ACTA ARITHMETICA
147.1 (2011)

Addendum to the papers "On polynomials taking small values at integral arguments I, II"

(Acta Arith. 42 (1983), 189–196; ibid. 106 (2003), 115–121)

by

R. Dvornicich and U. Zannier (Pisa)

1. Introduction. It has recently occurred to us that the main theorem proved in the paper [DZ] actually admits a more precise formulation, which in fact is contained in that proof. It is the purpose of this short note to state this and, as a consequence, to state more precise versions of the results of the paper [DTZ], which relied upon [DZ]. Also, we shall take this opportunity to correct an inaccurate assertion made in Remark (ii) of [DTZ] (see Remark 1 below).

Before stating the main result, we recall the definition of upper and lower asymptotic density of an increasing sequence $\mathcal{A}$ of natural numbers. By abuse of language, in this paper we shall identify a sequence of natural numbers with the corresponding subset of $\mathbb{N}$; this is justified because we consider only increasing sequences.

For such a sequence we put, for a real number $t \geq 0$,

$$\mathcal{A}(t) = \{m \in \mathcal{A} : m \leq t\}$$

and

$$\bar{d}(\mathcal{A}) = \limsup_{t\to\infty} \frac{|\mathcal{A}(t)|}{t}, \qquad \underline{d}(\mathcal{A}) = \liminf_{t\to\infty} \frac{|\mathcal{A}(t)|}{t}.$$

Theorem 1. *Let $F \in \mathbb{R}[X,Y]$ have $\deg_Y F > 0$. Assume that $\mathcal{A}$ is an increasing sequence of natural numbers with $\bar{d}(\mathcal{A}) > 0$ such that there exists a function $y : \mathcal{A} \to \mathbb{Z}$ satisfying*

$$|F(a, y(a))| = o\Big(\sup_{|\xi - y(a)|\leq 1} \Big|\frac{\partial F}{\partial Y}(a,\xi)\Big|\Big).$$

Then there exist polynomials $Q_1, \ldots, Q_r \in \mathbb{Q}[X]$ with $r \leq \deg_Y F$ and an increasing sequence $\mathcal{B} \subset \mathcal{A}$ such that:

2010 *Mathematics Subject Classification*: 11C08, 12E05.
Key words and phrases: polynomials, integral points.

DOI: 10.4064/aa147-1-4

(i) $\bar{d}(\mathcal{A}\setminus\mathcal{B})=0$.

(ii) *For every* $b\in\mathcal{B}$ *there exists an index* $i\in\{1,\ldots,r\}$ *with* $y(b)=Q_i(b)$.

We proceed to state a corresponding sharpening of Theorem 2 of [DTZ]; not only do we give a more precise conclusion, but also we allow the polynomials to have real coefficients, rather than rational.

THEOREM 2. *Let* $F\in\mathbb{R}[X,Y]$ *be such that* $\partial F/\partial Y$ *is nonconstant and such that, for every* $h\in\mathbb{R}[X]$, $F(X,Y+h(X))$ *is not a polynomial in* Y *only. Assume that* $\mathcal{A}$ *is an increasing sequence of natural numbers with* $\bar{d}(\mathcal{A})>0$ *such that there exists a function* $y:\mathcal{A}\to\mathbb{Z}$ *satisfying* $|F(a,y(a))|=o(\sqrt{a})$ *for every* $a\in\mathcal{A}$. *Then there exist an increasing sequence* $\mathcal{B}\subset\mathcal{A}$ *such that* $\bar{d}(\mathcal{A}\setminus\mathcal{B})=0$ *and polynomials* $Q_1,\ldots,Q_r\in\mathbb{Q}[X]$ *with* $r\le\deg_Y F$ *satisfying:*

(i) *for all* $i=1,\ldots,r$, $F(X,Q_i(X))$ *is a constant,*

(ii) *for every* $b\in\mathcal{B}$ *there exists* $i\in\{1,\ldots,r\}$ *with* $y(b)=Q_i(b)$.

The nonconstancy assumptions did not appear in [DTZ, Theorem 2], but it is very easy to see that they cannot be omitted in order that the present sharpened conclusion holds.

Also, note that if $F(X,Y+h(X))$ is constant in X for some $h\in\mathbb{R}[X]$, then the "algebraic function" solutions $Y=\xi_1(x),\ldots,\xi_d(x)$ of $F(x,Y)=0$ are polynomials which satisfy $\xi_i(x)=\xi_j(x)+c_{ij}$ for constants $c_{ij}\in\mathbb{C}$, and conversely.

