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Estimation of some exponential sums containing the fractional part function and some other "non-standard" exponential sums

by

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1. Introduction. Some problems in number theory and some other branches of mathematics can be reduced to the estimation of exponential sums

$$\sum_{X_1<x\le X_2} e(F(x)) \quad \text{with } X = X_2 - X_1 \le X_1.$$

If $F(x)$ is a polynomial or a function which can be reduced to a polynomial then the sum can be evaluated by using Vinogradov's method; if $F(x)$ is "van der Corput" type function then one uses van der Corput's method or Bombieri–Iwaniec method. Here by *van der Corput* (*v.d.c.*) *type function of order* k we mean a real-valued k times continuously differentiable function $F(x)$ such that $F^{(j)}(x) \asymp F_j(x)/x^j$ $(j = 1, \ldots, k)$ with piecewise monotone $F_j(x)$ such that if $k > 1$, then

$$1 \lll F_{j+1}(x)/F_j(x) \ll 1 \quad \text{and} \quad \overline{\lim}\, x^{1-2/K} F^{(k)}(x) \ll 1;$$

if $k = 1$, then

$$\lim_{x\to\infty} F_1(x) = \infty \quad \text{and} \quad \overline{\lim}\, |F'(x)| < 1$$

(see the notation below).

Note that if $k > 1$ is the smallest integer such that $F(x)$ is a v.d.c. function of order k and $K = 2^k$ then

$$F^{(k)}(x) \lll x^{2/K-1} \quad \text{and} \quad F^{(k-1)}(x) \ggg x^{4/K-1}$$

so that

$$x^{4/K-2} \lll F^{(k)}(x) \lll x^{2/K-1}. \tag{1}$$

If X is "not small", the above mentioned methods give non-trivial estimates. We call such sums *standard* exponential sums. If X is "small", the

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sum is called *short* and the well-known van der Corput's estimates may be larger than the trivial estimates. Also, if $F(x)$ contains an oscillating term, van der Corput's method cannot be used directly. We call such sums *non-standard* exponential sums. In the past we studied short sums [2] and sums containing an oscillating term [1], [2].

Wenguang Zhai has recently introduced [4] a method of evaluation of exponential sums with $F(x) = f(x) + g(x)\{h(x)\}$. He applied the method to prove that for any $k \neq 0$ and any $c > 0$ the sequence $\{[n^c]\log^k n\}$ is uniformly distributed modulo 1 by proving that the discrepancy of the sequence satisfies

$$D(X) \ll X^{-\delta(c)} \log X \quad \text{for some } \delta(c) > 0.$$

His result improved the result of Rieger [3] who proved the uniform distribution of the sequence for $1 < c < 3/2$ and $0 < k < 1$.

The method of Zhai gives a non-trivial estimate if $f(x), g(x)$ and $h(x)$ are v.d.c. functions and $g(x) \ll x^{3/4-\alpha}$ for any fixed $\alpha > 0$. One can evaluate such sums (and more general sums) with $g(x) \ll x^{1-\alpha}$ using our method of evaluation of short sums and

LEMMA 1. *Let $f(t,x)$ be a real-valued function such that*

$$|f(t_1,x) - f(t_2,x)| \le \lambda |t_1 - t_2|.$$

Then for any real function $g(x)$, any positive integer r and any $M > 0$ we have

$$\begin{aligned}(2)\qquad S &= \sum_x a(x) e(f(g(x), \{h(x)\})) \\ &= \frac{1}{M}\sum_{m=0}^{M-1}\sum_{j=0}^{\infty} b_{j,m} \sum_x a(x) e(f(g(x), m/M) + jh(x)) \\ &\quad + O\biggl(\frac{\lambda r + r}{M}\sum_x |a(x)|\biggr) \\ &\quad + O\biggl(\frac{r}{M}\sum_{j=0}^{\infty}\frac{\sin(2\pi rj/M)}{\sin(\pi j/M)}\, a_j \sum_x |a(x)| e(jh(x))\biggr),\end{aligned}$$

where

$$a_j = (\sin(\pi j/M)/(\pi j/M))^{r+1}, \qquad a_0 = 1,$$
$$b_{j,m} = a_j e(-(2m+1)j/(2M)).$$

This lemma is also simpler to use than the corresponding lemma of Zhai.

