ACTA ARITHMETICA
147.3 (2011)

A new kind of Diophantine equations

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1. Introduction. Among various kinds of Diophantine equations, a famous one is

$$\binom{n}{k} = m^l, \quad 2 \le k \le n-2,\ l \ge 2. \tag{1.1}$$

The complete solution of (1.1) was given by Erdős [3] for $4 \le k \le n-4$, and by Győry [5] for $k \le 3$ and $k \ge n-3$. In 1975, Erdős and Selfridge [4] proved that the product of consecutive positive integers is never a perfect power. Actually the Diophantine equation

$$n(n-1)\cdots(n-k+1) = bm^l, \quad 2 \le k \le n,\ l \ge 2, \tag{1.2}$$

under the assumption $P(b) < k$ was solved in [4], where $P(b)$ denotes the greatest prime divisor of b, with $P(1) = 1$. As a common generalization of the above two results, (1.2) was resolved under the assumption $P(b) \le k$ by Saradha [11] and Győry [6] for $k \ge 4$ and $k \le 3$ respectively. When $\gcd(n,d) = 1$, the Diophantine equation

$$n(n-d)\cdots(n-(k-1)d) = bm^l, \quad k \ge 2,\ (k-1)d < n,\ l \ge 2,$$

under the assumption $P(b) \le k$ has also been considered. For related results, we refer to [1], [2], [7], [9], [12].

For $t \ge 1$, let $(2t-1)!! = 1 \cdot 3 \cdot \ldots \cdot (2t-1)$, and define an analogue of the binomial coefficient

$$\binom{n}{k}_{!!} = \frac{(2n-1)(2n-3)\cdots(2n-2k+1)}{(2k-1)(2k-3)\cdots 1} = \frac{(2n-1)!!}{(2k-1)!!(2(n-k)-1)!!}$$

for $1 \le k \le n-1$. In this paper, we consider in which case $\binom{n}{k}_{!!}$ is a power of a rational number. We completely solve the equation

2010 *Mathematics Subject Classification*: Primary 11D99; Secondary 11A25, 11B65.
Key words and phrases: binomial coefficient, Diophantine equation, perfect power.

DOI: 10.4064/aa147-3-4

$$\binom{n}{k}_{!!} = \left(\frac{m}{M}\right)^l \tag{1.3}$$

in integers $n \geq 4$, $2 \leq k \leq n-2$, $m \geq 1$, $M \geq 1$, $\gcd(m, M) = 1$, $l \geq 2$.

THEOREM 1.1. *All the solutions of* (1.3) *are*

$$l = k = 2, \quad M = 1, \quad 2n + m\sqrt{3} = (2+\sqrt{3})^{2t+1} + 2 \quad (t \in \mathbb{N}^*)$$

and

$$l = 2, \quad k = n-2, \quad M = 1, \quad 2n + m\sqrt{3} = (2+\sqrt{3})^{2t+1} + 2 \quad (t \in \mathbb{N}^*).$$

2. Preliminaries. Due to the observation $\binom{n}{k}_{!!} = \binom{n}{n-k}_{!!}$, we assume $n \geq 2k$ in the following. Define

$$\varDelta = \varDelta(n, k) = (2n-1)(2n-3)\cdots(2n-2k+1).$$

We have

LEMMA 2.1. *Let $k \geq 9$. Then $\varDelta$ is divisible by a prime exceeding $2k$.*

Proof. Write $W(\varDelta)$ for the number of terms in $\varDelta$ divisible by a prime exceeding k and $\pi(x)$ for the number of primes not exceeding x. It is shown in [8, Theorem 1] that

$$W(\varDelta) \geq \pi(2k) - \pi(k) + 1, \quad k \geq 9. \tag{2.1}$$

On the other hand, we note that every prime exceeding k divides at most one term of $\varDelta$, and every odd prime less than k divides $\varDelta$. Hence

$$\omega(\varDelta) - \pi(k) + 1 \geq W(\varDelta), \tag{2.2}$$

where $\omega(\varDelta)$ denotes the number of distinct prime divisors of $\varDelta$. Combining (2.1) and (2.2), we have $W(\varDelta) \geq \pi(2k)$, which implies Lemma 2.1 as $2 \nmid \varDelta$. ∎

The next lemma is a consequence of [10, Theorem 1].

