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Comparison of L^1- and L^∞-norms of squares of polynomials

by

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1. Introduction. Let $\mathcal{P}(n)$ be the set of polynomials $P(X) = Q(X)^2$ where Q is a nonzero polynomial of degree $< n$ with nonnegative real coefficients. We are interested in

$$A(n) = n^{-1} \sup_{P \in \mathcal{P}(n)} |P|_1/|P|_\infty,$$

where $|P|_1$ is the sum, and $|P|_\infty$ the maximum of the coefficients of P. Let $\mathcal{F}$ be the set of functions $f = g * g$ where $*$ denotes convolution and g runs through nonnegative, not identically zero, integrable functions with support in $[0, 1]$. Functions in $\mathcal{F}$ have support in $[0, 2]$. We set

$$B = \sup_{f \in \mathcal{F}} |f|_1/|f|_\infty$$

where $|f|_1$ is the L^1-norm and $|f|_\infty$ the sup norm of f.

It is fairly obvious that

$$1 \le A(n) \le 2 - 1/n.$$

Indeed, the left inequality follows on taking $P = Q^2$ with $Q(X) = 1 + X + \ldots + X^{n-1}$, the right inequality is obtained by noting that $P \in \mathcal{P}(n)$ has at most $2n - 1$ nonzero coefficients, so that $|P|_1/|P|_\infty \le 2n - 1$. In a similar way one sees that

$$1 \le B \le 2.$$

Theorem 1. *For natural* n, l,

(i) $A(n) \le A(nl)$,
(ii) $A(n) \le B$,
(iii) $A(n) > B(1 - 6n^{-1/3})$.

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It follows that

$$B = \lim_{n\to\infty} A(n) = \sup_n A(n).$$

The determination of B appears to be difficult.

THEOREM 2. $4/\pi \le B < 1.7373$.

A slightly better upper bound will in fact be proved. We should mention that Ben Green [1] showed in effect that

$$(|f|_1/|f|_2)^2 < 7/4$$

for $f \in \mathcal{F}$, where $|f|_2$ denotes the L^2-norm. In fact he has the slightly better bound 1.74998... Since $|f|_2^2 \le |f|_1|f|_\infty$, this yields $B < 1.74998\ldots$, which is only slightly weaker than the upper bound in Theorem 2. However, Green's result is valid without the assumption $g \ge 0$.

On the other hand, Prof. Stanisław Kwapień (private communication) proved that

$$A(n) \ge B(1 - 3(B/4)^{1/3}n^{-1/3}).$$

2. Assertions (i), (ii) of Theorem 1. When R is a polynomial or power series $a_0 + a_1X + \ldots$, set $|R|_\infty$ for the maximum modulus of its coefficients. For such R, and for a polynomial S,

$$|RS|_\infty \le |R|_\infty |S|_1. \tag{2.1}$$

When $P \in \mathcal{P}(n)$, say $P = Q^2$, set

$$\widetilde{Q} = (1 + X + \ldots + X^{l-1})Q(X^l) \quad \text{and} \quad \widetilde{P} = \widetilde{Q}^2.$$

Then $\deg \widetilde{Q} \le l-1+l(n-1) = ln-1$, so that $\widetilde{P} \in \mathcal{P}(ln)$. Further $|\widetilde{Q}|_1 = l|Q|_1$, yielding

$$|\widetilde{P}|_1 = |\widetilde{Q}|_1^2 = l^2|Q|_1^2 = l^2|P|_1. \tag{2.2}$$

For polynomials or series $R = a_0 + a_1X + \ldots$, $S = b_0 + b_1X + \ldots$ with nonnegative coefficients, write $R \preceq S$ if $a_i \le b_i$ $(i = 0, 1, \ldots)$. Then

$$Q(X^l)^2 \preceq |Q^2|_\infty(1 + X^l + X^{2l} + \ldots) = |P|_\infty(1 + X^l + X^{2l} + \ldots).$$

Therefore

$$\begin{aligned}\widetilde{P} &= (1 + X + \ldots + X^{l-1})^2 Q(X^l)^2 \\ &\preceq |P|_\infty(1 + X^l + X^{2l} + \ldots)(1 + X + \ldots + X^{l-1})^2 \\ &= |P|_\infty(1 + X + X^2 + \ldots)(1 + X + \ldots + X^{l-1}).\end{aligned}$$

Now (2.1) gives $|\widetilde{P}|_\infty \le |P|_\infty l$. Together with (2.2) this yields $n^{-1}|P|_1/|P|_\infty \le (ln)^{-1}|\widetilde{P}|_1/|\widetilde{P}|_\infty \le A(nl)$. Assertion (i) follows. ∎