Using Lemma 1, we prove

THEOREM 1. *Let k be a sufficiently large positive integer such that $f(x)$, $g(x)$ and $h(x)$ are v.d.c. functions of order k and let $k_1 \in [2, k-1]$ and*

$k_2 \in [2, k-2]$ be the smallest integers such that $f(x)$, $g(x)$ and $h(x)$ are v.d.c. functions of orders k_1, 1 and k_2 respectively. Assume that $g(x) \ll x^{1-\alpha}$ for some $\alpha > 0$ and that for any m the functions $f_m(x)/h_m(x)$ are piecewise monotone on $\ll 1$ intervals and

$$|f_m(x)h_{m+1}(x)/(f_{m+1}(x)h_m(x)) - 1| \ggg 1.$$

Define

$$\varphi_j(y) = f^{(j)}(g^{-1}(y)), \qquad \phi_j(y) = h^{(j)}(g^{-1}(y))$$

and assume that for any m the functions $\varphi_j^{(m)}(y)/\phi_j^{(m)}(y)$ are piecewise monotone on $\ll 1$ intervals and

$$|\varphi_j^{(m)}(y)\phi_j^{(m+1)}(y)/(\varphi_j^{(m+1)}(y)\phi_j^{(m)}(y)) - 1| \ggg 1,$$

$$|\varphi_1^{(p)}(y)| \ll y^{2/P-3}, \qquad |\phi_1^{(p)}(y)| < y^{2/P-5} \qquad \text{for some integer } p > 1.$$

Then

$$S \equiv \sum_{X \le x \le 2X} e(f(x) + g(x)\{h(x)\}) \ll X\Delta_0,$$

where

$$\Delta_0 = X^{-\alpha/(3P)} + (G + X^{1/3})^{-1/(PK)}, \qquad G = g(X).$$

Also, if $f(x) = Ch(x)$ then the above estimate holds if $|C| > 1$ and

$$|S| \ll X\Delta_0 + X/G \qquad \text{if } |C| < 1.$$

THEOREM 2. *Let $f(x, y)$ be a real-valued function on $[X, 2X] \times [0, 1]$ such that for any y it is a v.d.c. function of order k. Assume that k is the smallest such integer. Assume also that $g(x)$ is a v.d.c. function of order 1 such that for some $a > 0$ we have $g(x) \ll x^{1-a}$ and, setting $h(n) = f(g^{-1}(n), n)$, assume that it is a v.d.c. function of order j. Let λ_k and μ_j be such that*

$$|\partial^k f(x, y)/\partial x^k| \asymp \lambda_k \quad \text{and} \quad |h^{(j)}(n)| \asymp \mu_j.$$

Then

$$S \equiv \sum_{X \le x \le 2X} e(f(x, \{g(x)\})) \ll X[\lambda_k^{1/(K-2)} + X^{-a/K} + G(X)^{-2/K} + \mu_j^{4/(KJ+K)}].$$

For the sequence $\{[n^\alpha] \log^\beta n\}$ considered by Rieger and Zhai,

$$f(x) = x^\alpha \log^\beta x, \qquad g(x) = -\log^\beta x, \qquad h(x) = x^\alpha,$$

so that if $\alpha\beta \neq 0$ the conditions of Theorem 1 are satisfied and one can use it to prove the uniform distribution of the sequence modulo 1 and to evaluate the discrepancy. One can do the same for $f(x) = x^\alpha$, $g(x) = x^\beta$ and $h(x) = x^\gamma$ with $\alpha \neq \gamma$ and $\beta < 1$, and some other functions.

2. Notation. We will use the following notation: $e(x) = \exp(2\pi i x)$; $f(x) \ll g(x)$ means that $f(x) = O(g(x))$; $f(x) \lll g(x)$ means that $f(x) \ll g(x)x^{\varepsilon}$; $f(x) \asymp g(x)$ means that $f(x) \ll g(x) \ll f(x)$; $\{x\}, [x]$ and $\|x\|$ are the fractional part, the integer part and the distance to the nearest integer functions; $|S|$ is the cardinality of the set S. For positive integers k, r etc., we write $K = 2^k$, $R = 2^r$ etc.

