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Corrigendum to "Stability aspects of arithmetic functions, II"

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by

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Introduction. As pointed out by A. Schinzel, there is an error in the proof of [5, Proposition 1.4] which is a crucial part of the stability theorem for arithmetic additive functions [5, Theorem 1.5]. Namely, we are considering a certain submeasure $\varphi\colon 2^S \to [0,\infty)$ and, following the proof of [3, Theorem 4.1] by N. J. Kalton and J. W. Roberts, we need to estimate the covering index of the family $\mathcal{C} = \{C \subset S : \varphi(C) \le 9\varepsilon/2\}$. To this end J. L. Kelley's theorem [4, Corollary 6] is used: we claim that there is no additive set function $\kappa\colon 2^S \to [0,\infty)$ satisfying $\kappa(S) = 1$ and $\kappa(A) < 1/2$ for all $A \in \mathcal{C}$. Unfortunately, it is not clear why this should be true in our situation.

In the first section we provide a modified version of [5, Theorem 1.5] in which we assume that a given mapping is not only almost additive but almost strongly additive. The results from [5] which concern the multiplicative case are suitably modified as well; this is done in the last section.

1. Stability results for strongly additive functions. The main result of this section will be proved by slightly modifying the proof of the following theorem on nearly additive set functions which is Theorem 4.1 from [3].

THEOREM 1.1 (Kalton, Roberts). *There is an absolute constant $K \le 89/2$ with the following property: If X is a non-empty set, $\mathcal{A}$ is an algebra of subsets of X and a function $\nu\colon \mathcal{A} \to \mathbb{R}$ satisfies*

$$(A, B \in \mathcal{A},\ A \cap B = \emptyset) \ \Rightarrow\ |\nu(A \cup B) - \nu(A) - \nu(B)| \le \varepsilon$$

with some $\varepsilon \ge 0$, then there exists an additive set function $\mu\colon \mathcal{A} \to \mathbb{R}$ such that $|\nu(A) - \mu(A)| \le K\varepsilon$ for $A \in \mathcal{A}$.

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Let us recall some definitions and results used in the proof by Kalton and Roberts.

Definition 1.2. Let X be a non-empty set and $\mathcal{A}$ be an algebra of subsets of X. A mapping $\lambda\colon \mathcal{A} \to \mathbb{R}$ is called a *submeasure* if:

(i) $\lambda(\emptyset) = 0$,
(ii) $\lambda(A) \leq \lambda(B)$ for $A, B \in \mathcal{A}$, $A \subset B$,
(iii) $\lambda(A \cup B) \leq \lambda(A) + \lambda(B)$ for $A, B \in \mathcal{A}$.

The following notion was introduced in [4], together with the next result which is Corollary 6 therein. For any set A we denote by $\mathbf{1}_A$ the characteristic function of A.

Definition 1.3 (Kelley). Let X be a non-empty set and $\emptyset \neq \mathcal{C} \subset 2^X$. The *covering index* $J(\mathcal{C})$ of $\mathcal{C}$ is the supremum of all $t \geq 0$ for which there are some $C_1, \ldots, C_n \in \mathcal{C}$ such that

$$t\mathbf{1}_X \leq \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{C_i}.$$

Theorem 1.4 (Kelley). *Let X be a non-empty set, let $\mathcal{A}$ be an algebra of subsets of X and let $\emptyset \neq \mathcal{C} \subset \mathcal{A}$. Then there exists an additive set function $\mu\colon \mathcal{A} \to [0, \infty)$ such that*

$$\mu(C) \leq J(\mathcal{C}) \quad \text{for } C \in \mathcal{C} \quad \text{and} \quad \mu(X) = 1.$$

Now, we recall the notion of concentrator and an existence result from [7] that is proved by probabilistic methods. For $m \in \mathbb{N}$ denote $[m] = \{1, \ldots, m\}$. If $m, p \in \mathbb{N}$ and $R\colon [m] \to 2^{[p]}$, then for each $E \subset [m]$ we define

$$R[E] = \bigcup_{i \in E} R(i).$$

For any set A we denote by $\#A$ the cardinality of A.

Definition 1.5. Let $m, p, q, r \in \mathbb{N}$ and $q \leq p \leq m$. A function $R\colon [m] \to 2^{[p]}$ is called an (m, p, q, r)-*concentrator* if:

(i) $m^{-1} \sum_{i=1}^{m} \#R(i) \leq r$,
(ii) $\#E \leq \#R[E]$ for $E \subset [m]$ with $\#E \leq q$.

Theorem 1.6 (Pippenger). *There exists a $(6m, 4m, 3m, 6)$-concentrator for each $m \in \mathbb{N}$.*

The following useful notation was introduced in [3].

Definition 1.7. For $r \in \mathbb{N}$ and $\delta, \eta \in (0, 1)$ we say that $H(r, \delta, \eta)$ holds true if there exist sequences $(m_k)_{k \in \mathbb{N}}$, $(p_k)_{k \in \mathbb{N}}$, $(q_k)_{k \in \mathbb{N}}$ of natural numbers such that $m_k \to \infty$ and for every $k \in \mathbb{N}$ we have $p_k/m_k \leq \delta$, $q_k/m_k \geq \eta$ and there exists a (m_k, p_k, q_k, r)-concentrator. For $r \in \mathbb{N}$ and $\eta \in (0,1)$ we put

$$\vartheta(r,\eta) = \inf\{\delta \in (0,1) : \ H(r,\delta,\eta) \text{ holds true}\}.$$

Theorem 1.6 translates now into the inequality

$$\vartheta(6, 1/2) \leq 2/3. \tag{1.1}$$

The next result is Lemma 3.1 from [3] which gives a connection between concentrators and some estimates for submeasures.

Lemma 1.8 (Kalton, Roberts). *Let X be a non-empty set and $\mathcal{A}$ be an algebra of subsets of X. Assume that $\lambda\colon \mathcal{A} \to \mathbb{R}$ is a submeasure and there are some constants $\alpha, \beta \geq 0$ such that for any pairwise disjoint sets $A_1, \dots, A_n \in \mathcal{A}$ we have*

$$\sum_{i=1}^{n} \lambda(A_i) \leq \alpha n + \beta.$$

If $\eta \in (0,1)$ and $B_1, \dots, B_m \in \mathcal{A}$ satisfy

$$\frac{1}{m}\sum_{i=1}^{m} \mathbf{1}_{B_i} \geq (1-\eta)\mathbf{1}_X,$$

then for each $r \in \mathbb{N}$ with $r \geq 3$ we have

$$\frac{1}{m}\sum_{i=1}^{m} \lambda(B_i) \geq \lambda(X) - \alpha r - \beta\vartheta(r,\eta).$$

Let $\mathbb{P}$ be the set of all primes, let $\mathbb{S} = \{p^k : p \in \mathbb{P},\ k \in \mathbb{N}\}$ and for each $x \in \mathbb{N}$ let $P_x = \{p \in \mathbb{P} : p \,|\, x\}$. Our result concerning stability for strongly additive functions reads as follows.

