

The Brauer–Manin obstruction on a general diagonal quartic surface

by

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1. Introduction. Our object of study is the diagonal quartic surface $X \subset \mathbb{P}_{\mathbb{Q}}^3$ defined by the equation

$$(1.1) \quad a_0X_0^4 + a_1X_1^4 + a_2X_2^4 + a_3X_3^4 = 0$$

where $a_0, a_1, a_2, a_3 \in \mathbb{Q}$ are non-zero rational coefficients.

Multiplying the equation (1.1) through by a constant, permuting the coefficients, or changing any of the coefficients by a fourth power gives rise to another equation defining a surface which is clearly isomorphic (over \mathbb{Q}) to the original one. Two diagonal quartic equations related by such operations will be called *equivalent*. In particular, after replacing X with an equivalent surface, we may assume that the coefficients a_i are integers with no common factor, and that none of them is divisible by a fourth power.

When we talk about the *reduction* of X modulo some prime p , we mean simply the variety in $\mathbb{P}_{\mathbb{F}_p}^3$ defined by reducing the equation (1.1) modulo p . Suppose that p is odd. Then, according to the number of coefficients divisible by p , the reduction at p will be either: a smooth diagonal quartic surface; a cone over a smooth diagonal quartic curve; (geometrically) a union of four planes; or a quadruple plane.

THEOREM 1.1. *Let X be the diagonal quartic surface over \mathbb{Q} given by (1.1), and let H be the subgroup of $\mathbb{Q}^{\times}/(\mathbb{Q}^{\times})^4$ generated by $-1, 4$ and the quotients a_i/a_j . Suppose that the following conditions are satisfied:*

- (i) $X(\mathbb{Q}_v) \neq \emptyset$ for all places v of \mathbb{Q} ;
- (ii) $H \cap \{2, 3, 5\} = \emptyset$;
- (iii) $|H| = 256$;

- (iv) *there is some odd prime p which divides precisely one of the coefficients a_i , and does so to an odd power; assume further that if $p \in \{7, 11, 17, 41\}$, then the reduction of X modulo p is not equivalent to $x^4 + y^4 + z^4 = 0$.*

Then $\text{Br } X/\text{Br } \mathbb{Q}$ has order 2, and there is no Brauer–Manin obstruction to the existence of rational points on X .

REMARK. It is easy to check that the group H may also be generated by $-1, 4$ and a_i/a_0 ($i = 1, 2, 3$). It follows that H has order dividing 256.

Theorem 1.1 combines several ingredients, many of which are already known. The deepest part is the result, due to Ieronymou, Skorobogatov and Zarhin [5], that condition (ii) above implies the vanishing of the transcendental part of the Brauer group of X , meaning that $\text{Br } X = \text{Br}_1 X$. The calculation that, under condition (iii) above, $\text{Br}_1 X/\text{Br } \mathbb{Q}$ has order 2 can be found in the tables contained in the author’s thesis [1]. The new ingredients in the present article are a more geometric description of the non-trivial class of Azumaya algebras on X and a proof that these algebras, under condition (iv) above, give no obstruction to the existence of rational points on X .

1.1. Background. Let us recall the definition of the Brauer–Manin obstruction; see Skorobogatov’s book [8] for more details. Fix a number field k and a smooth, projective, geometrically irreducible variety X over k . We define the *Brauer group* of X to be $\text{Br } X = H_{\text{ét}}^2(X, \mathbb{G}_m)$. If K is any field containing k , and $P \in X(K)$ a K -point of X , then there is an evaluation homomorphism $\text{Br } X \rightarrow \text{Br } K$, $\mathcal{A} \mapsto \mathcal{A}(P)$, which is the natural map coming from the morphism $P: \text{Spec } K \rightarrow X$. In particular, this applies if $K = k_v$ is a completion of k .

As X is projective, the set of adelic points of X is $X(\mathbb{A}_k) = \prod_v X(k_v)$, the product being over all places of k . The set $X(\mathbb{A}_k)$ is non-empty precisely when each $X(k_v)$ is non-empty, that is, X has points over every completion of k . Let $\text{inv}_v: \text{Br } k_v \rightarrow \mathbb{Q}/\mathbb{Z}$ be the invariant map. Define the following subset of the adelic points:

$$X(\mathbb{A}_k)^{\text{Br}} = \left\{ (P_v) \in X(\mathbb{A}_k) \mid \sum_v \text{inv}_v \mathcal{A}(P_v) = 0 \text{ for all } \mathcal{A} \in \text{Br } X \right\}.$$

Suppose that $X(\mathbb{A}_k)$ is non-empty. By class field theory, the diagonal image of $X(k)$ is contained in $X(\mathbb{A}_k)^{\text{Br}}$; if in fact $X(\mathbb{A}_k)^{\text{Br}}$ is empty, then we say there is a *Brauer–Manin obstruction* to the existence of k -rational points on X .