3. Proofs. To prove Lemma 1, we take

$$\chi_{r,m}(x) \equiv \chi_{r,m}(x;\delta) = (2\delta)^{-r} \int_{-\delta}^{\delta} \ldots \int_{-\delta}^{\delta} \chi_{0,m}(x + t_1 + \ldots + t_r)\, d\underline{t} \quad (m = 0, \ldots, M-1),$$

where $\chi_{0,m}(x)$ is the characteristic function of $[m/M, (m+1)/M)$ modulo 1. Expanding $\chi_{0,m}(x)$ into a Fourier series, we obtain

$$\chi_{r,m}(x) = (2\delta)^{-r} \int_{-\delta}^{\delta} \ldots \int_{-\delta}^{\delta} \bigg(\frac{1}{M} + \sum_{|j|=1}^{\infty} a_{j,m} e(x + t_1 + \ldots + t_r)\bigg)\, d\underline{t} = \frac{1}{M} + \sum_{|j|=1}^{\infty} a_{j,m} \bigg(\frac{\sin(2\pi j\delta)}{2\pi j\delta}\bigg)^{r} e(jx) \tag{3}$$

where

$$a_{j,m} = \frac{\sin(\pi j/M)}{\pi j}\, e\bigg(\frac{-(2m+1)j}{2M}\bigg).$$

We use (3) with $\delta = 1/M$ so that $a_{j,m}(\sin(2\pi j\delta)/(2\pi j\delta))^r = b_{j,m}/M$ from the lemma. Since $\sum_{m=0}^{M-1} \chi_{0,m}(x) = 1$, we have $\sum_{m=0}^{M-1} \chi_{r,m}(x) = 1$ and we obtain

$$\begin{aligned} S &= \sum_x a(x) \sum_{m=0}^{M-1} \chi_{r,m}(h(x)) e(f(g(x), \{h(x)\})) \\ &= \sum_x a(x) \sum_{m=0}^{M-1} \chi_{r,m}(h(x)) e(f(g(x), m/M)) \\ &\quad + \sum_x a(x) \sum_{m=0}^{M-1} \chi_{r,m}(h(x)) [e(f(g(x), \{h(x)\})) - e(f(g(x), m/M))]. \end{aligned}$$

The first sum is reduced to the first sum in (2) by using (3) with $\delta = 1/M$. To evaluate the second sum (which we denote with S_1), we divide it into two subsums: the first subsum, S_1', is over all m with $\|m/M\| > r/M$, and S_1'' is the remaining part of S_1.

If $\|m/M\| > r/M$ then $\chi_{r,m}(g(x)) = 0$ unless $|g(x) - m/M| < r/M$. Since $|e(a) - e(b)| = 2|\sin(\pi(b-a))| < 2\pi|a-b|$, we obtain

$$S_1' \ll \sum_x |a(x)| \sum_m \chi_{r,m}(h(x)) \frac{r\lambda}{M} = \frac{r\lambda}{M} \sum_x |a(x)|.$$

To evaluate S_1'', we write first

$$|S_1''| \le \sum_x |a(x)| \sum_m 2\chi_{r,m}(h(x)) \le 2 \sum_x |a(x)| \chi_1(h(x); 1/(2M))$$

where

$$\chi_1(t;\delta) \equiv (2\delta)^{-r} \int_{-\delta}^{\delta} \cdots \int_{-\delta}^{\delta} \chi(t + t_1 + \ldots + t_r)\, d\underline{t}$$

and $\chi(t)$ is the characteristic function of $[-r/M, r/M)$ modulo 1. Similarly to (3), we obtain

$$\chi_1(t; 1/(2M)) = \frac{2r}{M} + 2 \sum_{|j|=1}^{\infty} \frac{\sin(2\pi rj/M)}{\pi j} \left(\frac{\sin(\pi j/M)}{\pi j/M} \right)^r e(jt)$$

so that

$$|S_1''| \le \frac{4r}{M} \sum_x |a(x)| + 2 \sum_{j=1}^{\infty} \frac{\sin(2\pi rj/M)}{\sin(\pi j/M)}\, a_j e(jh(x)).$$

To prove the theorems, we need three more lemmas.