Finally, we state a modification of Theorem 1 of [DTZ]. For this, we recall a definition from [DTZ]. For a sequence $\mathcal{A}\subset\mathbb{N}$, a polynomial $F\in\mathbb{R}[X,Y]$ with $\deg_Y F>0$, and a positive real number t, we put

$$S_{\mathcal{A},F}(t):=\max_{x\in\mathcal{A}(t)}\min_{y\in\mathbb{Z}}|F(x,y)|.$$

We note that the minima are attained because F has positive degree in Y. With this notation we have:

THEOREM 3. *Let* $F\in\mathbb{R}[X,Y]$ *be such that* $\partial F/\partial Y$ *is nonconstant and, for every* $h\in\mathbb{R}[X]$, $F(X,Y+h(X))$ *is not a polynomial in* Y *only. Let* $\mathcal{A}$ *be a sequence with* $\underline{d}(\mathcal{A})>0$. *Then either* $S_{\mathcal{A},F}(t)\gg\sqrt{t}$ *for* $t\to\infty$ *or there exist an increasing sequence* $\mathcal{B}\subset\mathcal{A}$ *such that* $\underline{d}(\mathcal{A}\setminus\mathcal{B})=0$ *and polynomials* $Q_1,\ldots,Q_r\in\mathbb{Q}[X]$ *with* $r\le\deg_Y F$ *satisfying:*

(i) *for all* $i=1,\ldots,r$, $F(X,Q_i(X))$ *is a constant,*

(ii) *for every* $b\in\mathcal{B}$ *and every integer* l *such that*

$$|F(b,l)|=\min_{y\in\mathbb{Z}}|F(b,y)|$$

there exists $i\in\{1,\ldots,r\}$ *with* $l=Q_i(b)$.

Again, the nonconstancy assumptions did not appear in [DTZ, Theorem 1]; however these assumptions cannot be omitted here, as is easy to see. Also, in the cases where they are not both verified, the function $S_{\mathcal{A},F}(t)$ is bounded independently of $\mathcal{A}$ (see the proof of the Corollary below), so we have a tame behaviour. We leave to the interested reader the easy task of finding an "explicit" description of $S_{\mathbb{N},F}(t)$ in such cases.

On the other hand, in the cases covered by Theorem 3, the conclusion is more precise than [DTZ, Theorem 1], and also works for polynomials over $\mathbb{R}$ rather than over $\mathbb{Q}$ (which was not the case in [DTZ]).

Finally, we note that both the assumption and the conclusion concern lower density (unlike the previous statements).

Remark. In all the above three theorems, the assumption that the (upper or lower) density of $\mathcal{A}$ is positive may be omitted. In fact, if such a density is zero, the conclusion turns out to be trivially satisfied in each case, on taking $\mathcal{B} = \emptyset$ and $r = 0$.

We have decided to leave the assumption to focus on the relevant cases of the results.

We conclude this section with a corollary concerning the case $\mathcal{A} = \mathbb{N}$:

Corollary. *Let $F \in \mathbb{R}[X, Y]$. Either $S_{\mathbb{N},F}(t) \gg \sqrt{t}$ or $S_{\mathbb{N},F}(t) = O(1)$.*

2. Proofs. In all the proofs which follow we shall indicate only the variations with respect to the arguments in [DZ], [DTZ]. We start with Theorem 1.

Proof of Theorem 1. For large $x \in \mathbb{C}$ the roots of $F(x, Y) = 0$ are given by Puiseux expansions $\rho_1(x), \ldots, \rho_D(x)$, where $D = \deg_Y F$. As in (5), p. 193, of [DZ], the assumptions imply that

$$\min_j |y(a) - \rho_j(a)| \to 0 \quad \text{for } a \in \mathcal{A},\ a \to \infty.$$

Now, we may write the Puiseux series $\rho_j(x)$ as $P_j(x^{1/e}) + \delta_j(x^{-1/e})$, where the $P_j \in \mathbb{C}[x]$, $e = e_F > 0$ is an integer, and $\delta_j \in \mathbb{C}[[x]]$ vanishes at 0. These series converge for large $x \in \mathbb{C}$, and the determinations of the roots $x^{1/e}$ are irrelevant, since a change of determination merely permutes the series.

