{J_{18}^{12}}{J_{3,18}^8J_{6,18}^4J_{9,18}}+\frac{2qJ_{18}^{12}}{J_{3,18}^7J_{6,18}^4J_{9,18}^2}+\frac{4q^2J_{18}^{12}}{J_{3,18}^6J_{6,18}^4J_{9,18}^3}\right)=1. \tag{2.2}$$

Recalling (1.18), we have

$$\frac{(q;q)_\infty}{(-q;q)_\infty}=\frac{(q^9;q^9)_\infty}{(-q^9;q^9)_\infty}-2q(q^3,q^{15},q^{18};q^{18})_\infty=J_{9,18}-2qJ_{3,18}.$$

Therefore the left side of (2.2) is

$$\begin{aligned}(J_{9,18}-2qJ_{3,18})&\left(\frac{J_{18}^{12}}{J_{3,18}^8J_{6,18}^4J_{9,18}}+\frac{2qJ_{18}^{12}}{J_{3,18}^7J_{6,18}^4J_{9,18}^2}+\frac{4q^2J_{18}^{12}}{J_{3,18}^6J_{6,18}^4J_{9,18}^3}\right)\\
&=\frac{J_{18}^{12}}{J_{3,18}^8J_{6,18}^4}+\frac{2qJ_{18}^{12}}{J_{3,18}^7J_{6,18}^4J_{9,18}}+\frac{4q^2J_{18}^{12}}{J_{3,18}^6J_{6,18}^4J_{9,18}^2}\\
&\quad-\frac{2qJ_{18}^{12}}{J_{3,18}^7J_{6,18}^4J_{9,18}}-\frac{4q^2J_{18}^{12}}{J_{3,18}^6J_{6,18}^4J_{9,18}^2}-\frac{8q^3J_{18}^{12}}{J_{3,18}^5J_{6,18}^4J_{9,18}^3}\\
&=\frac{J_{18}^{12}}{J_{3,18}^8J_{6,18}^4}-\frac{8q^3J_{18}^{12}}{J_{3,18}^5J_{6,18}^4J_{9,18}^3}=\frac{J_{18}^{12}}{J_{3,18}^8J_{6,18}^4J_{9,18}^4}(J_{9,18}^4-8q^3J_{3,18}^3J_{9,18}).\end{aligned}$$

Therefore, it suffices to show that

$$J_{9,18}^4-8q^3J_{3,18}^3J_{9,18}=J_{3,18}^8J_{6,18}^4J_{9,18}^4.$$

This is equivalent to equation (3.2) of [24] with q replaced by q^3. ∎

LEMMA 2.2. *We have*

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+2n+1}}{1-q^{6n+3}}=\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+6n+3}}{1-q^{6n+3}} \tag{2.3}$$

$$=\frac{(q;q)_\infty(-q^9;q^9)_\infty}{(-q;q)_\infty(q^9;q^9)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{9n^2+18n+9}}{1-q^{18n+15}}+q\frac{(q^6;q^6)_\infty^2}{(q^3;q^6)_\infty^2}, \tag{2.4}$$

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+4n+2}}{1-q^{6n+3}} \tag{2.5}$$
$$= -\frac{(q;q)_\infty(-q^9;q^9)_\infty}{(-q;q)_\infty(q^9;q^9)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+18n+7}}{1-q^{18n+9}} + 2q^2 \frac{(q^6;q^6)_\infty^2}{(q^3;q^6)_\infty(q^9;q^{18})_\infty^3}.$$

Proof. Equation (2.3) is easily shown by replacing n with $-n-1$ in the summation index of the series on the left side of (2.3),

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+2n+1}}{1-q^{6n+3}} = \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1} q^{n^2}}{1-q^{-6n-3}} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+6n+3}}{1-q^{6n+3}}.$$

Next, we show (2.4). Splitting the series on the left side of (2.3) into three series according to the summation index n modulo 3, we find that

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+2n+1}}{1-q^{6n+3}} \tag{2.6}$$
$$= \sum_{\substack{n=-\infty \\ n\equiv 0 \,(\mathrm{mod}\, 3)}}^{\infty} \frac{(-1)^n q^{n^2+2n+1}}{1-q^{6n+3}} + \sum_{\substack{n=-\infty \\ n\equiv 1 \,(\mathrm{mod}\, 3)}}^{\infty} \frac{(-1)^n q^{n^2+2n+1}}{1-q^{6n+3}}$$
$$+ \sum_{\substack{n=-\infty \\ n\equiv 2 \,(\mathrm{mod}\, 3)}}^{\infty} \frac{(-1)^n q^{n^2+2n+1}}{1-q^{6n+3}}$$
$$= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+6n+1}}{1-q^{18n+3}} - \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+12n+4}}{1-q^{18n+9}}$$
$$+ \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+18n+9}}{1-q^{18n+15}}$$
$$=: S_0 - S_1 + S_2.$$

Next, we recall [22, (3.1)] with q, ζ^2, and z^2 replaced by q^9, ζ, and z, respectively,

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{9n^2+18n} \left[\frac{\zeta^{-n}}{1-zq^{18n}/\zeta} + \frac{\zeta^{n+2}}{1-z\zeta q^{18n}}\right] \tag{2.7}$$
$$= \frac{-2(\zeta^2, q^{18}/\zeta^2; q^{18})_\infty(-q^9;q^9)_\infty^2}{(-\zeta, -q^9/\zeta; q^9)_\infty(\zeta^{-1}, q^{18}\zeta; q^{18})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+18n}}{1-zq^{18n}}$$
$$+ \frac{(-z, -q^9/z; q^9)_\infty(\zeta^2, q^{18}/\zeta^2, \zeta, q^{18}/\zeta, q^{18}, q^{18}; q^{18})_\infty}{(-\zeta, -q^9/\zeta; q^9)_\infty(z\zeta, q^{18}/(z\zeta), z/\zeta, q^{18}\zeta/z, z, q^{18}/z; q^{18})_\infty}.$$

Replacing the summation index n by $n+1$ in the series S_1, and applying (2.7) with $z = q^{15}$ and $\zeta = q^{12}$, we arrive at

$$S_0 - S_1 = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+6n+1}}{1-q^{18n+3}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+30n+25}}{1-q^{18n+27}}$$
$$= -2q\frac{(-q^9;q^9)_\infty (q^3,q^{15},q^{18};q^{18})_\infty}{(q^9;q^9)_\infty} S_2 + q\frac{(q^6;q^6)_\infty^2}{(q^3;q^6)_\infty^2}.$$

Therefore

$$\text{(2.8)} \quad S_0 - S_1 + S_2$$
$$= \left\{ \frac{(q^9;q^9)_\infty}{(-q^9;q^9)_\infty} - 2q(q^3,q^{15},q^{18};q^{18})_\infty \right\} \frac{(-q^9;q^9)_\infty}{(q^9;q^9)_\infty} S_2 + q\frac{(q^6;q^6)_\infty^2}{(q^3;q^6)_\infty^2}$$
$$= \frac{(q;q)_\infty(-q^9;q^9)_\infty}{(-q;q)_\infty(q^9;q^9)_\infty} S_2 + q\frac{(q^6;q^6)_\infty^2}{(q^3;q^6)_\infty^2}$$

by applying (1.18) in the last equality. Substituting (2.8) into (2.6), we complete the proof of (2.4).

Similarly, by splitting the series on the left side of (2.5) into three series according to the summation index n modulo 3, we find that

$$\text{(2.9)} \quad \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+4n+2}}{1-q^{6n+3}}$$
$$= \sum_{\substack{n=-\infty \\ n\equiv 0 \,(\mathrm{mod}\, 3)}}^{\infty} \frac{(-1)^n q^{n^2+4n+2}}{1-q^{6n+3}} + \sum_{\substack{n=-\infty \\ n\equiv 1 \,(\mathrm{mod}\, 3)}}^{\infty} \frac{(-1)^n q^{n^2+4n+2}}{1-q^{6n+3}}$$
$$+ \sum_{\substack{n=-\infty \\ n\equiv 2 \,(\mathrm{mod}\, 3)}}^{\infty} \frac{(-1)^n q^{n^2+4n+2}}{1-q^{6n+3}}$$
$$= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+12n+2}}{1-q^{18n+3}} - \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+18n+7}}{1-q^{18n+9}}$$
$$+ \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+24n+14}}{1-q^{18n+15}}$$
$$=: S_3 - S_4 + S_5.$$

Applying (2.7) with $z = q^9$ and $\zeta = q^6$, we arrive at

$$S_3 + S_5 = 2q\frac{(-q^9;q^9)_\infty(q^3,q^{15},q^{18};q^{18})_\infty}{(q^9;q^9)_\infty} S_4 + 2q^2\frac{(q^6;q^6)_\infty^2}{(q^3;q^6)_\infty(q^9;q^{18})_\infty^3}.$$

Therefore,

$$\text{(2.10)} \quad S_3 - S_4 + S_5 = -\left\{ \frac{(q^9;q^9)_\infty}{(-q^9;q^9)_\infty} - 2q(q^3,q^{15},q^{18};q^{18})_\infty \right\} \frac{(-q^9;q^9)_\infty}{(q^9;q^9)_\infty} S_4$$
$$+ 2q^2\frac{(q^6;q^6)_\infty^2}{(q^3;q^6)_\infty(q^9;q^{18})_\infty^3}$$

$$= -\frac{(q;q)_\infty(-q^9;q^9)_\infty}{(-q;q)_\infty(q^9;q^9)_\infty}S_4 + 2q^2\frac{(q^6;q^6)_\infty^2}{(q^3;q^6)_\infty(q^9;q^{18})_\infty^3}$$

by applying (1.18) in the last equality. Substituting (2.10) into (2.9), we complete the proof of (2.5) ■

LEMMA 2.3. *We have the identity*

$$\phi(q) = \frac{(-q^9;q^9)_\infty}{(q^9;q^9)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{9n^2+18n+9}}{1-q^{18n+15}} + q\frac{(-q;q)_\infty(q^6;q^6)_\infty^2}{(q;q)_\infty(q^3;q^6)_\infty^2} - \frac{1}{2}\frac{(-q^9;q^9)_\infty}{(q^9;q^9)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{9n^2+18n+7}}{1-q^{18n+9}} + q^2\frac{(-q;q)_\infty(q^6;q^6)_\infty^2}{(q;q)_\infty(q^3;q^6)_\infty(q^9;q^{18})_\infty^3}. \tag{2.11}$$

Proof. First, note that

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(n+1)^2}}{1-q^{2n+1}} &= \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(n+1)^2}(1+q^{2n+1}+q^{4n+2})}{1-q^{6n+3}}\\ &= \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(n+1)^2}}{1-q^{6n+3}} + \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+4n+2}}{1-q^{6n+3}} + \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+6n+3}}{1-q^{6n+3}}\\ &= 2\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(n+1)^2}}{1-q^{6n+3}} + \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+4n+2}}{1-q^{6n+3}},\end{aligned}$$

where we applied (2.3) in the last equality.