Lemma 2. *Let $f(x) \in C^{(k+j)}[X_1, X_2]$ with $k > 1$, $j > 0$ and $1 \le X = X_2 - X_1 \le X_1$. Assume that*

$$f^{(k)}(x) \le \lambda_k \quad \text{and} \quad f^{(k+j)}(x) \asymp \lambda_{k+j}.$$

Then

$$\Big| \sum_{X_1 \le x \le X_2} e(f(x)) \Big| \ll X[\lambda_k^{1/(K-2)} + (X^{-j-2}\lambda_k/\lambda_{k+j})^{4/(K(j+4))} + (\lambda_{k+j} X^{4+j-8/K})^{-4/(K(j+2))}.$$

Lemma 2 is a simple generalization of van der Corput estimates (for the proof, see [1, Lemma 4.1]).

Lemma 3 [2, Lemma 4.2]. *Let $f(x) \in C^2[X_1, X_2]$ be such that $f''(x) \asymp \lambda_2$ for $X_1 \le x \le X_2 = X_1 + X \le 2X_1$. Assume that $\|f'(x)\| \ge X\lambda_2$. Then*

$$\sum_{X_1 \le x \le X_2} e(f(x)) \ll X\sqrt{\lambda_2} + 1 + \min\{X; 1/\sqrt{\lambda_2}; 1/\|f'(X_2)\|; 1/\|f'(X_1)\|\}.$$

Lemma 4. *Let $f(x,y)$ be a real-valued function on $\{(x,y) : Y \le y \le 2Y,\ X_1 \equiv X_1(y) \le x \le X_2(y) \equiv X_2\}$ such that $f(x,y)$ is a v.d.c. function* *of order k as a function of x and either $g_1(y) \equiv f^{(k-1)}(X_1, y)$ or $g_2(y) \equiv f^{(k-1)}(X_2, y)$ is a v.d.c. function of order j. Assume that*

$$\frac{\partial^k f}{\partial x^k}(x,y) \asymp \lambda_k \quad \text{and} \quad g_i(y)^{(j)} \asymp \mu_j \quad \text{for a v.d.c. function } g_i(y).$$

Then

$$S \equiv \sum_{Y \le y \le 2Y} \Big| \sum_{X_1 \le x \le X_2} e(f(x,y)) \Big|$$
$$\ll XY(\lambda_k^{1/(K-2)} + X^{-4/(3K)} + Y^{-2/K} + \mu_j^{4/(KJ+K-8)}) \quad \text{if } k > 1$$

and

$$S \ll XY(\mu_j^{1/(J-1)} + 1/Y + \log X/X) \quad \text{if } k = 1.$$

Proof. If $k = 1$, we use van der Corput's Lemma to get

$$S \ll \sum_y \min\{X; 1/\|f_y(X_1, y)\| + 1/\|f_y(X_2, y)\|\}$$
$$\ll \sum_y \min\{X; 1/\|f_y(X_1, y)\|; 1/\|f_y(X_2, y)\|\}$$

and proceed as below. If $k = 2$ then we use van der Corput's estimates (Lemma 2 with $j = 0$) to get

$$S \ll X\sqrt{\lambda_2} + 1/\sqrt{\lambda_2}.$$

If $\lambda_2 \gg X^{-4/3}$, the above implies $S \ll XY\sqrt{\lambda_2} + YX^{2/3}$.

If $X\lambda_2 \equiv \Delta_0 \le X^{-1/3}$, we can evaluate S differently. We define

$$Y_i(\Delta) \equiv Y(\Delta) = |\{y \in [Y, 2Y] : \|g_i(y)\| \le \Delta\}|.$$

Using Lemma 3, we obtain

$$S \ll XY(X\lambda_2) + XY\sqrt{\lambda_2} + \sum_r \min\{X; 1/\sqrt{\lambda_2}; 1/(2^r \Delta_0)\} Y(2^r \Delta_0). \tag{4}$$

Now we need to evaluate $Y(\Delta)$. If μ_1 is small, we divide the interval $[Y, 2Y]$ into $\ll Y\mu_1 + 1$ subintervals of length $\ll 1/\mu_1$ such that $[g(y)]$ remains constant for all y in a subinterval. Each of them contains $\ll \Delta/\mu_1 + 1$ integers y such that $\|g(y)\| < \Delta$ so that

$$Y(\Delta) \ll (Y\mu_1 + 1)(\Delta/\mu_1 + 1) \ll Y\Delta + Y\mu_1 + 1.$$

If μ_1 is not small but μ_k is small for some $k > 1$, we use (3) with $r = 1$, $M = 3/\delta$, $m = 0$ and $m = M - 1$ to obtain