Theorem 1.9. *There is an absolute constant $K^* \leq 89/2$ having the property: If a function $f\colon \mathbb{N} \to \mathbb{R}$ satisfies*

$$x, y \in \mathbb{N},\ (x,y) = 1 \ \Rightarrow \ |f(xy) - f(x) - f(y)| \leq \varepsilon \tag{1.2}$$

and

$$x, y \in \mathbb{N},\ P_x = P_y \ \Rightarrow \ |f(x) - f(y)| \leq 2\varepsilon \tag{1.3}$$

with some $\varepsilon \geq 0$, then there exists a strongly additive function $\widetilde{f}\colon \mathbb{N} \to \mathbb{R}$ such that $|f(x) - \widetilde{f}(x)| \leq K^*\varepsilon$ for $x \in \mathbb{N}$.

We will derive this theorem from the following lemma.

Lemma 1.10. *There is an absolute constant $K^* \leq 89/2$ having the property: Let X be a non-empty finite set and let $\mathcal{R} = \{X_1, \dots, X_N\}$ be a family of non-empty pairwise disjoint subsets of X whose sum is X. Put*

$$\mathcal{A} = \{A \subset X : \#(A \cap X_i) \leq 1 \textit{ for } 1 \leq i \leq N\} \tag{1.4}$$

and let $\simeq$ *be the relation in* $\mathcal{A}$ *defined by*

$$A \simeq B \;\Leftrightarrow\; \#(A \cap X_i) = \#(B \cap X_i) \text{ for } 1 \le i \le N.$$

If a function $\nu\colon \mathcal{A} \to \mathbb{R}$ *satisfies*

$$(A, B, A \cup B \in \mathcal{A} \text{ and } A \cap B = \emptyset) \;\Rightarrow\; |\nu(A \cup B) - \nu(A) - \nu(B)| \le \varepsilon \tag{1.5}$$

and

$$(A, B \in \mathcal{A} \text{ and } A \simeq B) \;\Rightarrow\; |\nu(A) - \nu(B)| \le 2\varepsilon \tag{1.6}$$

with some $\varepsilon \ge 0$*, then there exists a function* $\mu\colon \mathcal{A} \to \mathbb{R}$ *such that*

$$(A, B, A \cup B \in \mathcal{A} \text{ and } A \cap B = \emptyset) \;\Rightarrow\; \mu(A \cup B) = \mu(A) + \mu(B), \tag{1.7}$$

$$(A, B \in \mathcal{A} \text{ and } A \simeq B) \;\Rightarrow\; \mu(A) = \mu(B) \tag{1.8}$$

and

$$|\nu(A) - \mu(A)| \le K^*\varepsilon \quad \text{for } A \in \mathcal{A}. \tag{1.9}$$

Proof. We divide the proof into several steps.

Step 1. Each function $\mu\colon \mathcal{A} \to \mathbb{R}$ satisfying (1.7) and (1.8) will be called *strongly* $\mathcal{R}$*-additive*. For any function $f\colon \mathcal{A} \to \mathbb{R}$ put

$$V(f) = \max_{A,B \in \mathcal{A}} (f(A) - f(B)).$$

Let $\sim$ be the equivalence relation in $\mathbb{R}^{\mathcal{A}}$ given by

$$(f \sim g) \;\Leftrightarrow\; (f - g \text{ is a constant function});$$

then $(\mathbb{R}^{\mathcal{A}}/{\sim}, \rho)$ is a metric space with $\rho([f]_\sim, [g]_\sim) := V(f - g)$. Define

$$\mathcal{M} = \{[\mu]_\sim : \mu \text{ is strongly } \mathcal{R}\text{-additive}\}.$$

Take a sequence of $[\mu_n]_\sim \in \mathcal{M}$, $n \in \mathbb{N}$, such that

$$\operatorname{dist}([\nu]_\sim, \mathcal{M}) =: l \le \rho([\nu]_\sim, [\mu_n]_\sim) \le l + 1/n.$$

The set $\{\mu_n\}_{n\in\mathbb{N}}$ is pointwise bounded on $\mathcal{A}$, and hence it is contained in a compact subset of $\mathbb{R}^{\mathcal{A}}$ (with the topology of pointwise convergence). Let μ be the limit of some convergent subsequence of $(\mu_n)_{n\in\mathbb{N}}$. Then μ is strongly $\mathcal{R}$-additive and we have $V(\nu - \mu) = l$.

Define $g = \nu - \mu$ and

$$a = \max_{A \in \mathcal{A}} g(A), \qquad b = -\min_{A \in \mathcal{A}} g(A);$$

we may assume that $a \ge b$ (otherwise consider $-g$ instead of g). We are to prove that $a \le 89\varepsilon/2$.

Choose $S \in \mathcal{A}$ with $g(S) = a$. Since $S \in \mathcal{A}$, we have $2^S \subset \mathcal{A}$. Assumptions (1.5), (1.6), jointly with the fact that μ is strongly $\mathcal{R}$-additive, imply that

$$(A, B, A \cup B \in \mathcal{A} \text{ and } A \cap B = \emptyset) \;\Rightarrow\; |g(A \cup B) - g(A) - g(B)| \le \varepsilon$$

and

$$(A, B \in \mathcal{A} \text{ and } A \simeq B) \;\Rightarrow\; |g(A) - g(B)| \le 2\varepsilon.$$

Hence, the mapping $\lambda\colon 2^S \to \mathbb{R}$ defined by

$$\lambda(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ \varepsilon + \sup\limits_{C \subset A} g(C) & \text{if } A \ne \emptyset, \end{cases}$$

is a submeasure. In fact, for each $A, B \subset S$ and every $C \subset A \cup B$ we have

$$g(C) \le \varepsilon + g(C \cap A) + g(C \cap (B \setminus A)).$$

Moreover, for arbitrary non-empty pairwise disjoint $A_1, \ldots, A_n \subset S$ there exist $B_i \subset A_i$ $(1 \le i \le n)$ such that

$$\begin{aligned} \sum_{i=1}^{n} \lambda(A_i) &= \varepsilon n + \sum_{i=1}^{n} g(B_i) \\ &\le \varepsilon n + \varepsilon(n-1) + g(B_1 \cup \cdots \cup B_n) \le 2\varepsilon n + (a - \varepsilon). \end{aligned} \tag{1.10}$$

Note also that $\lambda(S) = a + \varepsilon$.