Also, note that

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+4n+2}}{1-q^{6n+3}} + \sum_{n=-\infty}^{-1}\frac{(-1)^n q^{n^2+4n+2}}{1-q^{6n+3}}\\ &\quad= \sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+4n+2}}{1-q^{6n+3}} + \sum_{n=1}^{\infty}\frac{(-1)^n q^{n^2-4n+2}}{1-q^{-6n+3}}\\ &\quad= \sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+4n+2}}{1-q^{6n+3}} - \sum_{n=1}^{\infty}\frac{(-1)^n q^{n^2+2n-1}}{1-q^{6n-3}} = 2\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+4n+2}}{1-q^{6n+3}}\end{aligned}$$

is divisible by 2.

Recalling (1.16), we have

$$\begin{aligned}\phi(q) &= \frac{1}{2}\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(n+1)^2}}{1-q^{2n+1}}\\ &= \frac{(-q;q)_\infty}{(q;q)_\infty}\left(\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(n+1)^2}}{1-q^{6n+3}} + \frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+4n+2}}{1-q^{6n+3}}\right)\end{aligned}$$

$$= \frac{(-q^9;q^9)_\infty}{(q^9;q^9)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+18n+9}}{1-q^{18n+15}} + q\frac{(-q;q)_\infty (q^6;q^6)_\infty^2}{(q;q)_\infty (q^3;q^6)_\infty^2}$$
$$- \frac{1}{2} \frac{(-q^9;q^9)_\infty}{(q^9;q^9)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+18n+7}}{1-q^{18n+9}} + q^2 \frac{(-q;q)_\infty (q^6;q^6)_\infty^2}{(q;q)_\infty (q^3;q^6)_\infty (q^9;q^{18})_\infty^3},$$

where in the last equality, we applied (2.8) and (2.10). ∎

Proof of (1.2). Taking congruence modulo 2, we find that

$$\frac{(-q;q)_\infty}{(q;q)_\infty} \equiv \frac{(q;q)_\infty}{(q;q)_\infty} \pmod 2 = 1.$$

Therefore, taking congruences modulo 2 in (2.11) gives

$$\phi(q) \equiv \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+18n+9}}{1-q^{18n+15}} + q\frac{(q^6;q^6)_\infty^2}{(q^3;q^6)_\infty^2}$$
$$- \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+18n+7}}{1-q^{18n+9}} + q^2 \frac{(q^6;q^6)_\infty^2}{(q^3;q^6)_\infty (q^9;q^{18})_\infty^3} \pmod 2.$$

Note that the two series and the second product on the right side do not have terms with powers of q congruent to 4 modulo 9. For the first product on the right side, we examine

$$q\frac{(q^6;q^6)_\infty^2}{(q^3;q^6)_\infty^2} \equiv q\frac{(q^{12};q^{12})_\infty}{(q^6;q^{12})_\infty} \pmod 2 = \sum_{n=0}^{\infty} q^{3n(n+1)+1},$$

by applying the Jacobi triple product identity (1.17) with $(a,b) = (q^6, q^{18})$. Since $3n(n+1)+1$ is never congruent to 4 (mod 9), we conclude that the coefficient of q^n in $\phi(q)$ is even when $n \equiv 4 \pmod 9$. This completes the proof of (1.2). ∎

Proof of (1.3). Taking congruences modulo 4, we find that

$$\frac{(-q;q)_\infty}{(q;q)_\infty} = \frac{(q^2;q^2)_\infty}{(q;q)_\infty^2} = \frac{(q^2;q^2)_\infty}{(-q;q)_\infty^2} \pmod 4 = \frac{(q;q)_\infty}{(-q;q)_\infty}.$$

Therefore, taking congruences modulo 4 in (2.11) gives

$$\phi(q) \equiv \frac{(-q^9;q^9)_\infty}{(q^9;q^9)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+18n+9}}{1-q^{18n+15}} + q\frac{(q;q)_\infty (q^6;q^6)_\infty^2}{(-q;q)_\infty (q^3;q^6)_\infty^2}$$
$$- \frac{1}{2} \frac{(-q^9;q^9)_\infty}{(q^9;q^9)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n^2+18n+7}}{1-q^{18n+9}}$$
$$+ q^2 \frac{(q;q)_\infty (q^6;q^6)_\infty^2}{(-q;q)_\infty (q^3;q^6)_\infty (q^9;q^{18})_\infty^3} \pmod 4.$$

Clearly the two series on the right side do not have terms with powers of q congruent to 10 modulo 18. Therefore, it suffices to examine the two products. For the first product, we dissect each factor by applications of (1.18) and (1.20), respectively, to give

$$q\frac{(q;q)_\infty}{(-q;q)_\infty}\frac{(q^6;q^6)_\infty^2}{(q^3;q^6)_\infty^2} = \left(q\frac{(q^9;q^9)_\infty}{(-q^9;q^9)_\infty} - 2q^2(q^3,q^{15},q^{18};q^{18})_\infty\right) \times \left((-q^9,-q^{18},q^{27};q^{27})_\infty + q^3\frac{(q^{54};q^{54})_\infty}{(q^{27};q^{54})_\infty}\right)^2.$$

The only factor that contributes to terms with powers of q congruent to 1 modulo 9 is

$$\begin{aligned} q\frac{(q^9;q^9)_\infty}{(-q^9;q^9)_\infty} \times (-q^9,-q^{18},q^{27};q^{27})_\infty^2 &= q\frac{(q^9;q^9)_\infty}{(-q^9;q^9)_\infty}\frac{(-q^9;q^9)_\infty^2(q^{27};q^{27})_\infty^2}{(-q^{27};q^{27})_\infty^2} \\ &= q\frac{(q^{18};q^{18})_\infty(q^{27};q^{27})_\infty^4}{(q^{54};q^{54})_\infty^2} \\ &\equiv q(q^{18};q^{18})_\infty \pmod 4. \end{aligned}$$

Therefore the terms in this product with powers of q congruent to 10 modulo 18 are divisible by 4.

For the second product, we find that the factor

$$q^2\frac{(q^6;q^6)_\infty^2}{(q^3;q^6)_\infty(q^9;q^{18})_\infty^3}$$

only contributes to terms with powers of q congruent to 2 modulo 3, while the factor

$$\frac{(q;q)_\infty}{(-q;q)_\infty} = \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}$$

only contributes to terms with powers of q congruent to 0 or 1 modulo 3. Therefore the second product does not have any term with powers of q congruent to 10 modulo 18. This completes the proof of (1.3). ∎

First proof of (1.5). Examining (2.11), we find that the two series on the right side do not contain any terms with q^n where n is congruent to 2 modulo 3. For the two products on the right side, we invoke (2.1) to obtain 3-dissections of each product, namely,

$$\left(\frac{J_{18}^{12}}{J_{3,18}^8J_{6,18}^4J_{9,18}} + \frac{2qJ_{18}^{12}}{J_{3,18}^7J_{6,18}^4J_{9,18}^2} + \frac{4q^2J_{18}^{12}}{J_{3,18}^6J_{6,18}^4J_{9,18}^3}\right) \times \left(q\frac{(q^6;q^6)_\infty^2}{(q^3;q^6)_\infty^2} + q^2\frac{(q^6;q^6)_\infty^2}{(q^3;q^6)_\infty(q^9;q^{18})_\infty^3}\right)$$

$$= \left(\frac{J_{18}^{12}}{J_{3,18}^{8} J_{6,18}^{4} J_{9,18}} + \frac{2q J_{18}^{12}}{J_{3,18}^{7} J_{6,18}^{4} J_{9,18}^{2}} + \frac{4q^2 J_{18}^{12}}{J_{3,18}^{6} J_{6,18}^{4} J_{9,18}^{3}} \right) \times \left(q \frac{J_{6,18}^{2} J_{18}^{3}}{J_{3,18}^{2} J_{9,18}} + q^2 \frac{J_{6,18}^{2} J_{18}^{3}}{J_{3,18} J_{9,18}^{2}} \right).$$

Extracting only terms with q^n where $n \equiv 2 \pmod 3$, we arrive at the following corollary.