Let \bar{X} denote the base change of X to an algebraic closure \bar{k} of k . There is a natural filtration on $\text{Br } X$, given by $\text{Br}_0 X \subseteq \text{Br}_1 X \subseteq \text{Br } X$, where

- $\text{Br}_0 X = \text{im}(\text{Br } k \rightarrow \text{Br } X)$ consists of the constant classes in $\text{Br } X$;
- $\text{Br}_1 X = \ker(\text{Br } X \rightarrow \text{Br } \bar{X})$ is the *algebraic* part of the Brauer group, consisting of those classes which are split by base change to \bar{k} .

If $X(\mathbb{A}_k) \neq \emptyset$, then the natural homomorphism $\text{Br } k \rightarrow \text{Br } X$ is injective, and we will think of $\text{Br } k$ as being contained in $\text{Br } X$. The elements of $\text{Br } k$ do not contribute to the Brauer–Manin obstruction, so in describing $X(\mathbb{A}_k)^{\text{Br}}$ it is enough to consider the quotient $\text{Br } X/\text{Br } k$.

Elements of $\text{Br } X \setminus \text{Br}_1 X$ are called *transcendental*. For certain types of varieties X , we know that $\text{Br } \bar{X} = 0$ and therefore that $\text{Br } X$ is entirely algebraic: this is true in particular if X is a curve or a rational surface. It is, however, certainly not true if X is a K3 surface, such as our diagonal quartic surface. The transcendental part of the Brauer group of a diagonal quartic surface has been studied by Ieronymou [4] and by Ieronymou, Skorobogatov and Zarhin [5]. The algebraic part of the Brauer group of a diagonal quartic surface has been studied by the present author [1, 2].

1.2. Outline of the proof. We will now describe an outline of the proof of Theorem 1.1, with the details postponed to Section 2. As mentioned above, the first ingredient is the following result of Ieronymou, Skorobogatov and Zarhin.

THEOREM 1.2 ([5, Corollary 3.3]). *Let X and H be as in the introduction, and suppose that $H \cap \{2, 3, 5\} = \emptyset$. Then $\text{Br } X = \text{Br}_1 X$.*

So any Brauer–Manin obstruction on X comes entirely from the algebraic Brauer group. The structure of $\text{Br}_1 X/\text{Br } \mathbb{Q}$ as an abstract group can be computed using the isomorphism

$$\text{Br}_1 X/\text{Br } \mathbb{Q} \cong H^1(\mathbb{Q}, \text{Pic } \bar{X}).$$

In the case of diagonal quartic surfaces, $\text{Pic } \bar{X}$ is generated by the classes of the obvious 48 straight lines on \bar{X} , and condition (iii) of Theorem 1.1 ensures that the Galois action on these lines is the most general possible. Lemma 2.2 below shows that $\text{Br}_1 X/\text{Br } \mathbb{Q}$ is of order 2.

It remains to compute the Brauer–Manin obstruction coming from the non-trivial class in $\text{Br}_1 X/\text{Br } \mathbb{Q}$. In Lemma 2.1, we describe explicitly an Azumaya algebra \mathcal{A} which may be defined on any diagonal quartic surface (1.1) for which $a_0 a_1 a_2 a_3$ is not a square. Condition (iii) implies in particular that $a_0 a_1 a_2 a_3$ is non-square, so the algebra \mathcal{A} is defined on our particular surface.

The proof is completed by Lemma 2.3. This states that, given a prime p satisfying condition (iv) of Theorem 1.1, the Azumaya algebra \mathcal{A} , evaluated at different points of $X(\mathbb{Q}_p)$, gives invariants of both 0 and 1/2. In particular, \mathcal{A} is not equivalent to a constant algebra, and provides no obstruction to the existence of rational points on X .