We now turn to (ii). Let $P \in \mathcal{P}(n)$ be given, say $P = Q^2$ with $Q = a_0 + a_1X + \ldots + a_{n-1}X^{n-1}$. Let g be the function with support in $[0,1)$ having

$$g(x) = a_i \quad \text{for } i/n \le x < (i+1)/n \ \ (i = 0, 1, \ldots, n-1),$$

i.e., for $\lfloor nx \rfloor = i$. Then $|g|_1 = n^{-1}|Q|_1$, so that $f = g * g$ has

$$|f|_1 = n^{-2}|Q^2|_1 = n^{-2}|P|_1. \tag{2.3}$$

Let x be given. The interval $I = [0,1)$ is the disjoint union of the intervals (possibly empty) $I_{i,j}(x)$ $(i = 0, 1, \ldots, n-1; j \in \mathbb{Z})$ consisting of numbers y with

$$\lfloor ny \rfloor = i, \qquad \lfloor n(x-y) \rfloor = j - i.$$

When $y \in I_{i,j}(x)$ and $0 \le i' < n$, then $y + (i'-i)/n \in I_{i',j}(x)$. Therefore $I_{i,j}(x)$ has length independent of i; denote this length by $L_j(x)$. Clearly $L_j(x) = 0$ unless $j = \lfloor nx \rfloor$ or $\lfloor nx - 1 \rfloor$. We have

$$1 = \sum_{i=0}^{n-1} \sum_j L_j(x) = n \sum_j L_j(x). \tag{2.4}$$

For $y \in I_{i,j}(x)$ with $0 \le i < n$,

$$g(y)g(x-y) = \begin{cases} a_i a_{j-i} & \text{when } j - n < i \le j, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$\int\limits_{I_{i,j}(x)} g(y)g(x-y)\,dy = \begin{cases} a_i a_{j-i} & \text{when } j - n < i \le j, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

Now

$$\sum_{i=0}^{j} a_i a_{j-i} = b_j \le |P|_\infty,$$

where b_j is the coefficient of X^j in P. Taking the sum of (2.5) over $i = 0, 1, \ldots, n-1$ and $j \in \mathbb{Z}$, and observing (2.4), we obtain

$$f(x) = \int g(y)g(x-y)\,dy \le |P|_\infty \sum_j L_j(x) = |P|_\infty/n.$$

Therefore $|f|_\infty \le |P|_\infty/n$, so that in conjunction with (2.3),

$$n^{-1}|P|_1/|P|_\infty \le |f|_1/|f|_\infty \le B.$$

Assertion (ii) follows. ∎

3. Assertion (iii) of Theorem 1. Pick $f \in \mathfrak{F}$ with $|f|_1/|f|_\infty$ close to B. We may suppose that $|f|_\infty = 1$ and $|f|_1$ is close to B, in particular that $|f|_1 \geq 1$. Say $f = g * g$. Then for $r < s$,

$$(3.1)\quad \Big(\int_r^s g(x)\,dx\Big)^2 \leq \iint_{2r\leq x+y\leq 2s} g(x)g(y)\,dx\,dy$$
$$= \int_{2r}^{2s} dz \int g(x)g(z-y)\,dy = \int_{2r}^{2s} f(z)\,dz \leq 2(s-r).$$

Setting $G(y) = \int_0^y g(y)\,dy$, so that $G(y) \leq \sqrt{2y}$, and using partial integration, we obtain

$$(3.2)\qquad \int_0^\delta (\delta - x)g(x)\,dx = \int_0^\delta G(y)\,dy \leq \int_0^\delta (2y)^{1/2}\,dy < \delta^{3/2}.$$

Similarly,

$$\int_{1-\delta}^1 (\delta - (1-x))g(x)\,dx < \delta^{3/2}.$$

With $c \in \frac{1}{2}\mathbb{Z}$ in $1 \leq c \leq (n-1)/2$ to be determined later, set

$$a_i = \frac{n}{2c} \int_{(i+1/2-c)/n}^{(i+1/2+c)/n} g(x)\,dx \qquad (0 \leq i < n)$$

and

$$Q(X) = \sum_{i=0}^{n-1} a_i X^i.$$

Then

$$|Q|_1 = \sum_{i=0}^{n-1} a_i = \frac{n}{2c}\int_0^1 \nu(x)g(x)\,dx$$

where $\nu(x)$ is the number of integers i, $0 \leq i < n$, having $(i + 1/2 - c)/n \leq x \leq (i + 1/2 + c)/n$. Then $\nu(x)$ is the number of integers i having