Corollary 2.4. *We have the identity*

$$\sum_{n=0}^{\infty} a(3n+2) q^{3n+2} = 3q^2 \frac{J_{18}^{15}}{J_{3,18}^{9} J_{6,18}^{2} J_{9,18}^{3}}.$$

Taking congruence modulo 3, we complete the proof of (1.5). ∎

From (1.15), we find that

$$\sum_{n=0}^{\infty} \frac{(-q;q)_{2n} q^{n+1}}{(xq;q^2)_{n+1} (q/x;q^2)_{n+1}} = \frac{x}{1+x} \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(n+1)^2}}{1 - xq^{2n+1}}.$$

Substituting $x = \omega$, a third root of unity, we find that

$$\sum_{n=0}^{\infty} \frac{(-q;q)_{2n} q^{n+1}}{(\omega q;q^2)_{n+1} (q/\omega;q^2)_{n+1}} = \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{-\omega^2 (-1)^n q^{(n+1)^2}}{1 - \omega q^{2n+1}}. \tag{2.12}$$

Rationalizing the denominator of the summand on the right side, we find that

$$\begin{aligned} r \sum_{n=-\infty}^{\infty} \frac{-\omega^2 (-1)^n q^{(n+1)^2}}{1 - \omega q^{2n+1}} \\ &= \sum_{n=-\infty}^{\infty} \frac{-\omega^2 (-1)^n q^{(n+1)^2} (1 + \omega q^{2n+1} + \omega^2 q^{4n+2})}{1 - q^{6n+3}} \\ &= \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1} q^{n^2+4n+2}}{1 - q^{6n+3}} + \omega \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1} q^{n^2+6n+3}}{1 - q^{6n+3}} \\ &\quad + (-1-\omega) \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1} q^{n^2+2n+1}}{1 - q^{6n+3}} \\ &= \sum_{n=-\infty}^{\infty} \frac{(-1)^{n} q^{n^2+2n+1}}{1 - q^{6n+3}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1} q^{n^2+4n+2}}{1 - q^{6n+3}} \end{aligned} \tag{2.13}$$

where in the last equality, we appealed to (2.3) for the second series.

With S_2 and S_4 defined in (2.6) and (2.9), respectively,

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} \frac{-\omega^2(-1)^n q^{(n+1)^2}}{1-\omega q^{2n+1}} \\
&\quad= \frac{(q;q)_\infty(-q^9;q^9)_\infty}{(-q;q)_\infty(q^9;q^9)_\infty}(S_2+S_4) + q\frac{(q^6;q^6)_\infty^2}{(q^3;q^6)_\infty^2} - 2q^2\frac{(q^6;q^6)_\infty^2}{(q^3;q^6)_\infty(q^9;q^{18})_\infty^3} \\
&\quad= \frac{(q;q)_\infty(-q^9;q^9)_\infty}{(-q;q)_\infty(q^9;q^9)_\infty}(S_2+S_4) \\
&\qquad+ q\frac{(q^6;q^6)_\infty^2(-q^9;q^9)_\infty}{(q^3;q^6)_\infty^2(q^9;q^9)_\infty}\left\{\frac{(q^9;q^9)_\infty}{(-q^9;q^9)_\infty} - 2q(q^3,q^{15},q^{18};q^{18})_\infty\right\} \\
&\quad= \frac{(q;q)_\infty(-q^9;q^9)_\infty}{(-q;q)_\infty(q^9;q^9)_\infty}(S_2+S_4) + q\frac{(q;q)_\infty(q^6;q^6)_\infty^2(-q^9;q^9)_\infty}{(-q;q)_\infty(q^3;q^6)_\infty^2(q^9;q^9)_\infty} \\
&\quad= \frac{(q;q)_\infty}{(-q;q)_\infty}\left\{\frac{(-q^9;q^9)_\infty}{(q^9;q^9)_\infty}(S_2+S_4) + q\frac{(q^6;q^6)_\infty^2(-q^9;q^9)_\infty}{(q^3;q^6)_\infty^2(q^9;q^9)_\infty}\right\},
\end{aligned}$$

where in the penultimate equality, we applied (1.18) to the products in the braces. The last equality follows from (2.6), (2.8), (2.9), and (2.10). Substituting this into the right side of (2.12) and simplifying, we arrive at the following rank difference type identity, similar to those in [15].

THEOREM 2.5. *Let ω be a third root of unity. Then*

$$\begin{aligned}
(2.14)\qquad &\sum_{n=0}^{\infty} \frac{(-q;q)_{2n}q^{n+1}}{(\omega q;q^2)_{n+1}(q/\omega;q^2)_{n+1}} \\
&\quad= q^9\frac{(-q^9;q^9)_\infty}{(q^9;q^9)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{9n^2+18n}}{1-q^{18n+15}} + q\frac{(q^{18};q^{18})_\infty(q^6;q^6)_\infty^4}{(q^3;q^3)_\infty^2(q^9;q^9)_\infty^2} \\
&\qquad+ q^7\frac{(-q^9;q^9)_\infty}{(q^9;q^9)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{9n^2+18n}}{1-q^{18n+9}}.
\end{aligned}$$

Second proof of (1.5). From (2.14), we see that the coefficient of q^n for $n \equiv 2 \pmod 3$ is zero. The argument given in [9, Sect. 8] allows us to deduce the congruence

$$\begin{aligned}
(2.15)\qquad &\sum_{n=0}^{\infty} \frac{(-q;q)_{2n}q^{n+1}}{(xq;q^2)_{n+1}(q/x;q^2)_{n+1}} \\
&\quad\equiv q^9\frac{(-q^9;q^9)_\infty}{(q^9;q^9)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{9n^2+18n}}{1-q^{18n+15}} + q\frac{(q^{18};q^{18})_\infty(q^6;q^6)_\infty^4}{(q^3;q^3)_\infty^2(q^9;q^9)_\infty^2} \\
&\qquad+ q^7\frac{(-q^9;q^9)_\infty}{(q^9;q^9)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{9n^2+18n}}{1-q^{18n+9}} \pmod{x+1+1/x}.
\end{aligned}$$

Setting $x = 1$ in the congruence above, we conclude that $a(3n+2) \equiv 0 \pmod 3$. This completes the proof. ∎

3. Proof of (1.4). We require the following lemma in our proof.

Lemma 3.1. *By dissecting the series, we have*

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+2n+1}}{1-q^{10n+5}} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+10n+5}}{1-q^{10n+5}} \tag{3.1}$$

$$= \frac{(q;q)_\infty(-q^{25};q^{25})_\infty}{(-q;q)_\infty(q^{25};q^{25})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+50n+25}}{1-q^{50n+45}} \tag{3.2}$$
$$+ q\frac{(q^{10},q^{40};q^{50})_\infty^3(q^{50};q^{50})_\infty^2}{(q^5,q^{45};q^{50})_\infty^3(q^{20},q^{30};q^{50})_\infty} - q^4\frac{(q^{10};q^{10})_\infty(q^{50};q^{50})_\infty}{(q^5;q^{10})_\infty(q^{25};q^{50})_\infty},$$

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+4n+2}}{1-q^{10n+5}} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+8n+4}}{1-q^{10n+5}} \tag{3.3}$$

$$= -\frac{(q;q)_\infty(-q^{25};q^{25})_\infty}{(-q;q)_\infty(q^{25};q^{25})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+50n+23}}{1-q^{50n+35}} \tag{3.4}$$
$$+ q^2\frac{(q^{20},q^{30};q^{50})_\infty^3(q^{50};q^{50})_\infty^2}{(q^{10},q^{40};q^{50})_\infty(q^{15},q^{35};q^{50})_\infty^3} q^4\frac{(q^{10};q^{10})_\infty(q^{50};q^{50})_\infty}{(q^5;q^{10})_\infty(q^{25};q^{50})_\infty},$$

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+6n+3}}{1-q^{10n+5}} \tag{3.5}$$
$$= \frac{(q;q)_\infty(-q^{25};q^{25})_\infty}{(-q;q)_\infty(q^{25};q^{25})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+50n+19}}{1-q^{50n+25}}$$
$$- 2q^{10}\frac{(q^{10},q^{40};q^{50})_\infty^2(q^{50};q^{50})_\infty^2}{(q^{15},q^{35};q^{50})_\infty(q^{25};q^{50})_\infty^4} + 2q^3\frac{(q^{20},q^{30};q^{50})_\infty^2(q^{50};q^{50})_\infty^2}{(q^5,q^{45};q^{50})_\infty(q^{25};q^{50})_\infty^4}.$$

We omit the proof of Lemma 3.1 and only briefly mention that the proof is similar to that of Lemma 2.2, and involves shifting of the summation indices, applications of (2.7) with q^9 replaced by q^{25}, and the Jacobi triple product identity (1.17).

Proof of (1.4). By an application of (1.19), we find that

$$\frac{(-q;q)_\infty}{(q;q)_\infty} \equiv \frac{(q;q)_\infty}{(-q;q)_\infty} \pmod 4 \tag{3.6}$$
$$= \frac{(q^{25};q^{25})_\infty}{(-q^{25};q^{25})_\infty} - 2q(q^{15},q^{35},q^{50};q^{50})_\infty + 2q^4(q^5,q^{45},q^{50};q^{50})_\infty.$$

From (1.16), we have

$$\begin{aligned}\phi(q) &= \frac{1}{2}\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+2n+1}(1+q^{2n+1}+q^{4n+2}+q^{6n+3}+q^{8n+4})}{1-q^{10n+5}}\\ &= \frac{1}{2}\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+2n+1}(2+2q^{2n+1}+q^{4n+2})}{1-q^{10n+5}},\end{aligned}$$

after invoking (3.1) and (3.3). Substituting (3.2)–(3.6), we find that the only products containing terms with powers of q congruent to 14 or 24 modulo 25 are

$$-2q(q^{15},q^{35},q^{50};q^{50})_\infty \times q^3\frac{(q^{20},q^{30};q^{50})^2_\infty(q^{50};q^{50})^2_\infty}{(q^5,q^{45};q^{50})_\infty(q^{25};q^{50})^4_\infty} =: P_1$$

and

$$-2q^4(q^5,q^{45},q^{50};q^{50})_\infty \times q^{10}\frac{(q^{10},q^{40};q^{50})^2_\infty(q^{50};q^{50})^2_\infty}{(q^{15},q^{35};q^{50})_\infty(q^{25},;q^{50})^4_\infty} =: P_2.$$