We now renumber the indices $j = 1, \ldots, D$ so that $P_j \in \mathbb{Q}[x^e]$ precisely for $j = 1, \ldots, r$.

Next, we define $Q_j(x) = P_j(x^{1/e})$ for $j = 1, \ldots, r$; note that $Q_j \in \mathbb{Q}[x]$. In [DZ, p. 194], it is proved that there exists a sequence $\mathcal{A}' \subset \mathcal{A}$ with $\bar{d}(\mathcal{A}') = 0$ such that, for all large $b \in \mathcal{A} \setminus \mathcal{A}'$,

$$\min_{i=1,\ldots,r} |y(b) - Q_i(b)| = 0.$$

Let us now define $\mathcal{A}''$ by adding to $\mathcal{A}'$ the finitely many $b \in \mathcal{A} \setminus \mathcal{A}'$ that do not satisfy this formula. Then, putting $\mathcal{B} := \mathcal{A} \setminus \mathcal{A}''$, we obtain precisely the statement of Theorem 1. ∎

Proof of Theorem 2. We follow [DTZ], and put

$$\sigma(a) = \sup_{|\xi - y(a)| \le 1} \left| \frac{\partial F}{\partial Y}(a, \xi) \right|.$$

As in that paper (pp. 118, 119 up to l. −9), the key point is to prove inequality (3) of p. 118, that is,

$$\sigma(a) \gg \sqrt{a} \quad \text{for all large } a \in \mathcal{A}. \tag{1}$$

To prove this step, we reduce as in [DTZ] to the case when the second coefficient of F with respect to Y vanishes. To do this, we choose $h \in \mathbb{R}[X]$ such that $F_1(X, Y) := F(X, Y + h(X))$ has vanishing second coefficient, replacing $y(a)$ with $y(a) - h(a)$. At this point we follow step-by-step the said arguments of [DTZ, pp. 118–119]. (We note that the exceptional situations of Case 2 in [DTZ, p. 119], i.e. $\varphi_0(X)$ constant or $D = \deg_Y F = 1$, in the notation of [DTZ], do not occur here, in view of the present assumptions.)

Also, an inspection shows that these arguments of [DTZ] lead to inequality (1) independently of our present assumption that the $y(a)$ are integers: (1) also holds if they are arbitrary real numbers.

Now, combining inequality (1) with the assumption $|F(a, y(a))| = o(\sqrt{a})$, we deduce

$$|F(a, y(a))| = o\Big(\sup_{|\xi - y(a)| \le 1} \left| \frac{\partial F}{\partial Y}(a, \xi) \right| \Big) \quad \text{for } a \in \mathcal{A}.$$

It finally suffices to apply Theorem 1 to reach the desired conclusion. ∎

Proof of Theorem 3. By assumption we have $\underline{d}(\mathcal{A}) > 0$, so there exists a number $c > 0$ such that $|\mathcal{A}(t)| > ct$ for all large integers t.

Let us assume that the first alternative $S_{\mathcal{A},F}(t) \gg \sqrt{t}$ is not true; this means that there exists an increasing infinite sequence of positive integers $t_1 < t_2 < \cdots$ such that $S_{\mathcal{A},F}(t_n) < \sqrt{t_n}/n$. In view of our definitions, this means that for all large n we have

$$\max_{x \in \mathcal{A}(t_n)} \min_{y \in \mathbb{Z}} |F(x, y)| \le \sqrt{t_n}/n. \tag{2}$$

We now define $\mathcal{A}'$ to be the union of the intersections of $\mathcal{A}$ with the intervals $[t_n/n, t_n]$:

$$\mathcal{A}' := \bigcup_{n \in \mathbb{N}} (\mathcal{A} \cap [t_n/n, t_n]).$$

Let us first note that $\underline{d}(\mathcal{A} \setminus \mathcal{A}') = 0$: in fact, note that in the interval $[1, t_n]$ the sequence $\mathcal{A}'$ has at least $|\mathcal{A}(t_n)| - t_n/n$ elements. Hence $(\mathcal{A} \setminus \mathcal{A}')(t_n)$ has at most t_n/n elements in that interval, which proves the claim.