$$\max(0, nx - 1/2 - c) \leq i \leq \min(n-1, nx - 1/2 + c).$$

When $(c+1/2)/n \leq x \leq 1-(c+1/2)/n$, this becomes the interval $nx-1/2-c \leq i \leq nx-1/2+c$, so that $\nu(x) \geq 2c$, as $c \in \frac{1}{2}\mathbb{Z}$. When $x < (c+1/2)/n$, the interval becomes $0 \leq i \leq nx - 1/2 + c$, and $\nu(x) \geq nx + c - 1/2 = 2c - (c + 1/2 - nx)$. On the other hand when $x > 1 - (c + 1/2)/n$, then $\nu(x) \geq 2c - (c + 1/2 - n(1-x))$. Therefore

$$(3.3)\qquad |Q|_1 \ge n\int_0^1 g(x)\,dx - \frac{n}{2c}\int_0^{(c+1/2)/n}(c+1/2-nx)g(x)\,dx - \frac{n}{2c}\int_{1-(c+1/2)/n}^{1}(c+1/2-n(1-x))g(x)\,dx.$$

Applying (3.2) with $\delta = (c+1/2)/n$ we obtain

$$\frac{n}{2c}\int_0^{(c+1/2)/n}(c+1/2-nx)g(x)\,dx < \frac{n^2}{2c}((c+1/2)/n)^{3/2} < n((c+1/2)/n)^{1/2}.$$

The same bound applies to the last term on the right hand side of (3.3), so that

$$|Q|_1 \ge n|g|_1(1-2((c+1/2)/n)^{1/2}/|g|_1).$$

Here $|g|_1 \ge 1$ since $|f|_1 \ge 1$.

The polynomial $P = Q^2$ lies in $\mathcal{P}(n)$ and has

$$(3.4)\qquad |P|_1 \ge n^2|f|_1(1-4((c+1/2)/n)^{1/2}).$$

The coefficients of P are

$$b_l = \sum_{i+j=l} a_i a_j = \left(\frac{n}{2c}\right)^2 \sum_{i+j=l}\int_{(i+1/2-c)/n}^{(i+1/2+c)/n}\int_{(j+1/2-c)/n}^{(j+1/2+c)/n} g(x)g(y)\,dx\,dy.$$

Setting $z = x+y$, so that $(l+1-2c)/n \le z \le (l+1+2c)/n$, we obtain

$$b_l = \left(\frac{n}{2c}\right)^2 \int_{(l+1-2c)/n}^{(l+1+2c)/n} dz \int \mu(z,x)g(x)g(z-x)\,dx$$

where $\mu(z,x)$ is the number of integers i in $0 \le i \le n-1$ with $(i+1/2-c)/n \le x \le (i+1/2+c)/n$ and $(l-i+1/2-c)/n \le z-x \le (l-i+1/2+c)/n$. Thus $h = i - nx + 1/2$ lies in the range

$$\max(-c, -c+l+1-nz) \le h \le \min(c, c+l+1-nz),$$

and $\mu(z,x) \le \lambda(z)$, which is the length of the "interval" (possibly empty)

$$(3.5)\qquad -c-1/2+\max(0,l+1-nz) \le h \le c+1/2+\min(0,l+1-nz).$$

Therefore

$$b_l \le \left(\frac{n}{2c}\right)^2 \int dz\, \lambda(z) \int g(x)g(z-x)\, dx$$
$$= \left(\frac{n}{2c}\right)^2 \int \lambda(z) f(z)\, dz \le \left(\frac{n}{2c}\right)^2 \int \lambda(z)\, dz.$$

But $\int \lambda(z)\, dz$ is the area of the domain in the (h,z)-plane given by (3.5). Here h is contained in an interval of length $2c+1$, and given h, the variable z lies in an interval of length $\le (2c+1)/n$, so that

$$b_l \le \left(\frac{n}{2c}\right)^2 \frac{(2c+1)^2}{n} = n\left(1+\frac{1}{2c}\right)^2.$$

Therefore $|P|_\infty \le n(1+1/(2c))^2$, and by (3.4),

$$A(n) \ge \frac{1}{n}|P|_1/|P|_\infty \ge |f|_1\left(1 - 4\left(\left(c+\frac{1}{2}\right)\Big/n\right)^{1/2}\right)\Big/\left(1+\frac{1}{2c}\right)^2.$$