Note that

$$\begin{aligned}P_1+P_2 &\equiv -P_1+P_2 \pmod 4\\ &= 2q^4\frac{(q^{50};q^{50})^3_\infty}{(q^5;q^{10})_\infty(q^{25};q^{50})^3_\infty}\\ &\quad\times\{(q^{15},q^{20},q^{30},q^{35};q^{50})^2_\infty - q^{10}(q^5,q^{10},q^{40},q^{45};q^{50})^2_\infty\}\\ &= 2q^4\frac{(q^{50};q^{50})^3_\infty}{(q^5;q^{10})_\infty(q^{25};q^{50})^3_\infty}(q^{10},q^{20},q^{30},q^{40};q^{50})_\infty(q^{25};q^{50})^4_\infty\\ &= 2q^4(q^{25};q^{25})_\infty(q^{50};q^{50})_\infty\frac{(q^{10};q^{10})_\infty}{(q^5;q^{10})_\infty},\end{aligned}$$

where in the second equality, we invoked [14, Theorem 1.1],

$$\begin{aligned}(3.7)\qquad &(A/b,qb/A,A/c,qc/A,A/d,qd/A,A/e,qe/A;q)_\infty\\ &\qquad - (b,q/b,c,q/c,d,q/d,e,q/e;q)_\infty\\ &= b(A,q/A,A/bc,qbc/A,A/bd,qbd/A,A/be,qbe/A;q)_\infty\end{aligned}$$

with q replaced by q^{50} and $(A,b,c,d,e) = (q^{45},q^{10},q^{30},q^{25},q^{25})$. By (1.17),

$$\frac{(q^{10};q^{10})_\infty}{(q^5;q^{10})_\infty} = \sum_{n=0}^{\infty} q^{5n(n+1)/2},$$

hence does not contain terms q^n where $n \equiv 10, 20 \pmod{25}$. This shows that the coefficient of q^n for $n \equiv 14, 24 \pmod{25}$ is divisible by 4. This completes the proof of (1.4). ∎

We state without proof the following 5-dissection modulo 5 of the rank type function for ϕ. As this dissection does not seem to lead to any congruence modulo 5, it is nonessential in the context of this article, and so we omit the proof. We briefly mention that the proof follows from Lemma 3.1 and various theta function identities.

THEOREM 3.2. *Let ζ be a fifth root of unity. Then*

$$\sum_{n=0}^{\infty} \frac{(-q;q)_{2n} q^{n+1}}{(\zeta q;q^2)_{n+1}(q/\zeta;q^2)_{n+1}} = \frac{1}{J_{1,2}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(n+1)^2}(1-q^{2n+1})}{1-q^{10n+5}} + \left(\zeta + \frac{1}{\zeta}\right) \frac{1}{J_{1,2}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(n+1)^2}(q^{2n+1}-q^{4n+2})}{1-q^{10n+5}}.$$

The two series on the right have the following 5*-dissections:*

$$\begin{aligned}
&\frac{1}{J_{1,2}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(n+1)^2}(1-q^{2n+1})}{1-q^{10n+5}} \\
&\qquad = 2\frac{q^5 J_{10,50} J_{50}^{15}}{J_{5,50}^6 J_{15,50}^4 J_{20,50}^3 J_{25,50}^2} + q^5 \frac{1}{J_{25,50}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+50n}}{1-q^{50n+5}} \\
&\qquad\quad + \frac{q J_{50}^{15}}{J_{5,50}^5 J_{10,50}^2 J_{15,50}^6 J_{25,50}} + \frac{q^2 J_{50}^{15}}{J_{5,50}^6 J_{15,50}^5 J_{20,50}^2 J_{25,50}} \\
&\qquad\quad + 2\frac{q^3 J_{20,50} J_{50}^{15}}{J_{5,50}^4 J_{10,50}^3 J_{15,50}^6 J_{25,50}^2} + q^{13} \frac{1}{J_{25,50}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+50n}}{1-q^{50n+15}} \\
&\qquad\quad + 2\frac{q^4 J_{50}^{15}}{J_{5,50}^5 J_{10,50} J_{15,50}^5 J_{20,50} J_{25,50}^2}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{J_{1,2}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(n+1)^2}(q^{2n+1}-q^{4n+2})}{1-q^{10n+5}} \\
&= 2\frac{q^5 J_{50}^{15}}{J_{5,50}^4 J_{10,50}^2 J_{15,50}^6 J_{25,50}^2} + 2\frac{q^6 J_{50}^{15}}{J_{5,50}^5 J_{15,50}^5 J_{20,50}^2 J_{25,50}^2} + \frac{q^2 J_{20,50} J_{50}^{15}}{J_{5,50}^4 J_{10,50}^3 J_{15,50}^7 J_{25,50}} \\
&\quad + 4\frac{q^8 J_{50}^{15}}{J_{5,50}^4 J_{10,50} J_{15,50}^5 J_{20,50} J_{25,50}^3} - q^{13} \frac{1}{J_{25,50}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+50n}}{1-q^{50n+15}} \\
&\quad + 4\frac{q^{14} J_{50}^{15}}{J_{5,50}^3 J_{15,50}^5 J_{20,50}^2 J_{25,50}^4} + \frac{q^4 J_{50}^{15}}{J_{5,50}^5 J_{10,50} J_{15,50}^5 J_{20,50} J_{25,50}^2} \\
&\quad - q^{19} \frac{1}{J_{25,50}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+50n}}{1-q^{50n+25}}.
\end{aligned}$$

4. Proof of (1.8). In this section, we prove the following theorem, from which (1.8) follows.

THEOREM 4.1. *We have*

$$2\phi(q) = \frac{1}{J_{49,98}}\Big\{2\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{49n^2+98n+49}}{1-q^{98n+91}} - 2\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{49n^2+98n+47}}{1-q^{98n+77}}$$
$$+ 2\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{49n^2+98n+43}}{1-q^{98n+63}} - \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{49n^2+98n+37}}{1-q^{98n+49}}\Big\}$$
$$+ 7q^2\frac{J_{14}^4}{J_{1,2}J_7^2} + 2q\frac{J_{14}J_{28,98}J_{42,98}}{J_7J_{49}} - q^2\frac{J_{14}^4}{J_7^2J_{49,98}} - 2q^5\frac{J_{14}J_{14,98}J_{42,98}}{J_7J_{49}}$$
$$- 2q^7\frac{J_{14}J_{14,98}J_{28,98}}{J_7J_{49}}.$$

Therefore the congruence (1.8) *is true, that is,*

$$a(7n+3) \equiv a(7n+4) \equiv a(7n+6) \equiv 0 \pmod 7. \tag{4.1}$$

Noting that from (1.16), we have

$$\begin{aligned}2J_{1,2}\phi(q) &= \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(n+1)^2}}{1-q^{14n+7}}(1+q^{2n+1}+q^{4n+2}+q^{6n+3}+q^{8n+4}+q^{10n+5}+q^{12n+6})\\ &= \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(n+1)^2}}{1-q^{14n+7}}(2+2q^{2n+1}+2q^{4n+2}+q^{6n+3})\end{aligned}$$

after replacing the summation index n by $-n-1$ in the last three series. The theorem follows immediately from the next two lemmas.

LEMMA 4.2. *Let*

$$P_0 := q\frac{J_{14,98}^3J_{35,98}J_{98}^3}{J_{7,98}^3J_{21,98}J_{42,98}} + q^9\frac{J_{14,98}J_{42,98}J_{98}^3}{J_{7,98}J_{35,98}J_{49,98}} - q^4\frac{J_{14,98}J_{28,98}^2J_{98}^3}{J_{7,98}^2J_{35,98}J_{42,98}},$$
$$P_1 := q^2\frac{J_{28,98}J_{42,98}J_{98}^3}{J_{7,98}J_{21,98}J_{49,98}} - q^7\frac{J_{7,98}J_{42,98}^3J_{98}^3}{J_{21,98}^3J_{28,98}J_{35,98}} + q^6\frac{J_{14,98}^2J_{42,98}J_{98}^3}{J_{7,98}J_{21,98}^2J_{28,98}},$$
$$P_2 := q^3\frac{J_{28,98}J_{42,98}^2J_{98}^3}{J_{14,98}J_{21,98}J_{35,98}^2} + q^5\frac{J_{21,98}J_{28,98}^3J_{98}^3}{J_{7,98}J_{14,98}J_{35,98}^3} - q^{16}\frac{J_{14,98}J_{28,98}J_{98}^3}{J_{21,98}J_{35,98}J_{49,98}},$$
$$P_3 := 2q^4\frac{J_{42,98}^2J_{98}^3}{J_{7,98}J_{49,98}^2} - 2q^{13}\frac{J_{28,98}^2J_{98}^3}{J_{21,98}J_{49,98}^2} + 2q^{24}\frac{J_{14,98}^2J_{98}^3}{J_{35,98}J_{49,98}^2}.$$

Then

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(n+1)^2}}{1-q^{14n+7}} = P_0 + \frac{J_{1,2}}{J_{49,98}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{49n^2+98n+49}}{1-q^{98n+91}},$$

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(n+1)^2+2n+1}}{1-q^{14n+7}}=P_1-\frac{J_{1,2}}{J_{49,98}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{49n^2+98n+47}}{1-q^{98n+77}},$$
$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(n+1)^2+4n+2}}{1-q^{14n+7}}=P_2+\frac{J_{1,2}}{J_{49,98}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{49n^2+98n+43}}{1-q^{98n+63}},$$
$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(n+1)^2+6n+3}}{1-q^{14n+7}}=P_3-\frac{J_{1,2}}{J_{49,98}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{49n^2+98n+37}}{1-q^{98n+49}}.$$

Sketch of proof. The proof is tedious but routine, and it is similar to that of Lemma 2.2. We split each series on the left side according to the summation index modulo 7. After shifting the summation indices where necessary, we apply (2.7) with q^{49} three times in each identity to obtain the products in P_0, P_1, P_2, P_3, respectively. Application of (1.21) then gives the factor $1/J_{49,98}$ for each of the series on the right side. ∎

Lemma 4.3.

$$2P_0+2P_1+2P_2+P_3=J_{1,2}\Big\{7q^2\frac{J_{14}^4}{J_{1,2}J_7^2}+2q\frac{J_{14}J_{28,98}J_{42,98}}{J_7J_{49}} -q^2\frac{J_{14}^4}{J_7^2J_{49,98}}-2q^5\frac{J_{14}J_{14,98}J_{42,98}}{J_7J_{49}}-2q^7\frac{J_{14}J_{14,98}J_{28,98}}{J_7J_{49}}\Big\}. \tag{4.2}$$

To prove Lemma 4.3, we require three identities below, (4.5)–(4.7), which are corollaries from Halphen's identity

$$\begin{aligned} H(a,b,c,q) &:= \frac{(ab,q/(ab),bc,q/(bc),ca,q/(ca);q)_\infty(q;q)_\infty^2}{(a,q/a,b,q/b,c,q/c,abc,q/(abc);q)_\infty} \\ &= 1+F(a,q)+F(b,q)+F(c,q)-F(abc,q), \end{aligned} \tag{4.3}$$

where

$$F(x):=F(x,q):=\sum_{k=0}^{\infty}\frac{xq^k}{1-xq^k}-\sum_{k=1}^{\infty}\frac{q^k/x}{1-q^k/x}\qquad(|q|<1). \tag{4.4}$$

As a side note, we mention that Halphen's identity in the form above was first discovered and presented in [3]. However, equivalent versions of (4.3) in terms of Weierstrassian elliptic functions appeared in the literature much earlier, in Whittaker and Watson [26, Examples 19 and 20, p. 458] and in G. H. Halphen [16, p. 187].