LEMMA 2.2. *Both of the equations*

$$9z_1^4 - 5z_2^2 = 4 \quad \textit{and} \quad 25z_1^4 - 21z_2^2 = 4$$

have the unique positive integer solution $(z_1, z_2) = (1, 1)$.

LEMMA 2.3. *Let $l \geq 3$ be an integer, $t \in \{1, 2, 3\}$. If the equation*

$$|z_1^l - 3z_2^l| = 2t, \tag{2.3}$$

where $|\ |$ denotes the absolute value, has a positive integer solution (z_1, z_2), then $z_1 = z_2 = t = 1$.

Proof. Let $x = \min(z_1^l, 3z_2^l)$, $y = z_1 z_2$. Then

$$x(x + 2t) = 3y^l. \tag{2.4}$$

If $t = 1, 2$, as x is odd, we have $x = 1$ according to [1, Theorem 1.1], which happens only when $z_1 = 1$ and consequently $z_2 = t = 1$.

If $t = 3$, we have $x = 3X$ for some positive odd integer X by virtue of (2.4). Then we deduce from

$$3X(X + 2) = y^l$$

that $y = 3Y$ for some positive integer Y. Thus (2.4) changes to

$$X(X + 2) = 3^{l-1}Y^l.$$

According to [1, Theorem 1.1], $X = 1$, but this gives no solution of (2.3) for odd z_1 as $3 = x \le z_1^l \le x + 6 = 9$. ∎

3. Proof of Theorem 1.1. Suppose (1.3) has solutions. If $k \ge 9$, then according to Lemma 2.1, there exists a prime $p > 2k$ such that $p \mid \Delta$. Noting that $p^l \mid \Delta$ from (1.3) and the fact that p divides only one term of Δ, we deduce that $2n > p^l > (2k)^l \ge (2k)^2$. If $2 \le k \le 8$ and $2k \le n < k^2$, one can easily check that (1.3) has no solution. Thus in the following we assume

$$n \ge k^2. \tag{3.1}$$

Write $2n - 2i - 1 = a_i m_i^l$ for $i = 0, 1, \ldots, k-1$, where a_i is lth power free. We claim that $a_0, a_1, \ldots, a_{k-1}$ are distinct. Otherwise there exist integers $0 \le i < j \le k-1$ such that $a_i = a_j$, from which we deduce $m_i > m_j$. Then it follows that

$$\begin{aligned} 2k > 2(j-i) &= a_j(m_i^l - m_j^l) \ge a_j((m_j+1)^l - m_j^l) \ge l a_j m_j^{l-1} \\ &= l a_j^{1/l}(a_j m_j^l)^{(l-1)/l} \ge 2(a_j m_j^l)^{1/2} \ge 2(2n - 2k + 1)^{1/2} > 2n^{1/2}, \end{aligned}$$

contradicting (3.1).

Now rewrite (1.3) as

$$a_0 a_1 \cdots a_{k-1}(m_0 m_1 \cdots m_{k-1} M)^l = (2k-1)!!m^l. \tag{3.2}$$

Let

$$u = \frac{m_0 m_1 \cdots m_{k-1} M}{\gcd(m_0 m_1 \cdots m_{k-1} M, m)},$$
$$v = \frac{m}{\gcd(m_0 m_1 \cdots m_{k-1} M, m)}.$$

Then (3.2) can be written as

$$a_0 a_1 \cdots a_{k-1} u^l = (2k-1)!!v^l. \tag{3.3}$$

Suppose v has a prime divisor p. Obviously, p is odd and $p \nmid u$. Therefore from (3.3) we infer that

$$
\begin{aligned}
(3.4)\qquad \operatorname{ord}_p(a_0a_1\cdots a_{k-1})
&\geq \operatorname{ord}_p((2k-1)!!)+l\\
&= \operatorname{ord}_p((2k-1)!)-\operatorname{ord}_p((2k-2)(2k-4)\cdots 2)+l\\
&= \operatorname{ord}_p((2k-1)!)-\operatorname{ord}_p((k-1)!)+l\\
&= \sum_{i=1}^{\infty}\left(\left\lfloor\frac{2k-1}{p^i}\right\rfloor-\left\lfloor\frac{k-1}{p^i}\right\rfloor\right)+l\\
&\geq \sum_{i=1}^{l-1}\left(\left\lfloor\frac{2k-1}{p^i}\right\rfloor-\left\lfloor\frac{k-1}{p^i}\right\rfloor\right)+l.
\end{aligned}
$$