Step 2. Let

$$\mathcal{C} = \{C \subset S : \lambda(C) \le 9\varepsilon/2\}.$$

We will show that $J(\mathcal{C}) \ge 1/2$. In the light of Theorem 1.4, we shall show that for every additive set function $\kappa\colon 2^S \to [0, \infty)$ satisfying $\kappa(S) = \varepsilon$ there exists a set $A \in \mathcal{C}$ with $\kappa(A) \ge \varepsilon/2$.

Suppose, on the contrary, that $\kappa\colon 2^S \to [0, \infty)$ is an additive set function satisfying $\kappa(S) = \varepsilon$ and $\kappa(A) < \varepsilon/2$ for each $A \in \mathcal{C}$. For any $A \in \mathcal{A}$ let $\pi_S(A)$ be the maximal (with respect to set inclusion) subset of S satisfying $\pi_S(A) \simeq C$ for some $C \subset A$ (notice that all the subsets of S which have this property form a finite chain with respect to inclusion, thus $\pi_S(A)$ is uniquely determined). For arbitrary $A, B \in \mathcal{A}$ the conditions $A \cup B \in \mathcal{A}$ and $A \cap B = \emptyset$ imply that $\pi_S(A) \cap \pi_S(B) = \emptyset$ and hence

$$\kappa(\pi_S(A) \cup \pi_S(B)) = \kappa(\pi_S(A)) + \kappa(\pi_S(B)),$$

whereas the condition $A \simeq B$ implies $\pi_S(A) = \pi_S(B)$ and hence $\kappa(\pi_S(A)) = \kappa(\pi_S(B))$. This shows that the function

$$\mathcal{A} \ni A \mapsto \kappa(\pi_S(A)) \tag{1.11}$$

is strongly $\mathcal{R}$-additive.

Define $h\colon \mathcal{A} \to \mathbb{R}$ by $h(A) = g(A) - \kappa(\pi_S(A))$. By the definition of μ and the fact that the mapping (1.11) is strongly $\mathcal{R}$-additive, we have $V(h) \ge V(g) = a + b$. However, we will show that

$$-b - \varepsilon/2 < h(A) < a - \varepsilon/2 \quad \text{for } A \in \mathcal{A},$$

which yields a contradiction.

Suppose $h(A) \geq a - \varepsilon/2$ for some $A \in \mathcal{A}$. Then $g(A) \geq a - \varepsilon/2$. Let A^* be the subset of A for which $A^* \simeq \pi_S(A)$. We have

$$g(A \setminus A^*) \leq \varepsilon + g((A \setminus A^*) \cup S) - g(S) \leq \varepsilon,$$

therefore

$$g(A^*) \geq g(A) - g(A \setminus A^*) - \varepsilon \geq a - 5\varepsilon/2.$$

Hence, for every $B \subset S \setminus \pi_S(A)$ we have

$$g(B) \leq \varepsilon + g(A^* \cup B) - g(A^*) \leq 7\varepsilon/2.$$

It follows that $\lambda(S \setminus \pi_S(A)) \leq 9\varepsilon/2$, i.e. $S \setminus \pi_S(A) \in \mathcal{C}$. Therefore, $\kappa(S \setminus \pi_S(A)) < \varepsilon/2$ and thus $\kappa(\pi_S(A)) > \varepsilon/2$, which proves that $h(A) < g(A) - \varepsilon/2 \leq a - \varepsilon/2$; a contradiction.

Now, suppose $h(A) \leq -b - \varepsilon/2$ for some $A \in \mathcal{A}$. Then $g(A) \leq -b + \varepsilon/2$. For every $B \subset A^*$ we have

$$-b + \varepsilon/2 \geq g(A) \geq g(B) + g(A \setminus B) - \varepsilon \geq g(B) - b - \varepsilon,$$

hence $g(B) \leq 3\varepsilon/2$. Consequently, if $C \in \mathcal{A}$ satisfies $B \simeq C$, then $g(C) \leq 7\varepsilon/2$. This shows that for each $C \subset \pi_S(A)$ we have $g(C) \leq 7\varepsilon/2$, i.e. $\lambda(\pi_S(A)) \leq 9\varepsilon/2$, thus $\pi_S(A) \in \mathcal{C}$ and $\kappa(\pi_S(A)) < \varepsilon/2$. Hence, $h(A) > g(A) - \varepsilon/2 \geq -b - \varepsilon/2$; a contradiction.

Step 3. Define

$$\gamma = \inf\Big\{\sum_{C\in\mathcal{C}} x_C : \sum_{C\in\mathcal{C}} x_C \mathbf{1}_C \geq \mathbf{1}_S \text{ and } x_C \geq 0 \text{ for } C \in \mathcal{C}\Big\}.$$

It follows from Definition 1.3 that $J(\mathcal{C}) = 1/\gamma$. The set

$$Z := \Big\{(x_C)_{C\in\mathcal{C}} : \sum_{C\in\mathcal{C}} x_C \mathbf{1}_C \geq \mathbf{1}_S \text{ and } x_C \geq 0 \text{ for } C \in \mathcal{C}\Big\}$$

is an unbounded polyhedron in $\mathbb{R}^{\#\mathcal{C}}$. Either there exists a unique point $\boldsymbol{x} = (x_C)_{C\in\mathcal{C}}$ of Z that minimizes the sum $\sum_{C\in\mathcal{C}} x_C$, and it is then an extreme point of Z, or the set of all such points is a polyhedron $Z' \subset Z$, and then every extreme point of Z' is also an extreme point of Z.

In both cases there is at least one extreme point $\boldsymbol{x} = (x_C)_{C\in\mathcal{C}}$ of Z such that

$$\gamma = \sum_{C\in\mathcal{C}} x_C.$$

Such a point is uniquely determined as the intersection of finitely many hyperspaces in $\mathbb{R}^{\#\mathcal{C}}$, defined by equations with rational coefficients. The only solution of the system of those equations (which is $\boldsymbol{x}$) has all coefficients rational. Therefore, $\gamma \in \mathbb{Q}$ and hence $J(\mathcal{C}) \in \mathbb{Q}$.