2. The algebraic Brauer–Manin obstruction. In this section we describe an explicit Azumaya algebra on our diagonal quartic surface. For this purpose we may replace \mathbb{Q} by an arbitrary number field k . Let $X \subset \mathbb{P}_k^3$ be the diagonal quartic surface (1.1), and let $Y \subset \mathbb{P}_k^3$ be the smooth quadric surface defined by

$$a_0Y_0^2 + a_1Y_1^2 + a_2Y_2^2 + a_3Y_3^2 = 0.$$

There is a morphism $\phi: X \rightarrow Y$ given by $Y_i = X_i^2$. If X is everywhere locally soluble, then so is Y ; and, since Y is a quadric, it follows that Y has a k -rational point.

LEMMA 2.1. *Suppose that X is everywhere locally soluble. Pick a point $P = [y_0, y_1, y_2, y_3] \in Y(k)$, and let $g \in k[Y_0, Y_1, Y_2, Y_3]$ be the linear form*

$$g = a_0y_0Y_0 + a_1y_1Y_1 + a_2y_2Y_2 + a_3y_3Y_3$$

defining the tangent plane to Y at P . Let

$$(2.1) \quad f = \phi^*g = a_0y_0X_0^2 + a_1y_1X_1^2 + a_2y_2X_2^2 + a_3y_3X_3^2$$

be the quadratic form obtained by pulling g back to X . Write $\theta = a_0a_1a_2a_3$. Then the quaternion algebra $\mathcal{A} = (\theta, f/X_0^2) \in \text{Br } k(X)$ is an Azumaya algebra on X . The class of \mathcal{A} in $\text{Br } X/\text{Br } k$ is independent of the choice of P .

REMARK. Since f is a quadratic form on X and not a rational function, we divide it by X_0^2 to obtain an element of $k(X)^\times$. As always, when defining a quaternion algebra over a field, (a, b) and (a, bc^2) give isomorphic algebras. So the choice of X_0 here is completely arbitrary; we could replace it with any other X_i or indeed any linear form.

REMARK. The coordinates of the point P , and therefore the linear form g , are only defined up to multiplication by a scalar. So the point P only determines the algebra \mathcal{A} up to an element of $\text{Br } k$.

REMARK. The idea of obtaining an Azumaya algebra by pulling back in this way is not new: see, for example, Section 7 of [6] for examples on del Pezzo surfaces of degree 2. The construction there similarly involves pulling back a quaternion algebra from a quadric.

Proof of Lemma 2.1. If θ is a square in k , then \mathcal{A} is isomorphic to the algebra of 2×2 matrices over $k(X)$, and the conclusions are trivially true. So suppose that θ is not a square in k .

As described for example in [9, Lemma 11], to show that \mathcal{A} is an Azumaya algebra we need to show that the principal divisor (f/X_0^2) is the norm of a divisor on X defined over $k(\sqrt{\theta})$. Recall that, over $\bar{\mathbb{Q}}$, \bar{Y} admits two pencils of straight lines, the classes of which generate $\text{Pic } \bar{Y} \cong \mathbb{Z}^2$. The tangent plane to Y at P intersects Y in two lines, L and L' , which are each defined over $k(\sqrt{\theta})$ and conjugate over k ; so the divisor of vanishing of g on Y is $L + L'$.

Let $D = \phi^*L$ be the divisor obtained by pulling L back to X , and similarly $D' = \phi^*L'$. Then $(f/X_0^2) = D + D' - 2D_0 = N_{k(\sqrt{\theta})/k}(D - D_0)$, where D_0 is the divisor on X defined by $X_0 = 0$.

Independence of P is a routine calculation, but we reproduce it for the sake of completeness. Let $P_1 \in Y(k)$ be another point, and let g_1 be the corresponding linear form defining the tangent plane to Y at P_1 . Then the divisor of vanishing of g_1 on Y is $L_1 + L'_1$, where L_1 is a line, defined over $k(\sqrt{\theta})$ and linearly equivalent to L , and L'_1 its conjugate over k . So there exists a rational function h on Y , defined over $k(\sqrt{\theta})$, such that $(h) = L - L_1$. Then

$$(g_1 N_{k(\sqrt{\theta})/k} h) = (L_1 + L'_1) + (L - L_1) + (L' - L'_1) = L + L'$$

and so $g_1 N_{k(\sqrt{\theta})/k} h$ is a constant multiple of g . Let $f_1 = \phi^*g_1$; then f/f_1 is a constant multiplied by the norm of a rational function defined over $k(\sqrt{\theta})$, so $(\theta, f_1/X_0^2)$ differs from $(\theta, f/X_0^2)$ only by a constant algebra. ■