On the other hand, $\operatorname{ord}_p(a_0a_1\cdots a_{k-1})$ can be evaluated in the following way:

$$
\begin{aligned}
&\operatorname{ord}_p(a_0a_1\cdots a_{k-1})\\
&\quad= \sum_{i=1}^{l-1}\sharp\{j: p^i \mid a_j,\ 0\leq j\leq k-1\}\\
&\quad\leq \sum_{i=1}^{l-1}\sharp\{j: p^i \mid (2n-2j-1),\ 0\leq j\leq k-1\}\\
&\quad= \sum_{i=1}^{l-1}(\sharp\{j: p^i \mid j,\ 1\leq j\leq 2n-1\}-\sharp\{j: p^i \mid 2j,\ 1\leq j\leq n-1\}\\
&\qquad -\sharp\{j: p^i \mid j,\ 1\leq j\leq 2n-2k-1\}+\sharp\{j: p^i \mid 2j,\ 1\leq j\leq n-k-1\})\\
&\quad= \sum_{i=1}^{l-1}\left(\left\lfloor\frac{2n-1}{p^i}\right\rfloor-\left\lfloor\frac{n-1}{p^i}\right\rfloor-\left\lfloor\frac{2n-2k-1}{p^i}\right\rfloor+\left\lfloor\frac{n-k-1}{p^i}\right\rfloor\right).
\end{aligned}
$$

Noting that

$$
\left\lfloor\frac{2n-1}{p^i}\right\rfloor-\left\lfloor\frac{2n-2k-1}{p^i}\right\rfloor\leq\left\lfloor\frac{2k}{p^i}\right\rfloor+1,\qquad \left\lfloor\frac{n-1}{p^i}\right\rfloor-\left\lfloor\frac{n-k-1}{p^i}\right\rfloor\geq\left\lfloor\frac{k}{p^i}\right\rfloor,
$$

we have

$$
\begin{aligned}
(3.5)\qquad \operatorname{ord}_p(a_0a_1\cdots a_{k-1}) &\leq \sum_{i=1}^{l-1}\left(\left\lfloor\frac{2k}{p^i}\right\rfloor+1-\left\lfloor\frac{k}{p^i}\right\rfloor\right)\\
&= \sum_{i=1}^{l-1}\left(\left\lfloor\frac{2k}{p^i}\right\rfloor-\left\lfloor\frac{k}{p^i}\right\rfloor\right)+l-1.
\end{aligned}
$$

However, in view of

$$
\left\lfloor\frac{2k}{p^i}\right\rfloor-\left\lfloor\frac{2k-1}{p^i}\right\rfloor=\left\lfloor\frac{k}{p^i}\right\rfloor-\left\lfloor\frac{k-1}{p^i}\right\rfloor,
$$

we see that (3.5) contradicts (3.4). Therefore $v = 1$, whence $a_0 a_1 \cdots a_{k-1} \mid (2k-1)!!$. This together with the assertion that $a_0, a_1, \ldots, a_{k-1}$ are distinct odd integers tells us that

$$\{a_0, a_1, \ldots, a_{k-1}\} = \{1, 3, \ldots, 2k-1\}. \tag{3.6}$$

I. The case $l \geq 3$, $k \geq 5$. Let $k \equiv \sigma \pmod 3$, where $\sigma \in \{-1, 0, 1\}$. According to (3.6),

$$2k - 2\sigma - 3 = a_i, \qquad \frac{2k - 2\sigma - 3}{3} = a_j$$

for some $0 \leq i, j \leq k-1$. Then

$$0 < |m_j^l - 3m_i^l| = \frac{3|a_j m_j^l - a_i m_i^l|}{2k - 2\sigma - 3} \leq \frac{3(2k-2)}{2k-5} < 5.$$

As m_i, m_j are odd, $|m_j^l - 3m_i^l| = 2, 4$, which implies $m_i = m_j = 1$ by Lemma 2.3. Hence $2n - 2k + 1 \leq a_i m_i^l = 2k - 2\sigma - 3 \leq 2k - 1$, contradicting $n \geq 2k$.