Corollary 4.4. *We have*

$$H(a,b,c,q^{98})-H(a,b,d,q^{98})=H(c,1/d,abd,q^{98}), \tag{4.5}$$
$$H(a,a,q^{49}/a,q^{98})+H(b,b,q^{49}/b,q^{98})=2H(a,q^{49}/a,b,q^{98}), \tag{4.6}$$
$$H(a,a,q^{49}/a,q^{98})-H(b,b,q^{49}/b,q^{98})=2H(a,q^{49}/a,b/q^{49},q^{98}). \tag{4.7}$$

Proof. From (4.4), we find that

$$1+F(x,q)=-F(1/x,q)=F(xq,q). \tag{4.8}$$

Therefore, by (4.3),

$$\begin{aligned}
&H(a,b,c,q)-H(a,b,d,q)\\
&=1+F(a)+F(b)+F(c)-F(abc)-1-F(a)-F(b)-F(d)+F(abd)\\
&=F(c)-F(d)+F(abd)-F(abc)\\
&=1+F(c)+F(1/d)+F(abd)-F(abc)=H(c,1/d,abd,q),
\end{aligned}$$

where we applied (4.8) with $x=d$ in the penultimate equality. Replacing q by q^{98}, we obtain (4.5).

Similarly,

$$\begin{aligned}
&H(a,a,q/a,q^2)+H(b,b,q/b,q^2)\\
&=2+2F(a,q^2)+F(q/a,q^2)-F(aq,q^2)+2F(b,q^2)+F(q/b,q^2)-F(bq,q^2)\\
&=2+2F(a,q^2)+2F(q/a,q^2)+2F(b,q^2)-2F(bq,q^2)=2H(a,q/a,b,q^2),
\end{aligned}$$

where we applied (4.8) twice in the second equality. Replacing q by q^{49}, we obtain (4.6).

Finally,

$$\begin{aligned}
&H(a,a,q/a,q^2)-H(b,b,q/b,q^2)\\
&=2F(a,q^2)+F(q/a,q^2)-F(aq,q^2)-2F(b,q^2)-F(q/b,q^2)+F(bq,q^2)\\
&=2+2F(a,q^2)+2F(q/a,q^2)+2F(b/q)-2F(b,q^2)=2H(a,q/a,b/q,q^2),
\end{aligned}$$

where we applied (4.8) twice in the second equality. Replacing q by q^{49}, we obtain (4.7). ∎

Proof of Lemma 4.3. Noting that

$$J_{1,2}=J_{49,98}-2qJ_{35,98}+2q^4J_{21,98}-2q^9J_{7,98},$$

expanding the right side and comparing both sides according to the powers of q modulo 7, we find that it suffices to prove the following seven identities:

$$2q\frac{J_{14,98}^3J_{35,98}J_{98}^3}{J_{7,98}^3J_{21,98}J_{42,98}}=2q\frac{J_{14}J_{28,98}J_{42,98}J_{49,98}}{J_7J_{49}}+4q^8\frac{J_{14}J_{14,98}J_{28,98}J_{35,98}}{J_7J_{49}}, \tag{4.9}$$

$$\begin{aligned}
&2q^2\frac{J_{28,98}J_{42,98}J_{98}^3}{J_{7,98}J_{21,98}J_{49,98}}+2q^9\frac{J_{14,98}J_{42,98}J_{98}^3}{J_{7,98}J_{35,98}J_{49,98}}-2q^{16}\frac{J_{14,98}J_{28,98}J_{98}^3}{J_{21,98}J_{35,98}J_{49,98}}\\
&\qquad=7q^2\frac{J_{14}^4}{J_7^2}-4q^2\frac{J_{14}J_{28,98}J_{35,98}J_{42,98}}{J_7J_{49}}-q^2\frac{J_{14}^4J_{49,98}}{J_7^2J_{49,98}}\\
&\qquad\quad-4q^9\frac{J_{14}J_{14,98}J_{21,98}J_{42,98}}{J_7J_{49}}+4q^{16}\frac{J_{14}J_{7,98}J_{14,98}J_{28,98}}{J_7J_{49}},
\end{aligned} \tag{4.10}$$

$$(4.11)\quad 2q^3\frac{J_{28,98}J_{42,98}^2J_{98}^3}{J_{14,98}J_{21,98}J_{35,98}^2}+2q^{24}\frac{J_{14,98}^2J_{98}^3}{J_{35,98}J_{49,98}^2} = 2q^3\frac{J_{14}^4J_{35,98}}{J_7^2J_{49,98}}-4q^{10}\frac{J_{14}J_{7,98}J_{28,98}J_{42,98}}{J_7J_{49}},$$

$$(4.12)\quad 2q^4\frac{J_{42,98}^2J_{98}^3}{J_{7,98}J_{49,98}^2}-2q^4\frac{J_{14,98}J_{28,98}^2J_{98}^3}{J_{7,98}^2J_{35,98}J_{42,98}} = -4q^{11}\frac{J_{14}J_{14,98}J_{21,98}J_{28,98}}{J_7J_{49}}+2q^{11}\frac{J_{14}^4J_{7,98}}{J_7^2J_{49,98}},$$

$$(4.13)\quad 2q^5\frac{J_{21,98}J_{28,98}^3J_{98}^3}{J_{7,98}J_{14,98}J_{35,98}^3} = 4q^5\frac{J_{14}J_{21,98}J_{28,98}J_{42,98}}{J_7J_{49}}-2q^5\frac{J_{14}J_{14,98}J_{42,98}J_{49,98}}{J_7J_{49}},$$

$$(4.14)\quad 2q^6\frac{J_{14,98}^2J_{42,98}J_{98}^3}{J_{7,98}J_{21,98}^2J_{28,98}}-2q^{13}\frac{J_{28,98}^2J_{98}^3}{J_{21,98}J_{49,98}^2} = 4q^6\frac{J_{14}J_{14,98}J_{35,98}J_{42,98}}{J_7J_{49}}-2q^6\frac{J_{14}^4J_{21,98}}{J_7^2J_{49,98}},$$

$$(4.15)\quad -2q^7\frac{J_{7,98}J_{42,98}^3J_{98}^3}{J_{21,98}^3J_{28,98}J_{35,98}} = -2q^7\frac{J_{14}J_{14,98}J_{28,98}J_{49,98}}{J_7J_{49}}+4q^{14}\frac{J_{14}J_{7,98}J_{14,98}J_{42,98}}{J_7J_{49}}.$$

Simplifying each of these seven identities, we see that to prove (4.9), it suffices to prove

$$(4.16)\quad \frac{J_{14,98}J_{49,98}^2J_{98}^3}{J_{7,98}^2J_{42,98}^2}=\frac{J_{28,98}J_{49,98}^2J_{98}^3}{J_{14,98}^2J_{35,98}^2}-2\frac{J_{7,98}J_{28,98}J_{49,98}J_{98}^3}{J_{14,98}J_{35,98}J_{-7,98}J_{42,98}},$$

and this follows from (4.6) with $(a,b)=(q^{14},q^{42})$.

To prove (4.10), it suffices to show that

$$\frac{J_{35,98}J_{98}^2}{J_{14,98}J_{49}}+q^7\frac{J_{21,98}J_{98}^2}{J_{28,98}J_{49}}-q^{14}\frac{J_{7,98}J_{98}^2}{J_{42,98}J_{49}}=\frac{J_{14}^2}{J_7},$$

which is equivalent to

$$(4.17)\quad \frac{J_{35,98}}{J_{14,98}}+q^7\frac{J_{21,98}}{J_{28,98}}-q^{14}\frac{J_{7,98}}{J_{42,98}}=\frac{J_{14,98}J_{28,98}J_{42,98}}{J_{7,98}J_{21,98}J_{35,98}}.$$

From (3.7) with q replaced by q^{98}, by setting $(A,b,c,d,e)=(q^{56},q^7,q^{28},q^{35},q^{42})$ and $(A,b,c,d,e)=(q^{63},q^{14},q^{28},q^{42},q^{42})$, respectively, we obtain

$$J_{7,98}J_{21,98}J_{28,98}J_{35,98}^2J_{42,98}+q^7J_{7,98}J_{14,98}J_{21,98}^2J_{35,98}J_{42,98} = J_{14,98}J_{21,98}^2J_{28,98}J_{35,98}J_{49,98}$$

and

$$J_{14,98}^2 J_{28,98}^2 J_{42,98}^2 + q^{14} J_{7,98}^2 J_{14,98} J_{21,98} J_{28,98} J_{35,98} = J_{14,98} J_{21,98}^2 J_{28,98} J_{35,98} J_{49,98}.$$

These two identities imply (4.17).