$$Y(\Delta) \le \min_{\delta \ge \Delta} Y(\delta),$$

where

$$Y(\delta) \ll Y\delta + \sum_{|l|=1}^{\infty} \min\{1/M; Ml^{-2}\} \Big| \sum_y e(lg(y)) \Big|$$
$$\ll Y\delta + \sum_l \min\{1/M; Ml^{-2}\}$$
$$\times [Y(l\mu_j)^{1/(J-2)} + Y^{1-2/J}\log Y + Y^{1-8/J+16J^{-2}} \mu_j^{-2/J}],$$

and $Y(\Delta) \ll Y\Delta + Y\mu_j^{1/(J-1)}$. We substitute this into (4) to obtain

$$S \ll XY\sqrt{\lambda_2} + X^2 Y\lambda_2 + XY\mu_j^{1/(J-1)} + X\sqrt{Y} \tag{5}$$
$$\ll XY\sqrt{\lambda_2} + X^{2/3}Y + XY\mu_j^{1/(J-1)} + X\sqrt{Y}.$$

This proves the lemma for $k = 2$. If $k > 2$, we apply H. Weyl–van der Corput inequality $m = k - 2$ times:

$$\Big|\frac{S}{XY}\Big|^M \ll Q^{-M/2} + \frac{1}{Q^{M-1}XY} \sum_{q_1=1}^{Q} \cdots \sum_{q_m=1}^{Q^{M/2}} \sum_y \Big| \sum_{X_1(\underline{q}) \le x \le X_2(\underline{q})} e(f_1(x,y)) \Big|,$$

where

$$M = 2^m, \qquad Q = \min\{\lambda_k^{-1/(2M-1)}; X^{2/M}; \mu_j^{-2/(M(J+1)-2)}\}$$

and

$$f_1(x,y) = q_1 \ldots q_m \int_0^1 \ldots \int_0^1 f_{x^m}(x + t_1 q_1 + \ldots + t_m q_m, y)\, d\underline{t}.$$

Using (5), we obtain

$$\Big|\frac{S}{XY}\Big|^M \ll Q^{-M/2} + \frac{1}{Q^{M-1}XY}$$
$$\times \sum_{q_1,\ldots,q_m} (XY\sqrt{q_1 \ldots q_m \lambda_k} + X^{2/3}Y + X\sqrt{Y} + XY(q_1 \ldots q_m \mu_j)^{1/(J-1)})$$
$$\ll Q^{-M/2} + \sqrt{Q^{M-1}\lambda_k} + X^{-1/3} + Y^{-1/2} + (Q^{M-1}\mu_j)^{1/(J-1)}$$
$$\ll \lambda_k^{M/(K-2)} + X^{-1/3} + Y^{-1/2} + \mu_j^{4M/(KJ+K-8)}.$$

To prove Theorem 1, we assume first that $G \equiv g(X) \ll X^{1/(3K)}$. We use Lemma 1 with $r = 3$ and $M = \max\{X^{1/(2K)}/G; GX^{1/(4K)}\}$ to obtain

$$S \ll \sum_{|j|=0}^{\infty} |a_j| \sum_{m=0}^{M-1} \Big| \sum_x e(f(x) + g(x)m/M + jh(x)) \Big|$$
$$+ \frac{XG}{M} + \sum_{j=1}^{\infty} |a_j| \Big| \sum_x e(jh(x)) \Big|.$$

Lemma 2 with $k = k_1 + 1$ and $m = 0$ shows that the last sum is

$$\ll X \sum_{j=1}^{\infty} \min\{1/M; M^3 j^{-4}\} \times [(j\lambda_{k_1+1})^{1/(2K_1-2)} + X^{-2/K} + (j\lambda_{k_1+1} X^{4-4/K_1})^{-1/K_1}]$$
$$\ll X[(M\lambda_{k_1+1})^{1/(2K_1-2)} + X^{-2/K} + (M\lambda_{k_1+1} X^{4-4/K_1})^{-1/K_1}] \ll X^{1-1/(4K)}.$$