Since $J(\mathcal{C}) \geq 1/2$, we have

$$\sum_{C\in\mathcal{C}} J(\mathcal{C})x_C \mathbf{1}_C \geq \frac{1}{2}\mathbf{1}_S.$$

Moreover, all the numbers $J(\mathcal{C})x_C$ $(C \in \mathcal{C})$ are rational. Denoting by m the least common multiple of their denominators and repeating each set $C \in \mathcal{C}$ as required, we may find a sequence $C_1, \dots, C_m \in \mathcal{C}$ such that

$$\frac{1}{m}\sum_{i=1}^{m} \mathbf{1}_{C_i} \geq \frac{1}{2}\mathbf{1}_S.$$

By Lemma 1.8 and inequality (1.10), for every $r \in \mathbb{N}$, $r \geq 3$, we have

$$9\varepsilon/2 \geq \frac{1}{m}\sum_{i=1}^{m} \lambda(C_i) \geq (a+\varepsilon) - 2\varepsilon r - (a-\varepsilon)\vartheta(r, 1/2).$$

Putting $r = 6$ and making use of (1.1) we arrive at $a \leq 89\varepsilon/2$. ∎

Proof of Theorem 1.9. Let K^* be the constant from the above lemma. Consider the topological space

$$\mathcal{Z} := \prod_{p\in\mathbb{P}} [f(p) - K^*\varepsilon, f(p) + K^*\varepsilon]$$

with the compact Tikhonov topology. If $x \in \mathbb{N}$, $x \geq 2$, has a factorization $x = p_1^{\alpha_1} \cdot \ldots \cdot p_m^{\alpha_m}$ (i.e. $p_i \in \mathbb{P}$ are pairwise different and $\alpha_i \in \mathbb{N}$), then we put

$$\mathcal{Z}_x = \{(\xi(p))_{p\in\mathbb{P}} \in \mathcal{Z} : |f(x) - (\xi(p_1) + \cdots + \xi(p_m))| \leq K^*\varepsilon\}.$$

Plainly, $\mathcal{Z}_x$ is a closed subspace of $\mathcal{Z}$.

Fix an arbitrary finite set of natural numbers $x_1, \dots, x_n \geq 2$ with factorizations $x_i = s_{i1} \cdot \ldots \cdot s_{im_i}$ (i.e. $s_{i1}, \dots, s_{im_i} \in \mathbb{S}$ are pairwise relatively prime for each $1 \leq i \leq n$), where $s_{ij} = p_{ij}^{\alpha_{ij}}$ with $p_{ij} \in \mathbb{P}$, $\alpha_{ij} \in \mathbb{N}$. Put

$$X = \{s_{ij} : i = 1, \dots, n,\ j = 1, \dots, m_i\}.$$

Enlarging the set $\{x_1, \dots, x_n\}$ if necessary, we may assume that $p_{ij} \in X$ for $1 \leq i \leq n$, $1 \leq j \leq m_i$. Let $\mathcal{R} = \{X_1, \dots, X_N\}$ be the partition of X into non-empty and pairwise disjoint subsets, defined by the condition that s_{ij}, s_{kl} belong to the same element of $\mathcal{R}$ if and only if $(s_{ij}, s_{kl}) > 1$. Let also $\mathcal{A}$ be defined by formula (1.4).

Since condition (1.2) implies (1.5), whereas (1.3) implies (1.6), we may apply Lemma 1.10 to the function $\nu\colon \mathcal{A} \to \mathbb{R}$ defined by

$$\nu(A) = f\Big(\prod_{a\in A} a\Big).$$

Thus there exists a function $\mu\colon \mathcal{A} \to \mathbb{R}$ satisfying (1.7)–(1.9). In particular, putting $A = \{s_{i1}, \dots, s_{im_i}\}$, for $1 \leq i \leq n$, we get

$$|f(x_i) - (\mu(\{s_{i1}\}) + \cdots + \mu(\{s_{im_i}\}))| \leq K^*\varepsilon,$$

and putting $A = \{p_{ij}\}$, for $1 \le i \le n$, $1 \le j \le m_i$, we infer that

$$\mu(\{p_{ij}\}) \in [f(p_{ij}) - K^*\varepsilon, f(p_{ij}) + K^*\varepsilon].$$

Moreover, in view of (1.8), for any $1 \le i \le n$, $1 \le j \le m_i$ we have $\mu(\{s_{ij}\}) = \mu(\{p_{ij}\})$. This shows that the sequence $(\xi(p))_{p\in\mathbb{P}}$ given by

$$\xi(p) = \begin{cases} \mu(\{p_{ij}\}) & \text{if } p = p_{ij} \text{ for some } 1 \le i \le n,\ 1 \le j \le m_i, \\ f(p) & \text{otherwise,} \end{cases}$$

belongs to the set $\bigcap_{i=1}^{n} \mathcal{Z}_{x_i}$.We have thus proved that the family $\{\mathcal{Z}_x : x \ge 2\}$ has the finite intersection property, hence there exists

$$(\xi(p))_{p\in\mathbb{P}} \in \bigcap_{x\ge 2} \mathcal{Z}_x. \tag{1.12}$$

Now, we define $\widetilde{f}$ as the unique strongly additive function satisfying $\widetilde{f}(p) = \xi(p)$ for every $p \in \mathbb{P}$. Then for each $x \in \mathbb{N}$, $x \ge 2$, the inequality $|f(x) - \widetilde{f}(x)| \le K^*\varepsilon$ follows directly from (1.12); for $x = 1$ this inequality follows from $\widetilde{f}(1) = 0$ and $|f(1)| \le \varepsilon$. ■

Let K be the least possible value of the constant from Theorem 1.1 and let K^* be the least possible value of the constant from Theorem 1.9 (equivalently: from Lemma 1.10). By considering the partition $\mathcal{R} = \{\{x\} : x \in X\}$, we see that Lemma 1.10 is a generalization of Theorem 1.1 in the case where X is finite. On the other hand, the finite case is enough to prove this theorem in its full generality (by some compactness arguments; cf. the proof of [3, Theorem 4.1]). Therefore, $K \le K^*$. Taking into account the counterexample of B. Pawlik [6], which shows that $K \ge 3/2$, we may write

$$3/2 \le K \le K^* \le 89/2.$$

It is not known whether any of these three inequalities is sharp.

For the time being I have not succeeded in fixing the proof of [5, Theorem 1.5] completely. Therefore, the following question remains open: Assume that $f\colon \mathbb{N} \to \mathbb{R}$ satisfies (1.2) with some $\varepsilon \ge 0$. Does there exist an additive arithmetic function $\widetilde{f}\colon \mathbb{N} \to \mathbb{R}$ such that $|f(x) - \widetilde{f}(x)| \le L\varepsilon$ for $x \in \mathbb{N}$, where $L < \infty$ is an absolute constant?

Let us now proceed to other stability results for additive functions from [5] which require revision. Remark 1.6 should be rewritten in the following form.

Theorem 1.11. *If a function $f\colon \mathbb{N} \to \mathbb{C}$ satisfies* (1.2) *and* (1.3) *with some $\varepsilon \ge 0$, then there exists a strongly additive function $\widetilde{f}\colon \mathbb{N} \to \mathbb{C}$ such that for each $\eta > 0$ the set*

$$\{x \in \mathbb{N} : |f(x) - \widetilde{f}(x)| < \varepsilon + \eta\}$$

contains a subset of the form $\mathbb{N} \setminus (P \cdot \mathbb{N})$ with some finite set $P \subset \mathbb{P}$. In particular, it has a positive natural density.