To prove (4.11), it suffices to show that

$$\frac{J_{49,98}^2 J_{28,98}}{J_{14,98}^2 J_{35,98}^2} - \frac{J_{21,98}^2 J_{14,98}}{J_{-21,98} J_{42,98}^2 J_{35,98}} = \frac{J_{14,98} J_{28,98}^2}{J_{7,98}^2 J_{21,98} J_{35,98}} + 2\frac{J_{7,98} J_{28,98} J_{49,98}}{J_{-7,98} J_{14,98} J_{35,98} J_{42,98}},$$

and this follows from (4.16) and

$$\frac{J_{14,98} J_{49,98}^2}{J_{7,98}^2 J_{42,98}^2} = \frac{J_{14,98} J_{28,98}^2}{J_{7,98}^2 J_{21,98} J_{35,98}} + \frac{J_{21,98}^2 J_{14,98}}{J_{-21,98} J_{42,98}^2 J_{35,98}}.$$

This last identity follows from (4.5) with $(a,b,c,d) = (q^7, q^7, q^{42}, q^{21})$.

To prove (4.12), it suffices to show that

$$\frac{J_{42,98} J_{35,98} J_{98}^3}{J_{7,98} J_{28,98}^2} - \frac{J_{14,98} J_{49,98}^2 J_{98}^3}{J_{7,98}^2 J_{42,98}^2} = -2q^7 \frac{J_{14,98} J_{49,98} J_{98}^3}{J_{7,98} J_{28,98} J_{42,98}} + q^7 \frac{J_{14,98}^2 J_{42,98} J_{98}^3}{J_{7,98} J_{21,98}^2 J_{35,98}}.$$

This follows from the two identities

$$\frac{J_{42,98} J_{49,98}^2 J_{98}^3}{J_{21,98}^2 J_{28,98}^2} - \frac{J_{42,98} J_{35,98}^2 J_{98}^3}{J_{7,98} J_{28,98}^2 J_{35,98}} = \frac{J_{14,98}^2 J_{42,98} J_{98}^3}{J_{-7,98} J_{21,98}^2 J_{35,98}},$$

$$\frac{J_{42,98} J_{49,98}^2 J_{98}^3}{J_{21,98}^2 J_{28,98}^2} - \frac{J_{14,98} J_{49,98}^2 J_{98}^3}{J_{7,98}^2 J_{42,98}^2} = 2\frac{J_{14,98} J_{21,98} J_{49,98} J_{98}^3}{J_{-7,98} J_{21,98} J_{28,98} J_{42,98}},$$

which are equivalent to (4.5) with $(a,b,c,d) = (q^{28}, q^{28}, q^{21}, q^7)$ and (4.7) with $(a,b) = (q^{21}, q^{42})$, respectively.

To prove (4.13), it suffices to show that

$$\frac{J_{28,98} J_{49,98}^2 J_{98}^3}{J_{14,98}^2 J_{35,98}^2} = 2\frac{J_{35,98} J_{42,98} J_{49,98} J_{98}^3}{J_{14,98} J_{21,98} J_{28,98} J_{35,98}} - \frac{J_{42,98} J_{49,98}^2 J_{98}^3}{J_{21,98}^2 J_{28,98}^2},$$

and this follows from (4.6) with $(a,b) = (q^{14}, q^{21})$.

To prove (4.14), it suffices to show that

$$\frac{J_{42,98} J_{49,98}^2 J_{98}^3}{J_{21,98}^2 J_{28,98}^2} - q^7 \frac{J_{7,98} J_{28,98} J_{98}^3}{J_{14,98}^2 J_{21,98}} = 2\frac{J_{42,98} J_{49,98} J_{98}^3}{J_{14,98} J_{21,98} J_{28,98}} - \frac{J_{28,98} J_{42,98}^2 J_{98}^3}{J_{7,98} J_{21,98} J_{35,98}^2}.$$

This follows from the two identities

$$\frac{J_{42,98}J_{49,98}^2J_{98}^3}{J_{21,98}^2J_{28,98}^2}+\frac{J_{28,98}J_{49,98}^2J_{98}^3}{J_{14,98}^2J_{35,98}^2}=2\frac{J_{35,98}J_{42,98}J_{49,98}J_{98}^3}{J_{14,98}J_{21,98}J_{28,98}J_{35,98}},$$

$$\frac{J_{28,98}J_{49,98}^2J_{98}^3}{J_{14,98}^2J_{35,98}^2}-\frac{J_{7,98}^2J_{28,98}J_{98}^3}{J_{-7,98}J_{14,98}^2J_{21,98}}=\frac{J_{28,98}J_{42,98}^2J_{98}^3}{J_{7,98}J_{21,98}J_{35,98}^2},$$

which are equivalent to (4.6) with $(a,b)=(q^{21},q^{14})$ and (4.5) with (a,b,c,d) $=(q^{14},q^{14},q^{35},q^{-7})$, respectively.

To prove (4.15), it suffices to show that

$$\frac{J_{42,98}J_{49,98}^2J_{98}^3}{J_{21,98}^2J_{28,98}^2}=\frac{J_{14,98}J_{49,98}^2J_{98}^3}{J_{7,98}^2J_{42,98}^2}+2\frac{J_{14,98}J_{21,98}J_{49,98}J_{98}^3}{J_{28,98}J_{21,98}J_{-7,98}J_{42,98}},$$

and this follows from replacing q by q^{49} and setting $(a,b)=(q^{21},q^{42})$ in (4.7).

This completes the proof of the lemma. ■

5. Proofs of (1.6)–(1.7). First we prove the following lemma which gives a simple representation for the odd coefficients of ϕ.

LEMMA 5.1. *The coefficients of the odd powers of ϕ have the generating function*

$$\sum_{n=0}^{\infty}a(2n+1)q^n=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty^7}. \tag{5.1}$$

Note this is the case $(j,p)=(1,2)$ of Theorem 1.2 since $\sum_{n=0}^{\infty}a_{1,2}(n)q^n$ $=2\phi(q)$. Here we present a different proof than the general proof of Theorem 1.2 in Section 7.

Proof. First, we note that from [7, Entry 25(v), p. 40], we have

$$\frac{(-q;q^2)_\infty^2(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2(-q^2;q^2)_\infty^2}-\frac{(q;q^2)_\infty^2(q^2;q^2)_\infty^2}{(-q;q^2)_\infty^2(-q^2;q^2)_\infty^2} \tag{5.2}$$
$$=\varphi^2(q)-\varphi^2(-q)=8q\psi^2(q^4)=8q\frac{(q^8;q^8)_\infty^2}{(q^4;q^8)_\infty^2}.$$

Setting $x=1$ in the right side of (1.14), we have

$$\sum_{n=0}^{\infty}\frac{(-1)^n(q^2;q^4)_nq^{2n^2}}{(-q^4;q^4)_n^2}+4\sum_{n=0}^{\infty}\frac{(-q;q)_{2n}q^{n+1}}{(q;q^2)_{n+1}^2}=\frac{(-q^2;q^4)_\infty^2(q^2;q^2)_\infty}{(q;q^2)_\infty^4(-q^4;q^4)_\infty^2}.$$

This gives, upon rearrangement,

$$4\phi(q)=\frac{(-q^2;q^4)_\infty^2(q^2;q^2)_\infty}{(q;q^2)_\infty^4(-q^4;q^4)_\infty^2}-\sum_{n=0}^{\infty}\frac{(-1)^n(q^2;q^4)_nq^{2n^2}}{(-q^4;q^4)_n^2}. \tag{5.3}$$

Noting that the series on the right side contains only terms with even powers of q, we find that

$$\begin{aligned}4\phi(q)-4\phi(-q)&=\frac{(-q^2;q^4)_\infty^2(q^2;q^2)_\infty}{(q;q^2)_\infty^4(-q^4;q^4)_\infty^2}-\frac{(-q^2;q^4)_\infty^2(q^2;q^2)_\infty}{(-q;q^2)_\infty^4(-q^4;q^4)_\infty^2}\\&=\frac{(-q^2;q^4)_\infty^2(-q^2;q^2)_\infty^2}{(-q^4;q^4)_\infty^2(q^2;q^4)_\infty^2(q^2;q^2)_\infty}\\&\quad\times\left\{\frac{(-q;q^2)_\infty^2(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2(-q^2;q^2)_\infty^2}-\frac{(q;q^2)_\infty^2(q^2;q^2)_\infty^2}{(-q;q^2)_\infty^2(-q^2;q^2)_\infty^2}\right\}\\&=8q\frac{(-q^2;q^4)_\infty^2(-q^2;q^2)_\infty^2}{(-q^4;q^4)_\infty^2(q^2;q^4)_\infty^2(q^2;q^2)_\infty}\frac{(q^8;q^8)_\infty^2}{(q^4;q^8)_\infty^2}=8q\frac{(q^4;q^4)_\infty}{(q^2;q^4)_\infty^7},\end{aligned}$$

where in the penultimate equality, we invoked (5.2).

A simple observation on the q-power series expansion of $4\phi(q)-4\phi(-q)$ gives

$$4\phi(q)-4\phi(-q)=8\sum_{n=0}^{\infty}a(2n+1)q^{2n+1}=8q\frac{(q^4;q^4)_\infty}{(q^2;q^4)_\infty^7}.$$

Replacing q^2 by q and dividing by $8q$, we obtain (5.1). This completes the proof of Lemma 5.1. ∎

Proof of (1.6). From (5.1), we find that

$$\begin{aligned}\sum_{n=0}^{\infty}a(2n+1)q^n&=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty^7}\equiv\frac{(q;q^2)_\infty^2(q^2;q^2)_\infty}{(q^9;q^{18})_\infty}\pmod 3\\&=\frac{1}{(q^9;q^{18})_\infty}\frac{(q;q)_\infty}{(-q;q)_\infty}\\&=\frac{1}{(q^9;q^{18})_\infty}\left(\frac{(q^9;q^9)_\infty}{(-q^9;q^9)_\infty}-2q(q^3,q^{15},q^{18};q^{18})_\infty\right)\end{aligned}$$

by an application of (1.18) in the last equality. Since there are no terms of the form q^{9k+3} and q^{9k+6}, we conclude that $a(18n+7)\equiv a(18n+13)\equiv 0 \pmod 3$. This completes the proof of (1.6). ∎