To evaluate the first sum, for a fixed j, we divide the interval $[X, 2X]$ into $\ll \log X$ subintervals with $|f^{(p)}(x) + jh^{(p)}(x)| \asymp \lambda_p \gg |f^{(p)}(X)| X^{-\varepsilon_1}$ and one interval (which we denote with I) on which the last inequality does not hold, where $\varepsilon_1 > 0$ is a sufficiently small number and p is the smallest integer such that

$$|f^{(p)}(X)| X^{1-1/P} \le 1 \quad \text{and} \quad |jh^{(p)}(X) X^{1-1/P}| \le 1.$$

Obviously, $p < k$. The conditions of the theorem imply that if $x \in I$ then

$$|f^{(p+1)}(x) + jh^{(p+1)}(x)| \ggg |f^{(p+1)}(X)|.$$

Using Lemma 2 with $k = p$ and $m = 1$ if $x \in I$ and $m = 0$ otherwise, we find that the first sum is

$$\ll \sum_j \min\{1; M^2 j^{-2}\} X^{1-1/K} \log X \ll X^{1-1/(4K)}$$

so that

$$S \ll X^{1-1/(4K)}.$$

Now we assume that $G \gg X^{1/(3K)}$. We take $\varepsilon_0 = (G')^{4/(9P)} + G^{-4/(PK)}$ and define $a(x) = 1 - \chi(x)$ where $G' = g'(X)$ and $\chi(x)$ is the characteristic function of $[-\varepsilon_0, \varepsilon_0]$ modulo 1. Then

$$S = \sum_{X \le x \le 2X} a(g(x)) e(f(x) + g(x)\{h(x)\}) + O\Big(\sum_{X \le x \le 2X} \chi(g(x)) \Big).$$

The O-term is $\ll |\{x \in [X, 2X] : \|g(x)\| \le 3\varepsilon_0\}|$. As above, we divide the interval $[X, 2X]$ into $\ll XG' + 1$ subintervals of length $\ll 1/G'$ each such that $[g(X)]$ remains constant on each subinterval. The number of x in each subinterval such that $\|g(x)\| \le 3\varepsilon_0$ is $\ll 1 + \varepsilon_0/G'$ so that the O-term is

$$\ll (XG' + 1)(1 + \varepsilon_0/G') \ll X\varepsilon_0 + XG'.$$

Now we apply Lemma 1 with $r = 3$ and $M = GX^{1/(4K)}$ to obtain

$$S \ll \sum_{|j|=0}^{\infty} |a_j| \Big| \sum_{m=0}^{M-1} \sum_x a(h(x)) e(f(x) + jh(x) + g(x)m/M + mj/M) \Big|$$
$$+ X\varepsilon_0 + XG' + X^{1-1/(4K)} + \sum_{j=1}^{\infty} |a_j| \Big| \sum_x e(jh(x)) \Big|.$$

As above, the last sum is $\ll X^{1-1/(4K)}$. Now we need to evaluate the first sum. We denote it by Σ and denote the sum over m and x by S_1; summing over m, we obtain

$$S \ll \sum_x a(g(x))e(f(x)+jh(x))\frac{e(g(x))-1}{e((j+g(x))/M)-1}.$$

Let G_1 and G_2 be the minimum and maximum of $g(x)$ on $[X,2X]$.

Setting

$$I(y)=\{x\in[X,2X]: y+\varepsilon_0\le g(x)\le y+1-\varepsilon_0\}\equiv[X_1(y),X(y)]$$

and writing $j=u+vM$ with $|u|<M/2$, we obtain

$$S_1 \ll \sum_{G_1\le y\le G_2}\Bigg|\sum_{x\in I(y)} e(f(x)+jh(x))\frac{e(g(x))-1}{e((u+g(x))/M)-1}\Bigg|.$$

If $X_1(y)\le x\le X(y)-1$ then $1/|e((u+g(x))/M)-1|\ll M/(|u+y|+\varepsilon_0)$ and

$$\bigg|\frac{1}{e((u+g(x))/M)-1}-\frac{1}{e((u+g(x+1))/M)-1}\bigg|\ll\frac{MG'}{(y+u)^2+\varepsilon_0^2}.$$