The proof remains the same except that instead of [5, Theorem 1.5] we use Theorem 1.9 above.

The proof of [5, Theorem 1.11] requires some modification. Instead of applying Theorem 1.5 therein, we first fix an arbitrary sequence $R = (x_m)_{m\in\mathbb{N}}$ of pairwise coprime natural numbers greater than 1 satisfying

$$\sup_{m\in\mathbb{N}} \omega(x_m) < \infty.$$

Next, we define

$$S_R = \{r \in \mathbb{S} : r \parallel x_m \text{ for some } m \in \mathbb{N}\}$$

and we let T_R be the set of all natural numbers having canonical factorizations with all factors belonging to S_R. Now, we consider a function $g\colon \mathbb{N} \to \mathbb{C}$ which is an extension of $f|_{T_R}$, and for which both conditions (1.2) and (1.3) are valid after replacing f by g. To see that such a function exists denote $Q(x) = \{r \in S_R : r \parallel x\}$ and put

$$g(x) = f\Big(\prod_{r\in Q(x)} r\Big) \quad \text{for } x \in \mathbb{N}.$$

Applying Theorem 1.9 to the real and imaginary parts of g we get a strongly additive complex-valued function $\widetilde{f}$ such that

$$|f(x) - \widetilde{f}(x)| \le \sqrt{2}\, K^*\varepsilon \quad \text{for } x \in T_R.$$

The rest of the proof needs only minor adjustments.

A correction should also be made to the proof of [5, Theorem 1.8]. Namely, in all estimates one has to replace (x_1, x_2) by p^k. Since $p^k \parallel x_1$ and $p^k \parallel x_2$, the numbers x_i/p^k and p^k are relatively prime for $i = 1, 2$, which is not necessarily true for $x_i/(x_1, x_2)$ and (x_1, x_2). The assumption of that theorem can be slightly weakened; the new version is the following.

We denote by $\mathcal{P}$ the set of all pairs of relatively prime natural numbers.

Theorem 1.12. *Given $\varphi\colon \mathcal{P} \to [0, \infty)$ assume that a function $f\colon \mathbb{N} \to \mathbb{R}$ satisfies*

$$x, y \in \mathbb{N},\ (x, y) = 1 \ \Rightarrow\ |f(xy) - f(x) - f(y)| \le \varphi(x, y).$$

Let $\psi\colon \mathbb{N} \to [0, \infty)$ be any function satisfying

$$\psi(xz) + \psi(yz) \ge \psi(x) + \psi(y) + \varphi(x, z) + \varphi(y, z)$$

for all $x, y, z \in \mathbb{N}$ such that $(x, z), (y, z) \in \mathcal{P}$. Then there is a real arithmetic additive function $\widetilde{f}$ such that $|f(x) - \widetilde{f}(x)| \le \psi(x)$ for $x \in \mathbb{N}$.

2. Stability results for strongly multiplicative functions. In this section we give a corrected version of the stability result [5, Theorem 2.2] concerning arithmetic multiplicative functions.

Theorem 2.1. *If a function* $f\colon \mathbb{N} \to \mathbb{C}\setminus\{0\}$ *satisfies*

$$x, y \in \mathbb{N},\ (x,y) = 1 \ \Rightarrow\ \left|\frac{f(xy)}{f(x)f(y)} - 1\right| \le \varepsilon \tag{2.1}$$

with some $\varepsilon \in [0,1)$ *and*

$$x, y \in \mathbb{N},\ P_x = P_y \ \Rightarrow\ \left|\frac{f(x)}{f(y)} - 1\right| \le \rho \tag{2.2}$$

with some $\rho \in [0, \sqrt{3}/2]$ *satisfying*

$$\rho \le \min\{2\varepsilon - \varepsilon^2, 2\varepsilon\sqrt{1-\varepsilon^2}\}, \tag{2.3}$$

then there exists a strongly multiplicative function $\widetilde{f}\colon \mathbb{N} \to \mathbb{C}\setminus\{0\}$ *such that*

$$\left|\frac{f(x)}{\widetilde{f}(x)} - 1\right| \le \delta(\varepsilon) \quad \textit{and} \quad \left|\frac{\widetilde{f}(x)}{f(x)} - 1\right| \le \delta(\varepsilon) \qquad \textit{for } x \in \mathbb{N},$$

where

$$\delta(\varepsilon) = \begin{cases} \sqrt{1 - \dfrac{2\cos(K^*\arcsin\varepsilon)}{(1-\varepsilon)^{K^*}} + \dfrac{1}{(1-\varepsilon)^{2K^*}}} & \textit{if } \varepsilon < \sin\dfrac{\pi}{K^*}, \\ 1 + (1-\varepsilon)^{-K^*} & \textit{if } \varepsilon \ge \sin\dfrac{\pi}{K^*}. \end{cases}$$

We will follow the methodology of the proof of a stability result for the exponential equation given in [1] by R. Ger and P. Šemrl. Let us start with a certain version of M. Hosszú's theorem [2].

Lemma 2.2. *If a function* $\psi\colon \mathcal{P} \to \mathbb{R}$ *satisfies*

$$\psi(x,y) = \psi(y,x) \quad \textit{for } (x,y) \in \mathcal{P}, \tag{2.4}$$

and

$$\psi(xy,z) + \psi(x,y) = \psi(x,yz) + \psi(y,z) \quad \textit{for } (x,y),(y,z),(z,x) \in \mathcal{P}, \tag{2.5}$$

then there exists a mapping $\beta\colon \mathbb{N} \to \mathbb{R}$ *such that*

$$\psi(x,y) = \beta(xy) - \beta(x) - \beta(y) \quad \textit{for } (x,y) \in \mathcal{P}. \tag{2.6}$$

Proof. Put $\beta(1) = -\psi(1,1)$ and for any $r \in \mathbb{S}$ let $\beta(r)$ be an arbitrarily chosen real number. We are going to define the values of β for arguments having at least two different prime factors.