Let us also note that $\min_{y\in\mathbb{Z}}|F(a,y)| = o(\sqrt{a})$ for $a \in \mathcal{A}'$. In fact, first, if $a \in \mathcal{A}'$ then a lies in some set $\mathcal{A}\cap[t_n/n, t_n]$, whence $\min_{y\in\mathbb{Z}}|F(a,y)| \le \max_{x\in\mathcal{A}(t_n)}\min_{y\in\mathbb{Z}}|F(x,y)| \le \sqrt{t_n}/n$. On the other hand, $a \ge t_n/n$, so $t_n \le na$ and we find (for $a \in \mathcal{A}'$)

$$\min_{y\in\mathbb{Z}}|F(a,y)| \le \frac{1}{n}\sqrt{t_n} \le \frac{1}{\sqrt{n}}\sqrt{a}. \tag{3}$$

For $a \in \mathcal{A}'$, let us now define $M(a)$ to be the finite set of integers z such that $|F(a,z)| = \min_{y\in\mathbb{Z}}|F(a,y)|$. Note that $|M(a)| \le 2D$ for all $a \in \mathcal{A}'$, except possibly for the finitely many $a \in \mathcal{A}'$ for which $F(a,Y)$ is constant.

We may construct sequences $(y_1(a))_{a\in\mathcal{A}'}, \dots, (y_{2D}(a))_{a\in\mathcal{A}'}$ with the property that $M(a) = \{y_1(a), \dots, y_{2D}(a)\}$ for all large $a \in \mathcal{A}'$. (If necessary, here we allow the same element of $M(a)$ to appear more than once.)

Observe that, for each $j = 1, \dots, 2D$, we have $|F(a, y_j(a))| = o(\sqrt{a})$ for $a \in \mathcal{A}'$, as follows from (3).

For each $j = 1, \dots, 2D$, we can therefore apply Theorem 2 with $\mathcal{A}'$ in place of $\mathcal{A}$, and $Y_j(a)$ in place of $y(a)$. We then obtain a sequence $\mathcal{B}_j$ with the properties of the sequence $\mathcal{B}$ therein, so in particular $\bar{d}(\mathcal{A}'\setminus\mathcal{B}_j) = 0$.

Let us define $\mathcal{B} := \bigcap_{j=1}^{2D}\mathcal{B}_j$; then $\bar{d}(\mathcal{A}'\setminus\mathcal{B}) \le \sum_{j=1}^{2D}\bar{d}(\mathcal{A}'\setminus\mathcal{B}_j)$.

Then, noting that $\underline{d}(\mathcal{A}\setminus\mathcal{B}) \le \underline{d}(\mathcal{A}\setminus\mathcal{A}') + \bar{d}(\mathcal{A}'\setminus\mathcal{B})$, we obtain $\underline{d}(\mathcal{A}\setminus\mathcal{B}) = 0$, proving claim (i) of Theorem 3.

We obtain the remaining assertion directly from Theorem 2(ii), and the present construction of our sequences $(y_j(a))$. ∎

Proof of Corollary. Let us first assume that F does not satisfy the assumptions of Theorem 3.

A first case occurs if $\partial F/\partial Y$ is constant, which means that $F(X,Y) = cY + P(X)$, where $c \in \mathbb{R}$ and $P \in \mathbb{R}[X]$. If $c = 0$ we have $\min_{y\in\mathbb{Z}}|F(x,y)| = |P(x)|$, hence $S_{\mathbb{N},F}(t) \asymp t^{\deg P}$, proving that we fall into one of the alternatives in the conclusion. If $c \ne 0$, then $\min_{y\in\mathbb{Z}}|F(x,y)|$ is bounded by $|c|$, and again we are done.

A second case occurs when, for some $h \in \mathbb{R}[X]$, $F(X, Y + h(X)) = Q(Y)$ is a polynomial in $\mathbb{R}[Y]$. Now $\min_{y\in\mathbb{Z}}|F(x,y)|$ is again bounded: it suffices to note that $F(x, [h(x)])$ is bounded.

Finally, let us assume that $F(X,Y)$ satisfies the assumptions for Theorem 3. Applying that theorem we find that either $S_{\mathbb{N},F}(t) \gg \sqrt{t}$ for $t \to \infty$ or there exist an increasing sequence $\mathcal{B} \subset \mathbb{N}$ such that $\bar{d}(\mathcal{B}) = 1$ and polynomials $Q_1, \dots, Q_r \in \mathbb{Q}[X]$ with $r \le \deg_Y F$ such that:

(i) $F(X, Q_i(X))$ is constant for all i,

(ii) for every $b \in \mathcal{B}$ and every integer l such that
$$|F(b,l)| = \min_{y\in\mathbb{Z}}|F(b,y)|$$
there exists $i \in \{1, \dots, r\}$ with $l = Q_i(b)$.