Abel's summation formula and the above inequalities yield

$$S_1\ll\sum_y\bigg\{\frac{M}{|y+u|+\varepsilon_0}\Big|\sum_{x\in I(y)}e(\psi(x))\Big| + \frac{MG'}{(y+u)^2+\varepsilon_0^2}\sum_{X_1(y)\le s\le X(y)}\Big|\sum_{s\le x\le X(y)}e(\psi(x))\Big|\bigg\},$$

where $\psi(x)=f(x)+jh(x)+ig(x)$ and $i=0$ or 1. We set $X_0=1/G'$. Then the second sum above is

$$\ll MG'\sum_{s\le X_0}\sum_y\frac{1}{(u+y)^2+\varepsilon_0^2}\Big|\sum_{X(y)-s\le x\le X(y)}e(\psi(x))\Big|$$

so we get

$$(6)\quad \Sigma\ll\sum_{|v|=0}^{\infty}\frac{1}{v^4+1}\sum_{u\le M/2}\sum_y\bigg\{\frac{1}{|y+u|+\varepsilon_0}\Big|\sum_{x\in I(y)}e(\psi(x))\Big| + \frac{G'}{(y+u)^2+\varepsilon_0^2}\sum_{s\le X_0}\Big|\sum_{X(y)-s\le x\le X(y)}e(\psi(y))\Big|\bigg\}$$

$$\ll\max_{s\le X_0}\sum_{|v|=0}^{V}\frac{1}{v^4+1}\sum_{|u|\le M/2}\sum_y\frac{1}{|y+u|+\varepsilon_0^2}\Big|\sum_{X(y)-s\le x\le X(y)}e(\psi(x))\Big|+R',$$

where

$$V = X^{1/(4K)} \quad \text{and} \quad R' \ll \sum_{v=V}^{\infty} v^{-4} X(\log X + \varepsilon_0^{-2}) \ll X^{1-1/(4K)}.$$

Let r be the smallest integer such that

$$|f^{(r)}(x)| \le X^{2/R-1} \quad \text{and} \quad |jh^{(r)}(x)| \le X^{2/R-1}.$$

Obviously, $1 < r < k$. To evaluate the sum in (6) we need to evaluate

$$Y(\varDelta) \equiv |\{y \in [G_1, G_2] : \|\varphi(y)\| \le \varDelta\}| \quad \text{where} \quad \varphi(y) = A\psi^{(r-1)}(X(y))$$

and $A \le X_0^{1-1/R}$ is a fixed number. Assume that t is the smallest integer such that

$$|(Af^{(r-1)}(X(y)))^{(t)}| \le G^{2/T-1} \quad \text{and} \quad |(Ajh^{(r-1)}(X(y)))^{(t)}| \le G^{2/T-1}.$$

We take a small constant $\varepsilon > 0$ and divide the set of all y into $\ll \log X$ intervals with

$$|\varphi^{(t)}(y)|\lambda_t \ge A(|(f^{(r-1)}(X(y)))^{(t)}| + |(jh^{(r-1)}(X(y)))^{(t)}|)X_0^{-\varepsilon_1}$$

and at most one interval, I, in which the above inequality is not satisfied. The conditions of the theorem imply that if $y \in I$ then

$$|\varphi^{(t+1)}(y)| \gg A|(f^{(r-1)}(X(y)))^{(t+1)}|G^{-\varepsilon_1}.$$

Using Lemma 2 with $k = t$ and $m = 0$ if $y \notin I$ and $m = 0$ otherwise as above we obtain

$$\begin{aligned}
(7) \qquad Y(\varDelta) &\le \min_{\delta \ge \varDelta} Y(\delta) \\
&\ll \min_{\delta} \Big(G\delta + \sum_{j=1}^{\infty} \min\{\delta; 1/(\delta j^2)\} \Big| \sum_{y} e(j\varphi(y)) \Big| \Big) \\
&\ll G \min_{\delta} \Big(\delta + \sum_{j=1}^{\infty} \min\{\delta; 1/(\delta j^2)\} \Big) (G^{-1/T} + (j\mu_t)^{1/(T-2)}) \\
&\ll G \min_{\delta} (\delta + G^{-1/T} + (\mu_t/\delta)^{1/(T-2)}) \\
&\le G(\varDelta + G^{-1/P} + \mu_t^{1/(T-1)}) \\
&\ll G(\varDelta + G^{-1/P}).
\end{aligned}$$

To evaluate the sum in (6) we assume first that $r = 2$. We divide the interval $[X(y) - s, X(y)]$ into $\ll \log X$ subintervals with $\varphi''(x) \asymp \lambda_2$ and consider one of them, corresponding to the largest subsum. We denote it by $S(u, v, y)$.