Let $p_1, p_2, \ldots$ be the increasing sequence of all primes. For each $x \in \mathbb{N}$ with a canonical factorization $x = p_{n_1}^{k_1} \cdot \ldots \cdot p_{n_m}^{k_m}$, where $m \ge 2$ and

$n_1 < \cdots < n_m$, we define

$$\beta(x) = \sum_{j=1}^{m} \beta(p_{n_j}^{k_j}) + \sum_{j=1}^{m-1} \psi\Big(p_{n_j}^{k_j}, \prod_{i=j+1}^{m} p_{n_i}^{k_i}\Big).$$

For simplicity we will write canonical factorizations in the form $r_1 \cdot \ldots \cdot r_m$ assuming implicitly that $r_i \in \mathbb{S}$ are labeled according to the natural order of primes of which they are powers. For an arbitrary $x \in \mathbb{N}$ with factorization $x = s_1 \cdot \ldots \cdot s_m$, where $m \geq 2$, we have

$$\beta(x) = \beta(s_m) + \beta(s_1 \cdot \ldots \cdot s_{m-1}) + \psi(s_1 \cdot \ldots \cdot s_{m-1}, s_m). \tag{2.7}$$

Indeed, by the definition of $\beta(x)$ and (2.5), we infer that

$$\begin{aligned}
\beta(x) &= \sum_{j=1}^{m} \beta(s_j) + \sum_{j=1}^{m-1} \psi\Big(s_j, \prod_{i=j+1}^{m} s_i\Big) \\
&= \sum_{j=1}^{m} \beta(s_j) + \sum_{j=1}^{m-3} \psi\Big(s_j, \prod_{i=j+1}^{m} s_i\Big) + \psi(s_{m-2}, s_{m-1}s_m) + \psi(s_{m-1}, s_m) \\
&= \sum_{j=1}^{m} \beta(s_j) + \sum_{j=1}^{m-3} \psi\Big(s_j, \prod_{i=j+1}^{m} s_i\Big) + \psi(s_{m-2}s_{m-1}, s_m) + \psi(s_{m-2}, s_{m-1}) \\
&\ \ \vdots \\
&= \sum_{j=1}^{m} \beta(s_j) + \psi(s_1 \cdot \ldots \cdot s_{m-1}, s_m) + \sum_{j=1}^{m-2} \psi\Big(s_j, \prod_{i=j+1}^{m-1} s_i\Big) \\
&= \beta(s_m) + \beta(s_1 \cdot \ldots \cdot s_{m-1}) + \psi(s_1 \cdot \ldots \cdot s_{m-1}, s_m).
\end{aligned}$$

Putting $x = y = 1$, $z = s \in \mathbb{S}$ in (2.5) yields $\psi(1, s) = \psi(1, 1) = -\beta(1)$, hence (2.7) is also valid in the case where $x \in \mathbb{S}$.

We shall prove equality (2.6) by induction on $\omega(x) + \omega(y)$. If $\omega(x) + \omega(y) = 0$, then $x = y = 1$ and the equality holds. So, assume that some relatively prime $x, y \in \mathbb{N}$ have factorizations $x = q_1 \cdot \ldots \cdot q_m$ and $y = r_1 \cdot \ldots \cdot r_n$, where $m + n > 0$. Let xy have a factorization $s_1 \cdot \ldots \cdot s_{m+n}$ with suitably numbered s_i's. Either $q_m = s_{m+n}$, or $r_n = s_{m+n}$; we may assume the former. Then, using (2.7), the inductive hypothesis and (2.4), (2.5), we get

$$\begin{aligned}
&\beta(xy) - \beta(x) - \beta(y) \\
&\qquad = \beta(s_{m+n}) + \beta(s_1 \cdot \ldots \cdot s_{m+n-1}) + \psi(s_1 \cdot \ldots \cdot s_{m+n-1}, s_{m+n}) \\
&\qquad\quad - \beta(s_{m+n}) - \beta(q_1 \cdot \ldots \cdot q_{m-1}) - \psi(q_1 \cdot \ldots \cdot q_{m-1}, s_{m+n}) \\
&\qquad\quad - \beta(r_1 \cdot \ldots \cdot r_n)
\end{aligned}$$

$$= \psi(q_1 \cdot \ldots \cdot q_{m-1} r_1 \cdot \ldots \cdot r_n, q_m) - \psi(q_1 \cdot \ldots \cdot q_{m-1}, q_m)$$
$$+ \psi(q_1 \cdot \ldots \cdot q_{m-1}, r_1 \cdot \ldots \cdot r_n)$$
$$= \psi\Big(\frac{x}{q_m} y, q_m\Big) - \psi\Big(\frac{x}{q_m}, q_m\Big) + \psi\Big(\frac{x}{q_m}, y\Big)$$
$$= \psi\Big(\frac{x}{q_m}, y\Big) + \psi\Big(q_m, \frac{x}{q_m} y\Big) - \psi\Big(q_m, \frac{x}{q_m}\Big) = \psi\Big(q_m \frac{x}{q_m}, y\Big) = \psi(x,y). \ \blacksquare$$

LEMMA 2.3. *If a function* $\alpha\colon \mathbb{N} \to \mathbb{R}$ *satisfies*

$$x, y \in \mathbb{N},\ (x,y) = 1 \ \Rightarrow\ \alpha(xy) - \alpha(x) - \alpha(y) \in [-\varepsilon, \varepsilon] + \mathbb{Z}$$

with some $\varepsilon \in [0, 1/4)$ *and*

$$x, y \in \mathbb{N},\ P_x = P_y \ \Rightarrow\ \alpha(x) - \alpha(y) \in [-\eta, \eta] + \mathbb{Z} \tag{2.8}$$

with some $\eta \in [0, 1/3)$ *satisfying* $\eta \le 2\varepsilon$ *and* $2\varepsilon + 3\eta < 1$, *then there exists a mapping* $\widetilde{\alpha}\colon \mathbb{N} \to \mathbb{R}$ *such that*

$$x, y \in \mathbb{N},\ (x,y) = 1 \ \Rightarrow\ \widetilde{\alpha}(xy) - \widetilde{\alpha}(x) - \widetilde{\alpha}(y) \in \mathbb{Z},$$
$$x, y \in \mathbb{N},\ P_x = P_y \ \Rightarrow\ \widetilde{\alpha}(x) - \widetilde{\alpha}(y) \in \mathbb{Z}$$

and

$$|\alpha(x) - \widetilde{\alpha}(x)| \le K^* \varepsilon \quad \textit{for } x \in \mathbb{N}.$$

Proof. For some functions $\psi\colon \mathcal{P} \to \mathbb{Z}$ and $\varphi\colon \mathcal{P} \to [-\varepsilon, \varepsilon]$ we have

$$\alpha(xy) - \alpha(x) - \alpha(y) = \psi(x,y) + \varphi(x,y) \quad \text{for } (x,y) \in \mathcal{P}. \tag{2.9}$$

Since the left-hand side is symmetric with respect to x and y, we infer that

$$\mathbb{Z} \ni \psi(x,y) - \psi(y,x) = \varphi(y,x) - \varphi(x,y) \in [-2\varepsilon, 2\varepsilon] \subset (-1/2, 1/2),$$

which implies (2.4). Moreover, for all $(x,y), (y,z), (z,x) \in \mathcal{P}$ we have

$$\begin{aligned}
\psi(x,yz) &+ \psi(y,z) + \varphi(x,yz) + \varphi(y,z) \\
&= \alpha(xyz) - \alpha(x) - \alpha(yz) + \alpha(yz) - \alpha(y) - \alpha(z) \\
&= \alpha(xyz) - \alpha(xy) - \alpha(z) + \alpha(xy) - \alpha(x) - \alpha(y) \\
&= \psi(xy,z) + \psi(x,y) + \varphi(xy,z) + \varphi(x,y).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{Z} \ni \psi(x,yz) &+ \psi(y,z) - \psi(xy,z) - \psi(x,y) \\
&= \varphi(x,yz) + \varphi(y,z) - \varphi(xy,z) - \varphi(x,y) \in [-4\varepsilon, 4\varepsilon] \subset (-1,1),
\end{aligned}$$

which implies (2.5). By Lemma 2.2, there exists a mapping $\beta\colon \mathbb{N} \to \mathbb{R}$ satisfying (2.6).