From (ii) we infer that the set of integers $m \in \mathbb{N}$ such that there exists an index $i = 1, \ldots, r$ with $Q_i(m) \in \mathbb{Z}$ contains $\mathcal{B}$. However this set is a union of arithmetic progressions, and since $\bar{d}(\mathcal{B}) = 1$, we conclude that this set equals $\mathbb{N}$.

Now, for an integer $m \in \mathbb{N}$ let us pick $i \in \{1, \ldots, r\}$ with $Q_i(m) \in \mathbb{Z}$. Then $\min_{y\in\mathbb{Z}} |F(m, y)| \le |F(m, Q_i(m))|$. However $F(m, Q_i(m))$ depends only on i, and hence is bounded independently of m, which in turn implies that $S_{\mathbb{N},F}(t)$ is a bounded function. ∎

3. Remarks. 1. We point out an inaccuracy in Remark (ii) of [DTZ], which concerned Theorems 1 and 2 therein, and a polynomial $f(x)$ such that $F(x, f(x))$ is constant. We stated that this polynomial could be chosen to take integral values on a sequence $\mathcal{B} \subset \mathcal{A}$ such that $\bar{d}(\mathcal{A} \setminus \mathcal{B}) = 0$. Such a strong conclusion is false in general, as the following example shows:

We take $\mathcal{A} = \mathbb{Z}$, $F(X, Y) = (2Y - X)(2Y - X - 1)$. It is readily checked that $S_{\mathbb{Z},F}(t) = 0$ for every t. However for every polynomial $f(x)$ which is integral-valued on $\mathbb{Z}$, clearly $F(x, f(x))$ is not constant.

The correct sharpening is given in the present Addendum, replacing a single polynomial by a finite number of suitable ones.

In the case of the above example, the two polynomials $f_0(x) = x/2$ and $f_1(x) = (x + 1)/2$ are such that: (i) for each integer $x \in \mathbb{Z}$ either $f_0(x)$ or $f_1(x)$ is integral, and (ii) $F(x, f_i(x)) = 0$ for $i = 0, 1$.

2. The arguments given for the proof of Theorem 2 (which originate in [DTZ]) yield a comparison result between the quantities

$$\sup_{|\xi - y(a)| \le 1} \left| \frac{\partial F}{\partial Y}(a, \xi) \right| \quad \text{and} \quad \min_{F(a,\eta)=0} |y(a) - \eta|.$$

In fact, we have the following general statement:

Let $f \in \mathbb{C}[x]$ be a polynomial of degree $d > 0$, and for $z \in \mathbb{C}$, define

$$q(z) := \frac{|f(z)|}{\sup_{|u-z|\le 1} |f'(u)|}.$$

Then, if $q(z) \le (d2^d)^{-1}$, we have

$$q(z) \ll_d \min_{f(\eta)=0} |z - \eta| \ll_d q(z)^{1/d}.$$

The result expresses the fact that if a value $f(z)$ is small compared to the derivative in a neighbourhood, then z must be near to a root of f, and conversely. The crucial point is that the implied constants do not depend on the coefficients of f but only on its degree.

We omit the proof, which can be given by following the above arguments in the proof of Theorem 2.

3. The result of the Corollary is best possible in the sense that $\sqrt{t}$ cannot be replaced by a function which grows faster. This follows from the example $F(X,Y) = Y^2 - X$.

References

[DTZ] R. Dvornicich, S. P. Tung and U. Zannier, *On polynomials taking small values at integral arguments II*, Acta Arith. 106 (2003), 115–121.

[DZ] R. Dvornicich and U. Zannier, *On polynomials taking small values at integral arguments*, ibid. 42 (1983), 189–196.

R. Dvornicich
Università di Pisa
Largo Pontecorvo, 5
56127 Pisa, Italy
E-mail: dvornic@dm.unipi.it

U. Zannier
Scuola Normale Superiore
Piazza dei Cavalieri, 7
56126 Pisa, Italy
E-mail: u.zannier@sns.it

Received on 19.1.2010 (6272)