REMARK. Even when θ is not a square in k , it is still possible for the class of \mathcal{A} in $\text{Br } X/\text{Br } k$ to be trivial. For example, taking $k = \mathbb{Q}$, the tables of [1] show that, for any integers c_1, c_2 , the diagonal quartic surface

$$X_0^4 + c_1 X_1^4 + c_2 X_2^4 - c_1^2 c_2^2 X_3^4$$

has $\text{Br}_1 X = \text{Br } \mathbb{Q}$. In particular, the algebra \mathcal{A} on this surface is equivalent to a constant algebra. Note that Lemma 2.3 below does not apply in this case, since no prime divides exactly one of the coefficients.

LEMMA 2.2. *Let X be a diagonal quartic surface over \mathbb{Q} . In the notation of Theorem 1.1, suppose that $|H| = 256$. Then $\text{Br}_1 X/\text{Br } \mathbb{Q}$ is of order 2.*

Proof. This calculation can be found in [1], and depends on the well-known isomorphism $\text{Br}_1 X/\text{Br } \mathbb{Q} \cong H^1(\mathbb{Q}, \text{Pic } \bar{X})$. What follows is a brief summary of the calculation. The variety X contains (at least) 48 straight lines: for example, setting

$$a_0 X_0^4 + a_1 X_1^4 = 0, \quad a_2 X_2^4 + a_3 X_3^4 = 0$$

and factorising each side over $\bar{\mathbb{Q}}$ gives equations for 16 lines; the other 32 are obtained by permuting the indices. The lines are all defined over the extension $K = \mathbb{Q}(i, \sqrt{2}, \sqrt[4]{a_1/a_0}, \sqrt[4]{a_2/a_0}, \sqrt[4]{a_3/a_0})$. The classes of the lines generate the Picard group of X over \mathbb{Q} , which is free of rank 20. By the inflation-restriction exact sequence, we have $H^1(\mathbb{Q}, \text{Pic } \bar{X}) = H^1(K/\mathbb{Q}, \text{Pic } X_K)$ and computing this cohomology group comes down to knowing the Galois group $\text{Gal}(K/\mathbb{Q})$ and its action on the 48 lines. Appendix A of [1] lists the result of this computation for all possible Galois groups $\text{Gal}(K/\mathbb{Q})$. In particular, case A222 there is where K/\mathbb{Q} is of maximal degree 256, so that the co-

efficients a_i are “as general as possible”. In that case, $H^1(K/\mathbb{Q}, \text{Pic } X_K)$ is computed to be of order 2.

We claim that $[K : \mathbb{Q}] = |H|$, so that condition (iii) of Theorem 1.1 implies that X falls into case A222 of [1]. Over $\mathbb{Q}(i)$, Kummer theory gives the degree $[K : \mathbb{Q}(i)] = |H'|$, where H' is the subgroup of $\mathbb{Q}(i)^\times / (\mathbb{Q}(i)^\times)^4$ generated by 4 and the a_j/a_0 . The kernel of the natural map

$$r : \mathbb{Q}^\times / (\mathbb{Q}^\times)^4 \rightarrow \mathbb{Q}(i)^\times / (\mathbb{Q}(i)^\times)^4$$

is of order 2, generated by the class of -4 ; so $H' = r(H)$, and

$$[K : \mathbb{Q}] = [K : \mathbb{Q}(i)][\mathbb{Q}(i) : \mathbb{Q}] = 2|H'| = |H|. \blacksquare$$

REMARK. Further cohomology calculations could show that, under the hypothesis that $|H| = 256$, the algebra \mathcal{A} of Lemma 2.1 represents the non-trivial class in $\text{Br } X/\text{Br } \mathbb{Q}$. However, there is no need for this, since in our situation non-triviality is also implied by the following lemma.

LEMMA 2.3. *Let X be a diagonal quartic surface over \mathbb{Q} given by equation (1.1). Let \mathcal{A} be the Azumaya algebra described in Lemma 2.1. Suppose that p is an odd prime such that:*

- (i) p divides precisely one of the coefficients a_0, a_1, a_2, a_3 , and does so to an odd power;
- (ii) $X(\mathbb{Q}_p)$ is not empty;
- (iii) if $p \in \{7, 11, 17, 41\}$, then the reduction of X modulo p is not equivalent to the cone over the quartic curve $x^4 + y^4 + z^4 = 0$.