We now pick $c \in \frac{1}{2}\mathbb{Z}$ with $n^{1/3} - 1 \le c < n^{1/3} - 1/2$. When $n \ge 8$, which we may clearly suppose in proving assertion (iii), then $1 \le n^{1/3}/2 \le c < (n-1)/2$. Since f may be chosen with $|f|_1$ arbitrarily close to B,

$$A(n) \ge B(1 - 4n^{-1/3})/(1 + n^{-1/3})^2 > B(1 - 6n^{-1/3}). \ \blacksquare$$

4. The lower bound in Theorem 2. Set $f = g * g$ where $g(x) = x^{-1/2}$ in $0 < x < 1$, and $g(x) = 0$ otherwise. Then $f \in \mathfrak{F}$, and $|f|_1 = |g|_1^2 = 4$. For $0 < z \le 2$,

$$f(z) = \int (z-x)^{-1/2} x^{-1/2}\, dx,$$

with the range of integration $\max(0, z-1) \le x \le \min(1, z)$. Setting $x = y^2 z$ we obtain

$$f(z) = 2\int \frac{dy}{(1-y^2)^{1/2}},$$

the integration being over $y \ge 0$ with $1 - 1/z \le y^2 \le \min(1/z, 1)$. When $0 < z \le 1$, this range is $0 \le y \le 1$, so that $f(z) = \pi$, whereas in $1 < z \le 2$ the range is smaller, and $f(z) < \pi$. We may conclude that $|f|_\infty = \pi$, and $B \ge |f|_1/|f|_\infty = 4/\pi$. $\blacksquare$

5. The upper bound $B \le 7/4$. The upper bound of Theorem 2 will be established in three stages. Here we will show that $B \le 7/4 = 1.75$, and in the following stages we will prove that $B \le 7/4 - 1/80 = 1.7375$, then that $B \le 1.7373$.

Our problem is invariant under translations. To exhibit symmetry, we therefore redefine $\mathfrak{F}$ to consist of functions $f = g * g$ with g nonzero, nonnegative and integrable, with support in $[-1/2, 1/2]$, so that f has support in $[-1,1]$. We will suppose throughout that $f \in \mathcal{F}$ with $|f|_\infty = 1$, and we will give upper bounds for $|f|_1$.

LEMMA 1.

$$\int_{1/2}^{1} f(z)f(-z)\,dz \le 1/4.$$

As a consequence of this lemma,

$$\begin{aligned}|f|_1 &= \int_{-1}^{1} f(z)\,dz = \int_{0}^{1} (f(z)+f(-z))\,dz \le 1 + \int_{1/2}^{1} (f(z)+f(-z))\,dz \\ &\le 1 + \int_{1/2}^{1} (1+f(z)f(-z))\,dz \le \frac{3}{2} + \frac{1}{4} = \frac{7}{4},\end{aligned}$$

so that indeed $B \le 7/4$.

Proof of Lemma 1.

$$f(z) = (g*g)(z) = \int g(x)g(z-x)\,dx = 2 \int_{\substack{x+y=z\\ x\le y}} g(x)g(y)\,dx. \tag{5.1}$$

(It is to exhibit symmetry that we write y for $z-x$.) Similarly

$$f(-z) = 2 \int_{\substack{u+v=-z\\ u\le v}} g(u)g(v)\,du. \tag{5.2}$$

Here x,y,u,v may be restricted to lie in $[1/2,-1/2]$. When $\delta \ge 0$ and $z \ge 1/2-\delta$, then $x = z-y \ge 1/2-\delta-1/2 = -\delta$, also $v = -u-z \le 1/2-1/2+\delta = \delta$, so that

$$u \le v \le \delta, \qquad -\delta \le x \le y.$$

We obtain

$$\int_{1/2-\delta}^{1} f(z)f(-z)\,dz \le 4 \int_{1/2-\delta}^{1} dz \iint_{\substack{u\le v\le\delta\\ -\delta\le x\le y\\ x+y=z\\ u+v=-z}} g(x)g(y)g(u)g(v)\,dx\,du.$$

In this integral $u \le -z/2 \le -1/4+\delta/2$, and $y \ge z/2 \ge 1/4-\delta/2$. Setting $w = u+y = -x-v$ we have $w \le u+1/2 \le 1/4+\delta/2$, and in fact $|w| \le 1/4+\delta/2$. Replacing the variables x,u,z in the above integral by

$x, y = z - x$, $w = u + z - x$, we obtain the bound

$$4 \int_{-1/4-\delta/2}^{1/4+\delta/2} dw \iint_{\substack{y+u=w \\ x+v=-w \\ -\delta \le x \le y \\ u \le v \le \delta \\ x+y \ge 1/2-\delta}} g(x)g(y)g(u)g(v)\,dx\,dy. \tag{5.3}$$