II. The case $l \geq 3$, $2 \leq k \leq 4$. Let $a_i = 3, a_j = 1$. Then $0 < |m_j^l - 3m_i^l| \leq 2k - 2 \leq 6$, which means $|m_j^l - 3m_i^l| = 2, 4, 6$ and thus $m_i = m_j = 1$ by Lemma 2.3. Hence $2n - 2k + 1 \leq a_j m_j^l = 1$, which is impossible.

III. The case $l = 2$, $k \geq 5$. This is impossible as, by (3.6), there exists some i with $a_i = 9$, but a_i must be square free.

IV. The case $l = 2$, $3 \leq k \leq 4$. As $2, -4$ are quadratic nonresidues modulo 3, we know that $x^2 - 3y^2 \neq 2, -4$ for any integers x, y. Similar argument can be applied to $3y^2 - 7w^2, x^2 - 5z^2, x^2 - 7w^2$ modulo $3, 5, 7$, respectively. Then we have

$$\begin{cases} x^2 - 3y^2 \neq 2, -4, \\ x^2 - 5z^2 \neq 2, -2, -6, \\ x^2 - 7w^2 \neq -2, -4, 6, \\ 3y^2 - 7w^2 \neq -2, 4. \end{cases} \tag{3.7}$$

When $k = 3$, noting that $a_i m_i^2 - a_j m_j^2 = \pm 2, \pm 4$ for $0 \leq i < j \leq 2$, we deduce from (3.6) and (3.7) that $(a_0, a_1, a_2) = (5, 3, 1)$. In fact, $(a_0, a_1, a_2) = (1, 3, 5)$ implies $m_1^2 - 3m_2^2 = 2$, which has no integer solution according to (3.7), but since $x^2 - 5z^2 \neq \pm 2$, (a_0, a_1, a_2) can only be $(1, 3, 5)$ or $(5, 3, 1)$, so $(a_0, a_1, a_2) = (5, 3, 1)$. Therefore,

$$9m_1^4 - 4 = (3m_1^2 + 2)(3m_1^2 - 2) = 5m_0^2 \cdot m_2^2 = 5(m_0 m_2)^2.$$

By Lemma 2.2, $m_0 = m_1 = m_2 = 1$, but this means $n = 3$, contradicting $n \geq 2k$.

When $k = 4$, we can deduce similarly that $(a_0, a_1, a_2, a_3) = (7, 5, 3, 1)$ or $(1, 7, 5, 3)$. Let $i = 1$ resp. 2. Then we have

$$25m_i^4 - 4 = (5m_i^2 + 2)(5m_i^2 - 2) = 21(m_{i-1}m_{i+1})^2.$$

By Lemma 2.2, $m_{i-1} = m_i = m_{i+1} = 1$, which implies $2n - 1 \le 7$, contradicting $n \ge 2k$.

V. The case $l = 2$, $k = 2$. As $M^2 \mid (2k-1)!!$, we have $M = 1$, whence what we are going to solve is

$$(2n-1)(2n-3) = 3m^2. \tag{3.8}$$

Let $2n - 2 = x$, with which (3.8) takes the form

$$x^2 - 3m^2 = 1. \tag{3.9}$$

All the positive integer solutions of the above Pell equation are given by

$$x_t + m_t\sqrt{3} = (2 + \sqrt{3})^t \quad (t \in \mathbb{N}^*).$$

This implies that all the positive integer solutions of (3.9) with $2 \mid x$ and $x \ge 6$ are given by

$$x + m\sqrt{3} = (2 + \sqrt{3})^{2t+1} \quad (t \in \mathbb{N}^*).$$

Hence all the solutions of (3.8) are given by

$$2n + m\sqrt{3} = (2 + \sqrt{3})^{2t+1} + 2 \quad (t \in \mathbb{N}^*).$$

This completes the proof of Theorem 1.1 as $\binom{n}{k}_{!!} = \binom{n}{n-k}_{!!}$. ■