Then $\text{inv}_p \mathcal{A}(Q)$ takes both values 0 and $1/2$ for $Q \in X(\mathbb{Q}_p)$. In particular, the class of \mathcal{A} in $\text{Br } X/\text{Br } \mathbb{Q}$ is non-trivial, and \mathcal{A} gives no Brauer–Manin obstruction to the existence of rational points on X .

REMARK. Condition (i) implies, in particular, that $\theta = a_0 a_1 a_2 a_3$ is not a square.

REMARK. Condition (ii), that $X(\mathbb{Q}_p)$ be non-empty, is automatic for $p \geq 37$: indeed, the reduction of X modulo p is a cone over a smooth quartic curve, which has a rational point by the Hasse–Weil bound. For $p < 37$, one can easily check by a computer search that the only smooth diagonal quartic curves over \mathbb{F}_p lacking a rational point are the following (up to equivalence):

- $x^4 + y^4 + z^4 = 0$ for $p = 5$ or 29 ;
- $x^4 + y^4 + 2z^4 = 0$ for $p = 5$ or 13 .

Proof of Lemma 2.3. Suppose, without loss of generality, that $p \mid a_0$.

In constructing \mathcal{A} , as described in Lemma 2.1, we may choose any point $P \in Y(\mathbb{Q})$ to start from. In particular, we may choose P such that y_1, y_2, y_3 are not all divisible by p , for the following reason. Recall that we

have assumed the coefficients a_i to be fourth-power-free, so that in particular $v_p(a_0) \leq 3$. The original surface X is locally soluble at p , so let $[x_0, x_1, x_2, x_3] \in X(\mathbb{Q}_p)$ with the x_i p -adic integers, not all divisible by p . If x_1, x_2, x_3 were all divisible by p , then we would have

$$v_p(a_1x_1^4 + a_2x_2^4 + a_3x_3^4) \geq 4 \quad \text{and} \quad v_p(a_0x_0^4) \leq 3,$$

and so the defining equation (1.1) could not be satisfied. Now $[x_0^2, x_1^2, x_2^2, x_3^2]$ is a point of $Y(\mathbb{Q}_p)$, with x_1, x_2, x_3 not all divisible by p , and so by weak approximation $Y(\mathbb{Q})$ contains a point with the desired property.

Looking at the equation of Y shows that, in fact, at most one of y_1, y_2, y_3 can be divisible by p . It would clarify the rest of the argument if none of y_1, y_2, y_3 were divisible by p , and the reader is encouraged to imagine this to be the case; but unfortunately if $p = 3$ it is not always possible.

Starting from such a P , we obtain f as in (2.1) where the coefficient of X_0^2 is divisible by p , but at least one of the other coefficients is not divisible by p . The reduction \tilde{f} of f modulo p is a non-zero diagonal quadratic form on \mathbb{F}_p^3 , with no term in X_0^2 .

We now reduce to a problem over \mathbb{F}_p . Let \tilde{X} denote the reduction of X modulo p . Let $\tilde{Q} \in \tilde{X}(\mathbb{F}_p)$ be a smooth point; then, by Hensel's Lemma, \tilde{Q} lifts to a point $Q \in X(\mathbb{Q}_p)$. Suppose that $\tilde{f}(\tilde{Q}) \neq 0$, and that $X_0(Q) \neq 0$. Since p divides θ to an odd power, the description of the Hilbert symbol at [7, III, Theorem 1] gives

$$(2.2) \quad \text{inv}_p \mathcal{A}(Q) = (\theta, f(Q)/X_0^2)_p = (\theta, f(Q))_p = \left(\frac{\tilde{f}(\tilde{Q})}{p} \right).$$

Here the leftmost equality is abusing notation slightly, since inv_p traditionally takes values in $\{0, 1/2\}$ whereas the Hilbert symbol $(\cdot, \cdot)_p$ takes values in $\{\pm 1\}$. Since f is of degree 2, the value $f(Q)$ is defined only up to squares, and likewise $\tilde{f}(\tilde{Q})$, but the expressions in (2.2) are well defined. The requirement that $X_0(Q) \neq 0$ is superfluous, since we can always replace \mathcal{A} by the isomorphic algebra $(\theta, f/X_i^2)$ for some $i \neq 0$ to show that the conclusion of (2.2) still holds.