Let us now take $\delta = 0$. In this case

$$\int_{1/2}^{1} f(z)f(-z)\,dz \le 4 \int_{-1/4}^{1/4} dw \iint_{\substack{x+v=-w \\ y+u=w \\ u \le v \le 0 \le x \le y}} g(x)g(y)g(u)g(v)\,dx\,dy.$$

Interchanging the rôles of the variables x, y, and as a result those of u, v, and replacing w by $-w$, we get an integral as before, except that the region $u \le v \le 0 \le x \le y$ is replaced by the region $v \le u \le 0 \le y \le x$. These regions are essentially disjoint, and are contained in $u \le 0 \le y$, $v \le 0 \le x$. We therefore obtain

$$\le 2 \int_{-1/4}^{1/4} dw \Big(\int_{\substack{x+v=-w \\ v \le 0 \le x}} g(x)g(v)\,dx \Big)\Big(\int_{\substack{y+u=w \\ u \le 0 \le y}} g(y)g(u)\,dy \Big)$$

$$= 2 \int_{-1/4}^{1/4} dw\, \widetilde{f}(w)\widetilde{f}(-w)$$

with

$$\widetilde{f}(w) = \int_{\substack{y+u=w \\ u \le 0 \le y}} g(y)g(u)\,dy. \tag{5.4}$$

Thus

$$\int_{1/2}^{1} f(z)f(-z)\,dz \le 4 \int_{0}^{1/4} \widetilde{f}(w)\widetilde{f}(-w)\,dw. \tag{5.5}$$

It is clear from (5.1) and (5.4) that $\widetilde{f}(w) \le f(w)/2 \le 1/2$, so that we obtain $\le 1/4$, and Lemma 1 follows. ■

6. The upper bound $B \le 1.7375$. With $f = g * g$ as above, and $\varepsilon = \pm 1$, set

$$I_\varepsilon = \int_{0}^{1/8} g(\varepsilon x)\,dx, \qquad J_\varepsilon = \iint_{\substack{\varepsilon y > 0,\, \varepsilon u > 0 \\ \varepsilon(y+u) \le 1/4}} g(y)g(u)\,dy\,du.$$

LEMMA 2. (i) $\int_{1/2}^{1} f(z)f(-z)\,dz \le 1/4 - J_\varepsilon$.
(ii) *For* $0 \le \delta \le 1/6$,

$$\int_{1/2-\delta}^{1} f(z)f(-z)\,dz \le \frac{1}{4} + \frac{\delta}{2} + \Big(\int_{-\delta}^{\delta} g(x)\,dx\Big)^2.$$

As a consequence,

$$(6.1)\qquad |f|_1 = \int_0^1 (f(z)+f(-z))\,dz = \int_0^{1/2-\delta} + \int_{1/2-\delta}^{1}$$
$$\le 1 - 2\delta + \int_{1/2-\delta}^{1} (1 + f(z)f(-z))\,dz$$
$$\le \frac{3}{2} - \delta + \int_{1/2-\delta}^{1} f(z)f(-z)\,dz \le \frac{7}{4} - \frac{\delta}{2} + \Big(\int_{-\delta}^{\delta} g(x)\,dx\Big)^2.$$

Setting $\delta = 1/8$ we obtain

$$(6.2)\qquad |f|_1 \le \frac{27}{16} + (I_1 + I_{-1})^2 \le \frac{27}{16} + 4M^2$$

with $M = \max(I_1, I_{-1})$. On the other hand by (i),

$$(6.3)\qquad |f|_1 \le \frac{3}{2} + \int_{1/2}^{1} f(z)f(-z)\,dz \le \frac{7}{4} - \max_{\varepsilon=\pm1} J_\varepsilon \le \frac{7}{4} - M^2.$$

In conjunction with (6.2) this gives $|f|_1 \le 7/4 - 1/80 = 1.7375$, so that indeed $B \le 1.7375$.

Proof of Lemma 2. When $w > 0$, we cannot have $y + u = w$ and $u \le y < 0$. Therefore $\widetilde{f}(w)$ as given by (5.4) is

$$\widetilde{f}(w) = \int_{\substack{y+u=w\\ u\le y}} g(y)g(u)\,dy - \int_{\substack{y+u=w\\ 0\le u\le y}} g(y)g(u)\,dy = \frac{1}{2}f(w) - \frac{1}{2}\widehat{f}(w)$$

with

$$\widehat{f}(w) = \int_{\substack{y+u=w\\ y,u\ge 0}} g(y)g(u)\,dy.$$

Now (5.5) yields

$$\int_{1/2}^{1} f(z)f(-z)\,dz \le \int_0^{1/4} (f(w) - \widehat{f}(w))f(-w)\,dw \le \int_0^{1/4} (1 - \widehat{f}(w))\,dw$$

$$= \frac{1}{4} - \int_0^{1/4} dw \int_{\substack{y+u=w\\ y,u\ge 0}} g(y)g(u)\,dy$$

$$= \frac{1}{4} - \iint_{\substack{y,u\ge 0\\ y+u\le 1/4}} g(y)g(u)\,dy\,du = \frac{1}{4} - J_1.$$

The bound $1/4 - J_{-1}$ is obtained similarly, so that assertion (i) is established.