Proof of (1.7). From (5.1), we find that

$$\begin{aligned}\sum_{n=0}^{\infty}a(2n+1)q^n&=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty^7}\equiv\frac{1}{(q^5;q^{10})_\infty^2}(q^2;q^2)_\infty(q;q^2)_\infty^3\\&\equiv\frac{1}{(q^5;q^{10})_\infty^2(q^{10};q^{10})_\infty}(q^2;q^2)_\infty^6(q;q^2)_\infty^3\\&\equiv\frac{1}{(q^5;q^{10})_\infty^2(q^{10};q^{10})_\infty}(q^2;q^2)_\infty^3(q;q)_\infty^3\pmod 5\end{aligned}$$

$$= \frac{1}{(q^5;q^{10})_\infty^2 (q^{10};q^{10})_\infty} \sum_{n,m=0}^{\infty} (-1)^{m+n}(2n+1)(2m+1)q^{n(n+1)/2+m(m+1)},$$

where in the last equality, we used Jacobi's identity [7, Entry 24(ii), p. 39]. Since $n(n+1)/2+m(m+1)$ is congruent to 4 modulo 5 only when both m and n are congruent to 2 modulo 5, we see that the coefficient of q^{5n+4} in the series on the right side is always divisible by 5. This in turn implies that

$$a(10n+9) \equiv 0 \pmod 5.$$

This completes the proof of (1.7). ∎

6. Proof of (1.9). First, we prove two lemmas.

Lemma 6.1. *We have the following* 3*-dissection:*

$$\frac{1}{(q;q^2)_\infty^3} = \frac{J_{18}^2}{J_{3,18}J_9} + \frac{12q^3 J_{18}^{17}}{J_{3,18}^6 J_{6,18}^4 J_9^7} + \frac{3qJ_{18}^{15}}{J_{3,18}^8 J_{6,18}^4 J_9^3} + \frac{6q^2 J_{18}^{16}}{J_{3,18}^7 J_{6,18}^4 J_9^5}. \tag{6.1}$$

Proof. Simplifying the right side, we rewrite (6.1) as

$$\begin{aligned}
\frac{1}{(q;q^2)_\infty^3} &= \frac{J_{18}^2}{J_{3,18}J_9} + \frac{12q^3 J_{18}^{17}}{J_{3,18}^6 J_{6,18}^4 J_9^7} + \frac{3qJ_{18}^{15}}{J_{3,18}^8 J_{6,18}^4 J_9^3} + \frac{6q^2 J_{18}^{16}}{J_{3,18}^7 J_{6,18}^4 J_9^5} \\
&= \frac{J_{18}^2}{J_{3,18}J_9} + 3q\frac{J_{18}^2}{J_9}\left\{ \frac{J_{18}^{13}}{J_{3,18}^8 J_{6,18}^4 J_9^2} + \frac{2qJ_{18}^{14}}{J_{3,18}^7 J_{6,18}^4 J_9^4} + \frac{4q^2 J_{18}^{15}}{J_{3,18}^6 J_{6,18}^4 J_9^6} \right\} \\
&= \frac{1}{(q^3;q^6)_\infty} + 3q\frac{(q^{18};q^{18})_\infty}{(q^9;q^{18})_\infty}\frac{(-q;q)_\infty}{(q;q)_\infty},
\end{aligned}$$

where we applied (2.1) in the last inequality. Therefore, it suffices to show that

$$\frac{1}{(q;q^2)_\infty^3} = \frac{1}{(q^3;q^6)_\infty} + 3q\frac{(q^{18};q^{18})_\infty}{(q^9;q^{18})_\infty}\frac{(-q;q)_\infty}{(q;q)_\infty}. \tag{6.2}$$

Upon rearrangement, we see that this is equivalent to [8, Entry 50(i), p. 202] with q replaced by $-q$. ∎

Lemma 6.2. *We have the following* 3*-dissection:*

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty^4} = \frac{J_{18}^3}{J_{3,18}^2} + \frac{18q^3 J_{18}^{18}}{J_{3,18}^7 J_{6,18}^4 J_9^6} + \frac{4qJ_{18}^{19}}{J_{3,18}^{10} J_9^8} + \frac{9q^2 J_{18}^{17}}{J_{3,18}^8 J_{6,18}^4 J_9^4}. \tag{6.3}$$

Proof. Note that by (1.20), we have

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{(q^9;q^9)_\infty (q^9;q^{18})_\infty}{(q^3;q^6)_\infty} + q\frac{(q^{18};q^{18})_\infty}{(q^9;q^{18})_\infty} = \frac{J_9 J_{18}}{J_{3,18}} + q\frac{J_{18}^2}{J_9}.$$

Therefore invoking (6.2), the left side of (6.3) becomes

$$\begin{aligned}\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty^4} &= \left(\frac{1}{(q^3;q^6)_\infty} + 3q\frac{(q^{18};q^{18})_\infty}{(q^9;q^{18})_\infty}\frac{(-q;q)_\infty}{(q;q)_\infty}\right)\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \\
&= \frac{1}{(q^3;q^6)_\infty}\left(\frac{J_9J_{18}}{J_{3,18}} + q\frac{J_{18}^2}{J_9}\right) + 3q\frac{(q^{18};q^{18})_\infty}{(q^9;q^{18})_\infty}\frac{1}{(q;q^2)_\infty^3} \\
&= \frac{J_{18}^3}{J_{3,18}^2} + \frac{qJ_{18}^4}{J_{3,18}J_9^2} \\
&\quad + 3q\frac{J_{18}^2}{J_9}\left(\frac{J_{18}^2}{J_{3,18}J_9} + \frac{12q^3J_{18}^{17}}{J_{3,18}^6J_{6,18}^4J_9^7} + \frac{3qJ_{18}^{15}}{J_{3,18}^8J_{6,18}^4J_9^3} + \frac{6q^2J_{18}^{16}}{J_{3,18}^7J_{6,18}^4J_9^5}\right) \\
&= \frac{J_{18}^3}{J_{3,18}^2} + \frac{qJ_{18}^4}{J_{3,18}J_9^2} + \frac{3qJ_{18}^4}{J_{3,18}J_9^2} + \frac{36q^4J_{18}^{19}}{J_{3,18}^6J_{6,18}^4J_9^8} + \frac{9q^2J_{18}^{17}}{J_{3,18}^8J_{6,18}^4J_9^4} + \frac{18q^3J_{18}^{18}}{J_{3,18}^7J_{6,18}^4J_9^6}.\end{aligned}$$

Comparing with the right side of (6.3), we find that it suffices to show

$$\frac{J_{18}^{19}}{J_{3,18}^{10}J_9^8} = \frac{J_{18}^4}{J_{3,18}J_9^2} + \frac{9q^3J_{18}^{19}}{J_{3,18}^6J_{6,18}^4J_9^8}.$$

Multiplying through by $J_{3,18}^7J_{6,18}^3J_9^6J_{18}^{-15}$, we see that this is equivalent to

$$\frac{J_{6,18}^3J_{18}^4}{J_{3,18}^3J_9^2} = \frac{J_{3,18}^6J_{6,18}^3J_9^4}{J_{18}^{11}} + 9q^3\frac{J_{3,18}J_{18}^4}{J_{6,18}J_9^2}. \tag{6.4}$$

This is [24, (3.6)] with q replaced by q^3. ∎

Proof of (1.9). From (5.1), we find that

$$\sum_{n=0}^{\infty} a(2n+1)q^n = \frac{1}{(q;q^2)_\infty^3}\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty^4}.$$

Invoking (6.1) and (6.3) on the right side and expanding, we find that

$$\begin{aligned}\sum_{n=0}^{\infty} a(6n+5)q^{3n+2} &= \frac{9q^2J_{18}^{19}}{J_{3,18}^9J_{6,18}^4J_9^5} + \frac{108q^5J_{18}^{34}}{J_{3,18}^{14}J_{6,18}^8J_9^{11}} + \frac{12q^2J_{18}^{34}}{J_{3,18}^{18}J_{6,18}^4J_9^{11}} \\
&\quad + \frac{6q^2J_{18}^{19}}{J_{3,18}^9J_{6,18}^4J_9^5} + \frac{108q^5J_{18}^{34}}{J_{3,18}^{14}J_{6,18}^8J_9^{11}} \\
&= \frac{15q^2J_{18}^{19}}{J_{3,18}^9J_{6,18}^4J_9^5} + \frac{216q^5J_{18}^{34}}{J_{3,18}^{14}J_{6,18}^8J_9^{11}} + \frac{12q^2J_{18}^{34}}{J_{3,18}^{18}J_{6,18}^4J_9^{11}}.\end{aligned} \tag{6.5}$$

Dividing (6.4) by $J_{3,18}^{15}J_{6,18}^7J_9^9J_{18}^{34}$ and multiplying by q^2 gives

$$\frac{q^2J_{18}^{34}}{J_{3,18}^{18}J_{6,18}^4J_9^{11}} = \frac{q^2J_{18}^{19}}{J_{3,18}^9J_{6,18}^4J_9^5} + \frac{9q^5J_{18}^{34}}{J_{3,18}^{14}J_{6,18}^8J_9^{11}}.$$

Substituting this into (6.5) gives

$$\sum_{n=0}^{\infty} a(6n+5)q^{3n+2} = \frac{27q^2 J_{18}^{19}}{J_{3,18}^9 J_{6,18}^4 J_9^5} + \frac{324 q^5 J_{18}^{34}}{J_{3,18}^{14} J_{6,18}^8 J_9^{11}} \equiv 0 \pmod{27}.$$

This completes the proof of (1.9). ∎

7. Conjectures and proof of Theorem 1.2. In this section, we first prove Theorem 1.2. Next, we state our conjectures for some similar functions.