Now let C be the smooth quartic curve in $\mathbb{P}_{\mathbb{F}_p}^2$ defined by

$$C : \tilde{a}_1X_1^4 + \tilde{a}_2X_2^4 + \tilde{a}_3X_3^4 = 0.$$

This is, of course, the same as the defining equation of \tilde{X} , but now considered as an equation in only three variables. Any point of $X(\mathbb{Q}_p)$ reduces to give us a point of $\tilde{X}(\mathbb{F}_p)$ and hence, forgetting the X_0 -coordinate, of $C(\mathbb{F}_p)$. Since the diagonal quadratic form \tilde{f} has no term in X_0^2 , we can consider it as a form on C . Note that \tilde{f} depends only on C , not on our original variety X , since we may also construct \tilde{f} as follows: the point $\tilde{P} = (\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$ lies on the smooth plane conic $Z : \tilde{a}_1Y_1^2 + \tilde{a}_2Y_2^2 + \tilde{a}_3Y_3^2 = 0$, and the linear form \tilde{g}

defines the tangent line to Z at \tilde{P} . Write $\tilde{\phi}$ for the map from C to Z given by $Y_i = X_i^2$; pulling \tilde{g} back under $\tilde{\phi}$ gives the form \tilde{f} . In particular, this shows that the divisor of \tilde{f} is a multiple of 2, for we have $(\tilde{g}) = 2\tilde{P}$ and therefore $(\tilde{f}) = 2(\tilde{\phi}^*\tilde{P})$. The geometric picture (which is only accurate as long as none of y_1, y_2, y_3 are divisible by p) is that \tilde{f} defines a plane conic which is tangent to C at four distinct points, which are the four points mapping to \tilde{P} under $\tilde{\phi}$.

Note also that the divisor $(\tilde{f})/2 = \tilde{\phi}^*\tilde{P}$ is not a plane section: as long as none of y_1, y_2, y_3 are divisible by p , this divisor consists of four distinct points of the form $[\pm\alpha, \pm\beta, \pm\gamma]$, with α, β, γ all non-zero; in characteristic $\neq 2$, such points can never be collinear. If one of y_1, y_2, y_3 is divisible by p , then we move to an extension of \mathbb{F}_p , replace \tilde{P} by some \tilde{P}' for which the above proof does work, and observe that \tilde{P}' is linearly equivalent to \tilde{P} , so $\tilde{\phi}^*\tilde{P}'$ is linearly equivalent to $\tilde{\phi}^*\tilde{P}$, but $\tilde{\phi}^*\tilde{P}'$ is not a plane section; therefore neither can $\tilde{\phi}^*\tilde{P}$ be a plane section.

By (2.2), it remains to show that the quadratic form \tilde{f} takes both square and non-square non-zero values on $C(\mathbb{F}_p)$. Equivalently, we need to show that, for any $c \in \mathbb{F}_p^\times/(\mathbb{F}_p^\times)^2$, the equations

$$(2.3) \quad T^2 = c\tilde{f}(X_1, X_2, X_3), \quad \tilde{a}_1X_1^4 + \tilde{a}_2X_2^4 + \tilde{a}_3X_3^4 = 0$$

have simultaneous solutions with T non-zero. These equations define a double cover E_c of C . As given, E_c is singular at the points with $T = 0$ (which are the points lying over the zeros of \tilde{f}), so we consider its normalisation $E'_c \rightarrow E_c$. This is a smooth double cover of C with the following properties:

- the morphism $E'_c \rightarrow E_c$ is an isomorphism outside eight (geometric) points lying over the points of E_c with $T = 0$;
- since the divisor (\tilde{f}) is a multiple of 2, the quadratic extension of function fields $\mathbb{F}_p(E'_c)/\mathbb{F}_p(C)$ is unramified and hence so is $E'_c \rightarrow C$;
- since the divisor $(\tilde{f})/2$ is not a plane section, this extension contains no non-trivial extension of \mathbb{F}_p and so E'_c is geometrically irreducible.

By the Riemann–Hurwitz formula, E'_c has genus 5. If $p > 114$, then the Hasse–Weil bounds show that E'_c has strictly more than eight points over \mathbb{F}_p , and so E_c has at least one point with $T \neq 0$, completing the argument in this case.