We will now suppose $\delta > 0$, and we return to the bound (5.3). We first deal with the part where $v \le x$ in the integral, so that

$$u \le v \le x \le y. \tag{6.4}$$

After interchanging the rôles of x and y, and of u and v, and replacing w by $-w$, the integrand will be the same, but now

$$v \le u \le y \le x. \tag{6.5}$$

The interiors of the domains (6.4), (6.5) are disjoint, and are contained in the region with $v \le x$ and $u \le y$, so that this part of (5.3) is

$$\le 2 \int_{-1/4-\delta/2}^{1/4+\delta/2} dw \Big(\int_{\substack{x+v=-w\\ v\le x}} g(x)g(v)\,dx \Big)\Big(\int_{\substack{y+u=w\\ u\le y}} g(y)g(u)\,dy \Big) \tag{6.6}$$

$$= \frac{1}{2} \int_{-1/4-\delta/2}^{1/4+\delta/2} dw\, f(-w)f(w) = \int_0^{1/4+\delta/2} f(w)f(-w)\,dw \le 1/4 + \delta/2.$$

It remains for us to deal with the part of (5.3) where $x \le v$ in the integral, so that $-\delta \le x \le v \le \delta$. This part is

$$\le 4 \int dw \int_{\substack{x+v=-w\\ -\delta\le x\le v\le\delta}} g(x)g(v)\,dx \int_{\substack{y+u=w\\ y\ge 1/2-\delta-x\\ u\le\delta}} g(y)g(u)\,dy.$$

When $0 < \delta \le 1/6$, then $y \ge 1/2 - 2\delta \ge \delta \ge u$, and the last integral is

$$\le \int_{\substack{y+u=w\\ u\le y}} g(y)g(u)\,dy = f(w)/2 \le 1/2.$$

Therefore the part in question of (5.3) becomes

$$\le 2 \int dw \int_{\substack{x+v=-w\\ -\delta\le x\le v\le\delta}} g(x)g(v)\,dx = \int dw \int_{\substack{x+v=-w\\ -\delta\le x,v\le\delta}} g(x)g(v)\,dx = \Big(\int_{-\delta}^{\delta} g(x)\,dx \Big)^2.$$

Together with (6.6) this gives the asserted bound for $\int_{1/2-\delta}^{1} f(z)f(-z)\,dz$. ■

7. The upper bound 1.7373. In fact we will show that

$$B \le 7/4 - 1/80 - \xi < 1.7373 \tag{7.1}$$

where $\xi = 0.000200513\ldots$ is a root of the transcendental equation

$$F(b(x)/a(x)) = 1/2,$$

where $a(x) = 1/10 - 2x$, $b(x) = (\sqrt{1/20 - x} - \sqrt{1/80 + x})^2/2$, and

$$F(x) = \sqrt{x^2 + x} + \log(\sqrt{x^2 + x} + \sqrt{x}).$$

The calculation of ξ has kindly been performed by Dr. A. Pokrzywa.

We will suppose that $f \in \mathcal{F}$, $|f|_\infty = 1$ and

$$|f|_1 > 7/4 - 1/80 - \xi, \tag{7.2}$$

and we will reach a contradiction, thereby establishing the truth of (7.1), and hence of Theorem 2.

Retaining earlier notation we now set $a = a(\xi)$,

$$u = I_1 + I_{-1}, \quad v = |I_1 - I_{-1}|, \quad m = \min(I_1, I_{-1}) = (u - v)/2,$$

and observe that $M = \max(I_1, I_{-1}) = (u + v)/2$. Also, u_0, u_1 will be the positive numbers with

$$u_0^2 = 1/20 - \xi = a/2, \quad u_1^2 = 1/20 + 4\xi.$$

We may suppose that

$$u \ge u_0, \tag{7.3}$$

for otherwise (6.2) yields $|f|_1 \le 27/16 + u_0^2 = 7/4 - 1/80 - \xi$, against (7.2). We further may suppose that

$$u + v \le u_1, \tag{7.4}$$

for otherwise (6.3) yields $|f|_1 \le 7/4 - u_1^2/4 = 7/4 - 1/80 - \xi$, contradicting (7.2). As a consequence,

$$\begin{aligned} 2u^2 - m^2/2 = 2u^2 - (u - v)^2/8 &= 3u^2/2 + u(u + v)/2 - (u + v)^2/8 \\ &\le 3u^2/2 + 3u(u + v)/8 \le 15u_1^2/8 < 1/10 - 2\xi = a, \end{aligned}$$

so that

$$0 = 2u_0^2 - a \le 2u^2 - a < m^2/2. \tag{7.5}$$

Lemma 3.