If $\lambda_2 \ge X_0^{-4/3}$, we use Lemma 2 with $k = 2$ and $m = 0$ and obtain

$$S(u, v, y) \ll X_0^{2/3}.$$

If $\lambda_2 \le X_0^{-4/3}$ we use Lemma 3 to evaluate $S(u,v,y)$ if

$$\|\psi'(X(y))\| \ge CX_0\lambda_2 \equiv \Delta_0 \tag{8}$$

with an appropriate C or evaluate it trivially otherwise.

Note that if (8) holds then for all $x \in [X(y)-s, X(y)]$ we have

$$\|\psi'(x)\| = \|\psi'(X(y)) + O(\lambda_2 X_0)\| \asymp \|\psi'(X(y))\|.$$

Summing over all u and y and using (7), we obtain

$$\begin{aligned}(9)\quad S(v) \equiv \sum_{u,y} S(u,v,y) &\ll (\log X + \varepsilon_0^{-2}) \\ &\times \Big[X_0^{2/3} + \sum_l \min\{X_0; 1/(\Delta_0 2^l)\} Y(\Delta_0 2^l) + X_0 Y(\Delta_0)\Big] \log X \\ &\ll (\log X + \varepsilon_0^{-2})(G X_0^{2/3} + X_0 G^{1-1/P}) \log^2 X.\end{aligned}$$

If $r > 2$, we apply Hölder's inequality to get

$$\begin{aligned}S(v) \ll \bigg(\sum_{u,y} \frac{1}{|y+u| + \varepsilon_0^2}\bigg)^{1-4/R} \\ \times \bigg(\sum_{u,y} \frac{1}{|y+u| + \varepsilon_0^2} \Big|\sum_x e(\psi(x))\Big|^{R/4}\bigg)^{4/R} \log X.\end{aligned}$$

Now we use H. Weyl–van der Corput inequality $r-2$ times with $Q = X_0^{4/R}$ to obtain

$$\begin{aligned}S(v) \ll [G(\log X + \varepsilon_0^{-2})]^{1-4/R} \bigg[\sum_{y,u} \frac{1}{|y+u| + \varepsilon_0^2} \Big(X_0^{R/4} Q^{-R/8} \\ + X_0^{R/4-1} Q^{1-R/4} \sum_{q_1=1}^{Q} \cdots \sum_{q_{r-2}=1}^{Q^{R/8}} \Big|\sum_x e(A\psi_1(x))\Big|\Big)\bigg]^{4/R}\end{aligned}$$

where

$$A = q_1 \ldots q_{r-2} \quad \text{and} \quad \psi_1(x) = \int_0^1 \ldots \int_0^1 \psi(x + q_1 t_1 + \ldots + q_{r-2} t_{r-2})\, d\underline{t}.$$

Using (7) we obtain, as in the proof of (9),

$$\begin{aligned}S(v) &\ll \varepsilon_0^{-2} X Q^{-1/2} \log^2 X + [\varepsilon_0^{-2} X]^{1-4/R} \log X \\ &\quad \times \Big[Q^{1-R/4} \sum_{q_1,\ldots,q_{r-2}} \varepsilon_0^{-2} (X_0^{2/3} G + X_0 G^{1-1/P})\Big]^{4/R} \\ &\ll \varepsilon_0^{-2} X [X_0^{-4/(3R)} + G^{-4/(PR)}] \log^2 X,\end{aligned}$$

and

$$\sum_v S(v) \ll \varepsilon_0^{-2} X[X_0^{-4/(3R)} + G^{-4/(PR)}] \log^2 X \ll X^{1-a/(3K)} + XG^{-1/(PK)}.$$

To prove Theorem 2 we set $n = [g(x)]$. Let G_1 and G_2 be the minimum and maximum of $g(x)$ on $[X, 2X]$ and $G' = g'(X)$. Using Lemma 4, we obtain

$$S \ll \Big(\sum_{x,n} 1\Big)(\lambda_k^{1/(K-2)} + (G')^{-4/(3K)} + G^{-2/K} + \mu_j^{4/(KJ+K-8)}).$$

Since $\sum_{x,n} 1 \ll X$ and $G' = g_1(X)/X \ggg X^{-a}$, this completes the proof.

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