4. A generalization of equation (1.3). As a generalization of $\binom{n}{k}$ and $\binom{n}{k}_{!!}$, we define

$$\binom{n}{k}_{a,b} = \frac{(an - a + b)(an - 2a + b)\cdots(an - ak + b)}{(ak - a + b)(ak - 2a + b)\cdots b},$$

$$1 \le b \le a, \ \gcd(a, b) = 1,$$

and ask whether $\binom{n}{k}_{a,b}$ is a power of a rational number when $2 \le k \le n - 2$.

In view of $\binom{n}{k}_{a,b} = \binom{n}{n-k}_{a,b}$, we only need to consider the following Diophantine equation:

$$\binom{n}{k}_{a,b} = \left(\frac{m}{M}\right)^l \tag{4.1}$$

in integers $n \ge 4$, $4 \le 2k \le n$, $m \ge 1$, $M \ge 1$, $\gcd(m, M) = 1$, $l \ge 2$.

When $(a, b) = (1, 1)$, (4.1) is (1.1), and when $(a, b) = (2, 1)$, (4.1) is (1.3). However, we cannot solve (4.1) using the method of this paper when $a \ge 3$.

Furthermore, for $1 \le b \le a$, $\gcd(a, b) = 1$, we can consider the quotient of two products of consecutive k terms in the arithmetic progression $b, a + b,$

$a + 2b, \ldots$, and ask for the solutions of the Diophantine equation

$$\frac{(an-a+b)(an-2a+b)\cdots(an-ak+b)}{(aN-a+b)(aN-2a+b)\cdots(aN-ak+b)} = \left(\frac{m}{M}\right)^l$$

in integers $|N-n| \geq k$, $m \geq 1$, $M \geq 1$, $\gcd(m, M) = 1$, $l \geq 2$.

Acknowledgements. This research was supported by the NNSFC Grant 10871169.

References

[1] M. A. Bennett, *Products of consecutive integers*, Bull. London Math. Soc. 36 (2004), 683–694.

[2] M. A. Bennett, N. Bruin, K. Győry and L. Hajdu, *Powers from products of consecutive terms in arithmetic progression*, Proc. London Math. Soc. 92 (2006), 273–306.

[3] P. Erdős, *On a diophantine equation*, J. London Math. Soc. 26 (1951), 176–178.

[4] P. Erdős and J. L. Selfridge, *The product of consecutive integers is never a power*, Illinois J. Math. 19 (1975), 292–301.

[5] K. Győry, *On the diophantine equation* $\binom{n}{k} = x^l$, Acta Arith. 80 (1997), 289–295.

[6] —, *On the diophantine equation* $n(n+1)\cdots(n-k+1) = bx^l$, ibid. 83 (1998), 87–92.

[7] K. Győry, L. Hajdu and Á. Pintér, *Perfect powers from products of consecutive terms in arithmetic progression*, Compos. Math. 145 (2009), 845–864.

[8] S. Laishram and T. N. Shorey, *Number of prime divisors in a product of terms of an arithmetic progression*, Indag. Math. 15 (2004), 505–521.

[9] S. Laishram, T. N. Shorey and Sz. Tengely, *Squares in products in arithmetic progression with at most one term omitted and common difference a prime power*, Acta Arith. 135 (2008), 143–158.

[10] W. Ljunggren, *On the diophantine equation* $Ax^4 - By^2 = C$ $(C = 1, 4)$, Math. Scand. 21 (1967), 149–158.

[11] N. Saradha, *On perfect powers in products with terms from arithmetic progressions*, Acta Arith. 82 (1997), 147–172.

[12] T. N. Shorey, *Diophantine approximations, diophantine equations, transcendence and applications*, Indian J. Pure Appl. Math. 37 (2006), 9–39.

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Received on 16.10.2009
and in revised form on 19.5.2010 (6175)