$$\frac{7}{4} - |f|_1 \ge \frac{1}{4}(u^2 + v^2) + \int_{2u^2-a}^{m^2/2} (\sqrt{(\eta + a)/2} - u)\,\frac{d\eta}{\sqrt{2\eta}}.$$

Proof. By (6.1) and (7.2),

$$1/80+\xi>\delta/2-\Big(\int_{-\delta}^{\delta} g(x)\,dx\Big)^2$$

for δ in $0<\delta<1/6$. Setting $\delta=1/8+\eta$ with $0<\eta<1/24$, this gives

$$\Big(\int_{-1/8-\eta}^{1/8+\eta} g(x)\,dx\Big)^2>\eta/2+1/20-\xi=(\eta+a)/2,$$

and

$$G(\eta):=\int_{1/8}^{1/8+\eta}(g(x)+g(-x))\,dx>\sqrt{(\eta+a)/2}-u. \tag{7.6}$$

On the other hand by (6.3) and (7.2), and since $m^2/2\le u^2/8\le u_1^2/8<1/24<1/8$,

$$\begin{aligned}\frac{1}{80}+\xi&>\frac{1}{2}\sum_{\varepsilon=\pm1}J_\varepsilon=\frac{1}{2}\Big(I_1^2+I_{-1}^2+2\sum_{\varepsilon=\pm1}\int_{1/8}^{1/4}g(\varepsilon x)\,dx\int_0^{1/4-x}g(\varepsilon y)\,dy\Big)\\&\ge\frac{1}{2}\Big(\frac{u^2+v^2}{2}+2\sum_{\varepsilon=\pm1}\int_{1/8}^{1/8+m^2/2}g(\varepsilon x)\,dx\int_0^{1/4-x}g(\varepsilon y)\,dy\Big)\\&=\frac{1}{4}(u^2+v^2)+\sum_{\varepsilon=\pm1}\int_0^{m^2/2}g(\varepsilon/8+\varepsilon\eta)\,d\eta\int_0^{1/8-\eta}g(\varepsilon y)\,dy.\end{aligned}$$

By (3.1) with $r=1/8-\eta$, $s=1/8$,

$$\int_0^{1/8-\eta}g(\varepsilon y)\,dy=I_\varepsilon-\int_{1/8-\eta}^{1/8}g(\varepsilon y)\,dy\ge I_\varepsilon-\sqrt{2\eta}\ge m-\sqrt{2\eta}.$$

Thus

$$\begin{aligned}\frac{1}{80}+\xi&>\frac{1}{4}(u^2+v^2)+\sum_{\varepsilon=\pm1}\int_0^{m^2/2}g(\varepsilon/8+\varepsilon\eta)(m-\sqrt{2\eta})\,d\eta\\&=\frac{1}{4}(u^2+v^2)+\int_0^{m^2/2}(g(1/8+\eta)+g(-1/8-\eta))(m-\sqrt{2\eta})\,d\eta.\end{aligned}$$

Integrating by parts we represent the last integral as

$$\int_0^{m^2/2}G(\eta)\,\frac{d\eta}{\sqrt{2\eta}}\ge\int_{2u^2-a}^{m^2/2}G(\eta)\,\frac{d\eta}{\sqrt{2\eta}}.$$

Since $m^2/2<1/24$ we may apply (7.6) to obtain the lemma. ■

Lemma 4. *In the domain of points* (u,v) *with* (7.3), (7.4), $v \ge 0$, *the function*

$$H(u,v)=\frac{1}{4}(u^2+v^2)+\int\limits_{2u^2-a}^{\frac{1}{2}(\frac{u-v}{2})^2}(\sqrt{(\eta+a)/2}-u)\,\frac{d\eta}{\sqrt{2\eta}}$$

satisfies $H(u,v)\ge H(u_0,u_1-u_0)$.