Let $\mathcal{Q}$ stand for the set of all pairs of natural numbers $x, y > 1$ such that $P_x = P_y$. Assumption (2.8) implies that there are some functions $\psi_1\colon \mathcal{Q} \to \mathbb{Z}$ and $\varphi_1\colon \mathcal{Q} \to [-\eta, \eta]$ satisfying,

$$\alpha(x) - \alpha(y) = \psi_1(x,y) + \varphi_1(x,y) \quad \text{for } (x,y) \in \mathcal{Q}. \tag{2.10}$$

An inspection of the proof of Lemma 2.2 shows that the values of β for arguments from the set $\mathbb{S}$ may be defined arbitrarily (this is also seen from condition (2.6) which remains valid if we add to β any additive function). Fix $p \in \mathbb{P}$; we may assume that for each $k \in \mathbb{N}$ we have $\beta(p^k) = \psi_1(p^k, p) + \beta(p)$. For all $k, l, m \in \mathbb{N}$ we then have

$$\mathbb{Z} \ni \psi_1(p^k, p^l) + \psi_1(p^l, p^k) = -\varphi_1(p^k, p^l) - \varphi_1(p^l, p^k) \in [-2\eta, 2\eta],$$

thus

$$\psi_1(p^k, p^l) = -\psi_1(p^l, p^k). \tag{2.11}$$

Moreover,

$$\begin{aligned}\mathbb{Z} \ni \psi_1(p^k, p^m) &- \psi_1(p^k, p^l) - \psi_1(p^l, p^m) \\ &= -\varphi_1(p^k, p^m) + \varphi_1(p^k, p^l) + \varphi_1(p^l, p^m) \in [-3\eta, 3\eta],\end{aligned}$$

thus

$$\psi_1(p^k, p^l) + \psi_1(p^l, p^m) = \psi_1(p^k, p^m). \tag{2.12}$$

By (2.11) and (2.12), we have

$$\begin{aligned}\beta(p^k) - \beta(p^l) &= \psi_1(p^k, p) - \psi_1(p^l, p) = \psi_1(p^k, p) + \psi_1(p, p^l) \\ &= \psi_1(p^k, p^l).\end{aligned} \tag{2.13}$$

Now, we will show that

$$\psi_1(x, y) = \beta(x) - \beta(y) \quad \text{for } (x, y) \in \mathcal{Q}. \tag{2.14}$$

Equality (2.13) means precisely that the above equality holds true when x and y are powers of the same prime. Fix natural numbers $m \geq 2$, x and y satisfying $P_x = P_y$ and having canonical factorizations $x = q_1 \cdot \ldots \cdot q_m$, $y = r_1 \cdot \ldots \cdot r_m$; assume also that (2.14) holds true for numbers having less than m prime divisors. Define $x' = q_1 \cdot \ldots \cdot q_{m-1}$ and $y' = r_1 \cdot \ldots \cdot r_{m-1}$. By (2.7), we have

$$\begin{aligned}&\beta(x) - \beta(y) \\ &\quad = \beta(q_m) - \beta(r_m) + \beta(x') - \beta(y') + \psi(x', q_m) - \psi(y', r_m) \\ &\quad = \psi_1(q_m, r_m) + \psi_1(x', y') + \psi(x', q_m) - \psi(y', r_m) \in \mathbb{Z}.\end{aligned} \tag{2.15}$$

Observe also that

$$\begin{aligned}\mathbb{Z} \ni \psi_1(x, y) &= \alpha(x) - \alpha(y) - \varphi_1(x, y) \\ &= (\alpha(x) - \alpha(x') - \alpha(q_m)) - (\alpha(y) - \alpha(y') - \alpha(r_m)) \\ &\quad + (\alpha(x') - \alpha(y')) + (\alpha(q_m) - \alpha(r_m)) - \varphi_1(x, y) \\ &= \psi(x', q_m) + \varphi(x', q_m) - \psi(y', r_m) - \varphi(y', r_m) \\ &\quad + \psi_1(x', y') + \varphi_1(x', y') + \psi_1(q_m, r_m) + \varphi_1(q_m, r_m) - \varphi_1(x, y) \\ &\in \psi_1(q_m, r_m) + \psi_1(x', y') + \psi(x', q_m) - \psi(y', r_m) \\ &\qquad\qquad + [-2\varepsilon - 3\eta, 2\varepsilon + 3\eta],\end{aligned}$$

which, jointly with (2.15), yields formula (2.14).

Define $\gamma\colon \mathbb{N} \to \mathbb{R}$ by $\gamma = \alpha - \beta$. Then, in view of (2.6) and (2.9),

$$\beta(xy) - \beta(x) - \beta(y) \in \mathbb{Z} \quad \text{for } (x,y) \in \mathcal{P}$$

and

$$\gamma(xy) - \gamma(x) - \gamma(y) = \varphi(x,y) \in [-\varepsilon, \varepsilon] \quad \text{for } (x,y) \in \mathcal{P}.$$

Moreover, by (2.10) and (2.14), we have

$$\beta(x) - \beta(y) \in \mathbb{Z} \quad \text{for } (x,y) \in \mathcal{Q}$$

and

$$\gamma(x) - \gamma(y) = \varphi_1(x,y) \in [-\eta, \eta] \quad \text{for } (x,y) \in \mathcal{Q}.$$

By Theorem 1.9, there exists a strongly additive function $\delta\colon \mathbb{N} \to \mathbb{R}$ such that $|\gamma(x) - \delta(x)| \le K^*\varepsilon$ for $x \in \mathbb{N}$. It remains to define $\widetilde{\alpha}\colon \mathbb{N} \to \mathbb{R}$ by $\widetilde{\alpha} = \beta + \delta$. ∎