It remains to check the cases with $p < 114$. For each prime p , we can take $\tilde{a}_1 = 1$ and let \tilde{a}_2, \tilde{a}_3 run through $\mathbb{F}_p^\times/(\mathbb{F}_p^\times)^4$. A straightforward computer search shows that the only cases when some E_c fails to have points are those listed in the statement of the lemma. ■

REMARK. With a slightly longer argument, we could avoid having to throw away the points on E_c with $T = 0$. By taking two different \tilde{P} s to start with, we obtain two different \tilde{f} s with no common zeros on C . The

ratio of the \tilde{f} s is a square, and the corresponding equations (2.3) patch together to give a description of E'_c with no singularities. The sophisticated reader will recognise E'_c as a torsor under μ_2 corresponding to the 2-torsion divisor class $(f/X_0^2) \in \text{Pic } C[2]$.

3. A counterexample. In this section we present a counterexample showing that Theorem 1.1 can fail when condition (iv) is not met. We begin by giving an infinite family of diagonal quartics satisfying conditions (i)–(iii) of Theorem 1.1, but not condition (iv).

LEMMA 3.1. *Let p, q be odd primes with the following properties:*

- $p \equiv q \equiv 3 \pmod{4}$;
- p and q are both fourth powers modulo 17;
- $\left(\frac{p}{q}\right) = 1$.

Then the diagonal quartic surface

$$X_0^4 + qX_1^4 = pX_2^4 + 17pqX_3^4$$

satisfies conditions (i)–(iii) of Theorem 1.1.

Proof. Conditions (ii) and (iii) are clear, since there are no non-obvious relations between the generators for $H = \langle -1, 4, p, q, 17 \rangle$. It remains to prove local solubility. For \mathbb{R} this is clear. For primes $\ell \geq 23$ of good reduction, the Weil conjectures guarantee a point modulo ℓ and hence a point over \mathbb{Q}_ℓ by Hensel’s Lemma. At $\ell = 3, 7, 11, 13, 17, 19$, a computer search shows that every smooth diagonal quartic surface has a rational point modulo ℓ . At $\ell = 5$, the only smooth diagonal quartic surface lacking a point over \mathbb{F}_5 is the Fermat quartic $X_0^4 + X_1^4 + X_2^4 + X_3^4 = 0$, so for local solubility to fail we would need $q \equiv -p \equiv -17pq \equiv 1 \pmod{5}$, which is impossible.

Since p and q are both congruent to 3 (mod 4), the fourth powers modulo p or q are the same as the squares. At q , the condition $\left(\frac{p}{q}\right) = 1$ guarantees local solubility; at p , we have $\left(\frac{-q}{p}\right) = -\left(\frac{q}{p}\right) = \left(\frac{p}{q}\right) = 1$ and so again the surface is locally soluble. Finally, at 17, the fact that p , hence $-p$, and q are fourth powers means that the reduction at 17 is isomorphic to the cone over the Fermat quartic curve $x^4 + y^4 + z^4 = 0$, which has smooth points over \mathbb{F}_{17} . ■

However, choosing p and q to be fourth powers modulo 17 means that condition (iv) of Theorem 1.1 is not satisfied.

We will show that the Azumaya algebra \mathcal{A} described in Section 2 can give an obstruction to the existence of rational points on X , at least for some values of p and q . Recall that multiplying the form f by a constant changes \mathcal{A} by a constant algebra. To avoid contributions at unnecessary primes, we choose our representation $\mathcal{A} = (\theta, f)$ such that the coefficients

of f are integers with no common factor. (This is equivalent to writing our point $P = [y_0, y_1, y_2, y_3]$ with the y_i integers having no common factor.)

LEMMA 3.2. *Let X be the surface of Lemma 3.1, and let $\mathcal{A} = (\theta, f)$ be normalised as above. Then, for all places $v \neq 17$, $\text{inv}_v \mathcal{A}(Q) = 0$ for all $Q \in X(\mathbb{Q}_v)$. The invariant is constant on $X(\mathbb{Q}_{17})$.*

Proof. Our normalisation of f ensures that, at all places of good reduction for X , the algebra \mathcal{A} also has good reduction and so the invariant is zero at these places. See [3, Corollary 4] for one explanation of why this is true.