Proof.

$$2H(u,v)=\frac{1}{2}(u^2+v^2)+\int\limits_{2u^2-a}^{\frac{1}{2}(\frac{u-v}{2})^2}\sqrt{\frac{\eta+a}{\eta}}\,d\eta-u(u-v)+2u\sqrt{4u^2-2a}.$$

Hence

$$\begin{aligned}2\,\frac{\partial H(u,v)}{\partial v}&=v+u+\left(\frac{(u-v)^2+8a}{(u-v)^2}\right)^{1/2}\cdot\frac{v-u}{4}\\&=v+u-\frac{1}{4}((u-v)^2+8a)^{1/2}.\end{aligned}$$

We claim that this partial derivative is ≤ 0 in our domain. For otherwise $16(u+v)^2-((u-v)^2+8a)>0$, or $15(u+v)^2+4uv-8a>0$. But $u+v\le u_1$ and $4uv\le 4u(u_1-u)\le 4u_0(u_1-u_0)$ since $u\ge u_0>u_1/2$. Therefore $15u_1^2+4u_0u_1-4u_0^2-8a>0$. Substituting the values for a,u_0,u_1 gives

$$4u_0u_1\ge 1/4-80\xi.$$

Squaring, we get

$$16(1/20+4\xi)(1/20-\xi)>(1/4-80\xi)^2,$$

which is not true. Thus our claim is proven, and

$$H(u,v)\ge H(u,u_1-u). \tag{7.7}$$

Next,

$$2H(u,u_1-u)=-u^2+\frac{1}{2}u_1^2+\int\limits_{2u^2-a}^{\frac{1}{2}(\frac{2u-u_1}{2})^2}\sqrt{\frac{\eta+a}{\eta}}\,d\eta+2u\sqrt{4u^2-2a},$$

so that

$$\begin{aligned}2\,\frac{d}{du}H(u,u_1-u)=&-2u+\left(\frac{(2u-u_1)^2+8a}{(2u-u_1)^2}\right)^{1/2}\cdot\frac{2u-u_1}{2}\\&-\left(\frac{2u^2}{2u^2-a}\right)^{1/2}\cdot 4u\\&+2(4u^2-2a)^{1/2}+8u^2(4u^2-2a)^{-1/2}\\=&-2u+\frac{1}{2}\sqrt{(2u-u_1)^2+8a}+2\sqrt{4u^2-2a}.\end{aligned}$$

We claim that this derivative is ≥ 0 for $u_0 \le u \le u_1$. For otherwise $16u^2 \ge (2u-u_1)^2 + 8a$, so that $12u^2 + 4uu_1 - u_1^2 > 8a$. But this entails $15u_1^2 > 8a$, i.e.,

$$15(1/20 + 4\xi) > 4/5 + 16\xi,$$

which is not true. Thus our claim is correct, and

$$H(u, u_1 - u) \ge H(u_0, u_1 - u_0),$$

which together with (7.7) establishes the lemma. ■

It is now easy to arrive at the desired contradiction to (7.2). By Lemmas 3 and 4,

$$\begin{aligned} 7/4 - |f|_1 &\ge H(u_0, u_1 - u_0) \\ &= \frac{1}{4}(u_0^2 + (u_1 - u_0)^2) + \int\limits_{2u_0^2 - a}^{\frac{1}{2}(u_0 - \frac{1}{2}u_1)^2} \left(\frac{1}{2}\sqrt{\frac{\eta + a}{\eta}} - \frac{u_0}{\sqrt{2\eta}}\right) d\eta. \end{aligned}$$

Here $2u_0^2 - a = 0$ and $\frac{1}{2}\left(u_0 - \frac{1}{2}u_1\right)^2 = b(\xi) = b$, say, and

$$\int\limits_0^x \sqrt{\frac{\eta + a}{\eta}}\, d\eta = aF(x/a), \qquad \int\limits_0^x \frac{d\eta}{\sqrt{2\eta}} = \sqrt{2x}.$$

Therefore

$$\begin{aligned} 7/4 - |f|_1 &\ge \frac{1}{4}(2u_0^2 - 2u_0u_1 + u_1^2) + \frac{a}{2}F(b/a) - u_0(u_0 - u_1/2) \\ &= -u_0^2/2 + u_1^2/4 + \frac{a}{2}F(b/a) = -\frac{1}{80} + \frac{3}{2}\xi + \frac{a}{2}F(b/a) = 1/80 + \xi, \end{aligned}$$

contrary to (7.2). ■

Added in proof. Dr. Erik Bajalinov has checked that for $n \le 26$ and $n = 31, 36, 41, 46, 51$: $A(n) < 4/\pi$, which suggests that $B = 4/\pi$.

References

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