Proof of Theorem 2.1. For every $x \in \mathbb{N}$, $f(x) = |f(x)| \exp(i \arg(f(x)))$, where $-\pi < \arg(f(x)) \le \pi$. Inequality (2.1) implies for all relatively prime $x, y \in \mathbb{N}$ that

$$\bigl|\log|f(xy)| - \log|f(x)| - \log|f(y)|\bigr| = \left|\log\left|\frac{f(xy)}{f(x)f(y)}\right|\right| \le -\log(1-\varepsilon).$$

Similarly, for $x, y \in \mathbb{N}$ satisfying $P_x = P_y$ inequality (2.2) yields

$$\bigl|\log|f(x)| - \log|f(y)|\bigr| = \left|\log\left|\frac{f(x)}{f(y)}\right|\right| \le -\log(1-\rho).$$

It follows from $\rho \le 2\varepsilon - \varepsilon^2$ that

$$-\log(1-\rho) \le -2\log(1-\varepsilon),$$

which means that the function $\mathbb{N} \ni x \mapsto \log|f(x)|$ satisfies the assumptions of Theorem 1.9 with $-\log(1-\varepsilon)$ instead of ε. Hence, there exists a strongly additive function $g\colon \mathbb{N} \to \mathbb{R}$ such that

$$\bigl|\log|f(x)| - g(x)\bigr| \le -K^* \log(1-\varepsilon) \quad \text{for } x \in \mathbb{N}. \tag{2.16}$$

Define $\alpha\colon \mathbb{N} \to \mathbb{R}$ by the formula $\alpha(x) = \arg(f(x))$ (where $-\pi < \arg(f(x)) \le \pi$) and observe that, in view of (2.1), for all relatively prime $x, y \in \mathbb{N}$ we have

$$\begin{aligned}\alpha(xy) - \alpha(x) - \alpha(y) &\in \arg\left(\frac{f(xy)}{f(x)f(y)}\right) + 2\pi\mathbb{Z}\\ &\subset [-\arcsin\varepsilon, \arcsin\varepsilon] + 2\pi\mathbb{Z}.\end{aligned}$$

Similarly, for $x, y \in \mathbb{N}$ satisfying $P_x = P_y$ inequality (2.2) yields

$$\alpha(x) - \alpha(y) \in \arg\left(\frac{f(x)}{f(y)}\right) + 2\pi\mathbb{Z} \subset [-\arcsin\rho, \arcsin\rho] + 2\pi\mathbb{Z}.$$

To apply Lemma 2.3 to the function $(2\pi)^{-1}\alpha$ notice that: $(2\pi)^{-1}\arcsin\varepsilon < 1/4$ and $(2\pi)^{-1}\arcsin\rho < 1/3$, which is obvious; further,

$$\arcsin\rho \le 2\arcsin\varepsilon,$$

which is equivalent to the assumed inequality $\rho \le 2\varepsilon\sqrt{1-\varepsilon^2}$; finally,

$$2\arcsin\varepsilon + 3\arcsin\rho < 2\pi,$$

which follows from the assumption $\rho \le \sqrt{3}/2$.

By Lemma 2.3, there exists a function $\widetilde{\alpha}\colon \mathbb{N} \to \mathbb{R}$ satisfying the following conditions:

$$x, y \in \mathbb{N}, (x, y) = 1 \;\Rightarrow\; \widetilde{\alpha}(xy) - \widetilde{\alpha}(x) - \widetilde{\alpha}(y) \in 2\pi\mathbb{Z},$$
$$x, y \in \mathbb{N},\; P_x = P_y \;\Rightarrow\; \widetilde{\alpha}(x) - \widetilde{\alpha}(y) \in 2\pi\mathbb{Z},$$

and

$$|\alpha(x) - \widetilde{\alpha}(x)| \le K^* \arcsin\varepsilon \quad \text{for } x \in \mathbb{N}. \tag{2.17}$$

Define $\widetilde{f}\colon \mathbb{N} \to \mathbb{C}\setminus\{0\}$ by $\widetilde{f}(x) = \exp(g(x) + i\widetilde{\alpha}(x))$. Then $\widetilde{f}$ is strongly multiplicative and for every $x \in \mathbb{N}$ inequalities (2.16) and (2.17) imply that

$$\left|\frac{f(x)}{\widetilde{f}(x)}\right| = \exp(\log|f(x)| - g(x)) \in [(1-\varepsilon)^{K^*}, (1-\varepsilon)^{-K^*}]$$

and

$$\left|\arg\left(\frac{f(x)}{\widetilde{f}(x)}\right)\right| \le |\alpha(x) - \widetilde{\alpha}(x)| \le K^* \arcsin\varepsilon.$$

The same estimates are of course valid for the quotient $\widetilde{f}(x)/f(x)$. In other words, for each $x \in \mathbb{N}$ both $f(x)/\widetilde{f}(x)$ and $\widetilde{f}(x)/f(x)$ belong to the set

$$Z = \{z \in \mathbb{C} : (1-\varepsilon)^{K^*} \le |z| \le (1-\varepsilon)^{-K^*} \text{ and } |\arg(z)| \le K^* \arcsin\varepsilon\}.$$

Let $c = \sup_{z\in Z}|z - 1|$. It is easily seen that independently of the value of ε we have $c \le 1 + (1-\varepsilon)^{-K^*}$. In the case where $\varepsilon < \sin(\pi/K^*)$ we have $K^* \arcsin\varepsilon < \pi$ and hence

$$c = |(1-\varepsilon)^{-K^*}\exp(iK^*\arcsin\varepsilon) - 1| = \delta(\varepsilon),$$

which completes the proof. ∎

Remark 2.3 from [5] should now be rewritten in the following form.

Theorem 2.4. *If a function* $f\colon \mathbb{N} \to \mathbb{C}\setminus\{0\}$ *satisfies* (2.1) *with some* $\varepsilon \in [0, 1)$ *and* (2.2) *with some* $\rho \in [0, \sqrt{3}/2]$ *satisfying* (2.3)*, then there exists a strongly multiplicative function* $\widetilde{f}\colon \mathbb{N} \to \mathbb{C}\setminus\{0\}$ *such that for each* $\eta \in (0, \pi - \arcsin\varepsilon)$ *the set*

$$\left\{x \in \mathbb{N} : \left|\frac{f(x)}{\widetilde{f}(x)} - 1\right| \le \delta(\varepsilon, \eta) \text{ and } \left|\frac{\widetilde{f}(x)}{f(x)} - 1\right| \le \delta(\varepsilon, \eta)\right\},$$

where

$$\delta(\varepsilon,\eta)=\sqrt{1-\frac{2e^{\eta}\cos(\eta+\arcsin\varepsilon)}{1-\varepsilon}+\frac{e^{2\eta}}{(1-\varepsilon)^2}},$$

contains a subset of the form $\mathbb{N}\setminus(P\cdot\mathbb{N})$ *with some finite set* $P\subset\mathbb{P}$. *In particular, it has a positive natural density.*

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