The primes of bad reduction for X are $2, 17, p, q$. Observe that $\theta = 17p^2q^2$ is a square in $\mathbb{R}, \mathbb{Q}_2, \mathbb{Q}_p$ and \mathbb{Q}_q (the last two follow by quadratic reciprocity from the fact that $-p$ and q are fourth powers, hence squares, modulo 17). So the conclusion is true at each of these places.

At 17, the argument used in the proof of Lemma 2.3 shows that the invariant $\text{inv}_{17} \mathcal{A}(Q)$ is constant for $Q \in X(\mathbb{Q}_v)$. We give the details. The reduction of X modulo 17 is isomorphic to the cone $X_0^4 + X_1^4 + X_2^4 = 0$. The corresponding quadric is $Y_0^2 + Y_1^2 + Y_2^2 = 0$. To show simply that the invariant is constant, we can change \mathcal{A} by a constant algebra and so may as well replace P by any point which is convenient. So pick $\tilde{P} = [5, 5, 1, 0]$ and hence $\tilde{f} = 5X_0^2 + 5X_1^2 + X_2^2$. (This choice of \tilde{P} has the advantage that it does not lift to a point of the quartic, so \tilde{f} is never zero on \mathbb{F}_{17} -rational points of the quartic.) Now the solutions to $X_0^4 + X_1^4 + X_2^4$ over \mathbb{F}_{17} are all of the form $[\epsilon, 1, 0], [\epsilon, 0, 1]$ or $[0, \epsilon, 1]$ where $\epsilon^4 = -1$, and it turns out that evaluating \tilde{f} at any of these points gives a square in \mathbb{F}_{17} . ■

We do not yet know whether the invariant at 17 will be 0 or 1/2. If it is 0, then there is no Brauer–Manin obstruction on X (not even to weak approximation). If it is 1/2, then there is a Brauer–Manin obstruction to the existence of rational points. To determine which, we only need to evaluate the invariant at one point. A simple calculation reveals that the first example satisfying the conditions of Lemma 3.1 does indeed give a counterexample to the Hasse principle:

PROPOSITION 3.3. *Let X be the diagonal quartic surface given by*

$$X_0^4 + 47X_1^4 = 103X_2^4 + (17 \cdot 47 \cdot 103)X_3^4.$$

Then X has points in each completion of \mathbb{Q} , but the algebra \mathcal{A} gives a Brauer–Manin obstruction to the existence of a rational point on X .

Proof. Since X satisfies the conditions of Lemma 3.1, it only remains to evaluate the obstruction at 17. On the quadric

$$Y : Y_0^2 + 47Y_1^2 = 103Y_2^2 + (17 \cdot 47 \cdot 103)Y_3^2,$$

we can take $P = [20 : 13 : -9 : 0] \in Y(\mathbb{Q})$, and so obtain the Azumaya algebra

$$\mathcal{A} = (17, (20X_0^2 + (47 \cdot 13)X_1^2 + (103 \cdot 9)X_2^2)/X_0^2).$$

Evaluating the quadratic form $20X_0^2 + (47 \cdot 13)X_1^2 + (103 \cdot 9)X_2^2$ at any point of $X(\mathbb{F}_{17})$ gives a non-square value modulo 17, and therefore $\text{inv}_{17} \mathcal{A}(Q) = 1/2$ for all $Q \in X(\mathbb{Q}_{17})$. Combining this with the fact that the invariant is 0 at each other place, we deduce that $\sum_v \text{inv}_v \mathcal{A}(Q_v) = 1/2$ for all $(Q_v) \in X(\mathbb{A}_{\mathbb{Q}})$, and therefore that \mathcal{A} gives a Brauer–Manin obstruction to the existence of a rational point on X . ■

REMARK. It was not *a priori* clear that starting from different points P of the quadric Y should always give the same invariant at 17. Different points P might give algebras \mathcal{A} differing by a constant algebra. After all, in performing the verification of Lemma 3.2, we could have replaced the point $\tilde{P} = [5, 5, 1, 0]$ by a scalar multiple, say $[1, 1, 7, 0]$, and \tilde{f} would have been non-square at all points instead of square. However, our insistence that P should be given by coordinates which are coprime integers fixes the invariants at all places other than 17, and therefore (by the product rule) fixes the invariant at 17 as well. A somewhat surprising conclusion is this: given any point $\tilde{P} \in \tilde{Y}(\mathbb{F}_{17})$, at most half of the scalar multiples of \tilde{P} lift to rational points of Y with coprime integer coordinates.

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