Proof of Theorem 1.2. Recall from [20, Lemma 3], [12, p. 610]

$$\begin{aligned}\text{(7.1)}\quad &\frac{(a, q/a, -b, -q/b; q)_\infty (q;q)_\infty^2}{2(-a, -q/a, ab, q/(ab); q)_\infty (-q;q)_\infty^2} \\ &= \sum_{n=-\infty}^{\infty} \frac{(-1)^n b^{-n} q^{n(n+1)/2}}{1+aq^n} + \frac{b(1/b, qb; q)_\infty}{2(-q;q)_\infty^2} \sum_{n=-\infty}^{\infty} \frac{q^{n(n+1)/2}}{1-abq^n}.\end{aligned}$$

Replacing q by q^p, setting $(a,b) = (-x, 1/x)$ in (7.1), multiplying by $4x$, and dividing by $(x, q^p/x, q^p; q^p)_\infty$, we obtain

$$\begin{aligned}\frac{(-x, -q^p/x; q^p)_\infty^2 (q^p;q^p)_\infty}{(x, q^p/x; q^p)_\infty^2 (-q^p;q^p)_\infty^4} &= 4\frac{x}{(x, q^p/x, q^p; q^p)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n x^n q^{pn(n+1)/2}}{1-xq^{pn}} \\ &\quad + 2\frac{(q^p;q^{2p})_\infty}{(q^{2p};q^{2p})_\infty} \sum_{n=-\infty}^{\infty} \frac{q^{pn(n+1)/2}}{1+q^{pn}}.\end{aligned}$$

Setting $x = q^j$, we arrive at

$$\begin{aligned}\text{(7.2)}\quad &\frac{(-q^j, -q^{p-j}; q^p)_\infty^2 (q^p;q^p)_\infty}{(q^j, q^{p-j}; q^p)_\infty^2 (-q^p;q^p)_\infty^4} \\ &= 4\sum_{n=0}^{\infty} a_{j,p}(n) q^n + 2\frac{(q^p;q^{2p})_\infty}{(q^{2p};q^{2p})_\infty} \sum_{n=-\infty}^{\infty} \frac{q^{pn(n+1)/2}}{1+q^{pn}}.\end{aligned}$$

We observe that the second series on the right side only contains terms of the form q^n where n is a multiple of p. Therefore it suffices to examine the p-dissection of the product on the left side.

Recall that from Ramanujan's famous ${}_1\psi_1$ summation formula, we have the following corollary [12, (3.1)]:

$$\text{(7.3)}\qquad \frac{(xy, q/xy, q, q; q)_\infty}{(x, q/x, y, q/y)_\infty} = \sum_{n=-\infty}^{\infty} \frac{x^n}{1-yq^n}.$$

Replacing q by q^p and substituting $(x,y) = (q^j, -1)$, we find that the formula becomes

$$\text{(7.4)}\qquad \frac{(-q^j, -q^{p-j}, q^p, q^p; q^p)_\infty}{(q^j, q^{p-j}, -1, -q^p; q^p)_\infty} = \sum_{n=-\infty}^{\infty} \frac{q^{jn}}{1+q^{pn}}.$$

Separating the series on the right side according to the summation index modulo p, we find that

$$\begin{aligned}\frac{(-q^j,-q^{p-j},q^p,q^p;q^p)_\infty}{(q^j,q^{p-j},-1,-q^p;q^p)_\infty} &= \sum_{k=0}^{p-1}\sum_{n=-\infty}^{\infty}\frac{q^{pjn+jk}}{1+q^{p^2n+pk}}\\ &= \sum_{k=0}^{p-1} q^{jk}\frac{(-q^{p(j+k)},-q^{p^2-p(j+k)},q^{p^2},q^{p^2};q^{p^2})_\infty}{(q^{pj},q^{p^2-pj},-q^{pk},-q^{p^2-pk};q^{p^2})_\infty},\end{aligned}$$

where in the last equality, we applied (7.3) to each series. Applying this to the left side of (7.2), we arrive at

$$\begin{aligned}&\frac{1}{(q^p;q^p)_\infty^3}\left(\frac{(-q^j,-q^{p-j};q^p)_\infty(q^p;q^p)_\infty^2}{(q^j,q^{p-j};q^p)_\infty(-q^p;q^p)_\infty^2}\right)^2\\ &\quad = 4\frac{1}{(q^p;q^p)_\infty^3}\left(\sum_{k=0}^{p-1} q^{jk}\frac{(-q^{p(j+k)},-q^{p^2-p(j+k)},q^{p^2},q^{p^2};q^{p^2})_\infty}{(q^{pj},q^{p^2-pj},-q^{pk},-q^{p^2-pk};q^{p^2})_\infty}\right)^2.\end{aligned}$$

Extracting only terms with q^n where $n\equiv pj-j^2 \pmod p$, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty} a_{j,p}(pn+pj-j^2)q^{pn+pj-j^2}\\ &\quad = \frac{1}{(q^p;q^p)_\infty^3}\sum_{k=0}^{p-1} q^{jk}\frac{(-q^{p(j+k)},-q^{p^2-p(j+k)},q^{p^2},q^{p^2};q^{p^2})_\infty}{(q^{pj},q^{p^2-pj},-q^{pk},-q^{p^2-pk};q^{p^2})_\infty}\\ &\qquad \times q^{j(p-j-k)}\frac{(-q^{p(p-k)},-q^{p^2-p(p-k)},q^{p^2},q^{p^2};q^{p^2})_\infty}{(q^{pj},q^{p^2-pj},-q^{p(p-j-k)},-q^{p^2-p(p-j-k)};q^{p^2})_\infty}\\ &\quad = \frac{1}{(q^p;q^p)_\infty^3}\sum_{k=0}^{p-1} q^{j(p-j)}\frac{(q^{p^2};q^{p^2})_\infty^4}{(q^{pj},q^{p^2-pj};q^{p^2})_\infty^2}\\ &\quad = pq^{j(p-j)}\frac{(q^{p^2};q^{p^2})_\infty^4}{(q^p;q^p)_\infty^3(q^{pj},q^{p^2-pj};q^{p^2})_\infty^2}.\end{aligned}$$

Thus we obtain (1.11). ∎

In addition to the congruences satisfied in (1.12), we also conjecture the following congruences.

We list the following congruences observed based on q-expansions up to q^{900}.

Conjecture 7.1. *For any nonnegative integer n,*

$$\begin{aligned}a(50n+19)\equiv a(50n+39)\equiv a(50n+49)&\equiv 0 \pmod{25},\\ a_{1,3}(5n+3)\equiv a_{1,3}(5n+4)&\equiv 0 \pmod 5,\\ a_{1,6}(2n)&\equiv 0 \pmod 2,\end{aligned}$$

$$a_{1,6}(6n+3) \equiv 0 \pmod 3,$$
$$a_{1,10}(2n) \equiv a_{3,10}(2n) \equiv 0 \pmod 2,$$
$$a_{1,10}(10n+5) \equiv a_{3,10}(10n+5) \equiv 0 \pmod 5.$$

Replacing q by q^2 and setting $(a,b) = (-q, 1/q)$ in (7.1), we obtain

$$\begin{aligned} L(q) &:= \frac{(q^2;q^4)_\infty^9 (q^4;q^4)_\infty}{(q;q^2)_\infty^8} \\ &= 4\frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(n+1)^2}}{1-q^{2n+1}} + 2\frac{(q^2;q^4)_\infty}{(q^4;q^4)_\infty} \sum_{n=-\infty}^{\infty} \frac{q^{n(n+1)}}{1+q^{2n}}. \end{aligned}$$

Note that the left side is

$$\begin{aligned} \sum_{n=0}^{\infty} b(n) q^n := L(q) &= (q^2;q^2)_\infty (-q;q^2)_\infty^8 \\ &\equiv (-q^3;q^6)_\infty^2 (-q,-q,q^2;q^2)_\infty \pmod 3 \\ &= (-q^3;q^6)_\infty^2 \sum_{n=-\infty}^{\infty} q^{n^2}, \end{aligned}$$

where in the last equality, we applied (1.17) with $(a,b) = (-q,-q)$. Since n^2 is only congruent to 0 or 1 modulo 3, we conclude that $b(n) \equiv 0 \pmod 3$ for $n \equiv 2 \pmod 3$. The congruence (1.5) then leads to the following congruence.

THEOREM 7.2. *Let*

$$\sum_{n=0}^{\infty} c(n) q^n := 2\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{q^{n(n+1)/2}}{1+q^n}.$$

Then for any nonnegative integer n,

$$c(3n+1) \equiv 0 \pmod 3.$$

Following the approach in the proof of Corollary 2.4, one could also prove the stronger result,

$$\sum_{n=0}^{\infty} c(3n+1) q^n = 3\frac{(q;q^2)_\infty^7 (q^6;q^6)_\infty^3}{(q^2;q^2)_\infty^2}.$$

Similarly, by taking congruences modulo 7,

$$\begin{aligned} \sum_{n=0}^{\infty} b(n) q^n &= (q^2;q^2)_\infty (-q;q^2)_\infty^8 \\ &\equiv (-q^7;q^{14})_\infty (-q;-q)_\infty \pmod 7 \\ &= (-q^7;q^{14})_\infty \sum_{n=-\infty}^{\infty} (-q)^{n(3n+1)/2}, \end{aligned}$$

where in the last equality, we applied (1.17) with $(a,b) = (-q,q^2)$. Since $n(3n+1)/2$ is only congruent to $0,1,2$, or 5 modulo 7, we conclude that $b(n) \equiv 0 \pmod 7$ for $n \equiv 3,4,6 \pmod 7$. The congruence (1.8) then leads to the following congruence.

THEOREM 7.3. *For any nonnegative integer n,*

$$c(7n+2) \equiv c(7n+3) \equiv c(7n+5) \equiv 0 \pmod 7.$$

We remark that the results in Theorems 7.2 and 7.3 were also independently discovered and proved by M. Waldherr in [25].

Finally, by examining a function that closely resembles ϕ but with an extra factor of q^n in the summand, we offer the following conjecture.

CONJECTURE 7.4. *Let*

$$\sum_{n=1}^{\infty} d(n)q^n := \sum_{n=0}^{\infty} \frac{(-q;q)_{2n}q^{2n}}{(q;q^2)_{n+1}^2}.$$

We conjecture that for any nonnegative integer n,

$$\begin{aligned} d(18n+5) &\equiv d(18n+8) \equiv d(18n+11) \equiv d(18n+14) \\ &\equiv d(18n+17) \equiv 0 \pmod 3. \end{aligned}$$

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