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On a mixed problem in Diophantine approximation

by

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1. Introduction. In analogy with the Littlewood conjecture, de Mathan and Teulié [7] proposed recently a "mixed Littlewood conjecture". For any prime number p, the usual p-adic absolute value $|\cdot|_p$ is normalized in such a way that $|p|_p = p^{-1}$. We denote by $\|\cdot\|$ the distance to the nearest integer.

DE MATHAN–TEULIÉ CONJECTURE. *For every real number α and every prime number p, we have*

$$\inf_{q\geq 1} q\cdot \|q\alpha\|\cdot |q|_p = 0. \tag{1.1}$$

Obviously, the above conjecture holds if α is rational or has unbounded partial quotients in its continued fraction expansion. Thus, it only remains to consider the case when α is an element of the set $\boldsymbol{Bad}_1$ of badly approximable real numbers, that is,

$$\boldsymbol{Bad}_1 = \{\alpha\in\mathbb{R} : \inf_{q\geq 1} q\cdot\|q\alpha\| > 0\}.$$

De Mathan and Teulié [7] proved that (1.1) holds for every quadratic real number α (recall that such a number is in $\boldsymbol{Bad}_1$) but, despite several recent results [4, 3], the general conjecture is still unsolved.

If we rewrite (1.1) in the form

$$\inf_{a,q\geq 1,\,\gcd(a,q)=1} q^2\cdot\left|\alpha-\frac{a}{q}\right|\cdot|q|_p = 0,$$

then we have $|q|_p = \min\{|\mathrm{Norm}(q/a)|_p, 1\}$. Hence, upon replacing α by $1/\alpha$, the de Mathan–Teulié conjecture can be reformulated as follows: For every irrational real number α, for every prime number p and every positive real number ε, there exists a non-zero rational number ξ satisfying

$$|\alpha-\xi|\cdot\min\{|\mathrm{Norm}(\xi)|_p, 1\} < \varepsilon H(\xi)^{-2}.$$

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Throughout this paper, the *height* $H(P)$ of an integer polynomial $P(X)$ is the maximal of the absolute values of its coefficients. The *height* $H(\xi)$ of an algebraic number ξ is the height of its minimal defining polynomial over the rational integers $a_0 + a_1X + \cdots + a_dX^d$, and the *norm* of ξ, denoted by $\mathrm{Norm}(\xi)$, is the rational number $(-1)^d a_0/a_d$.

The above reformulation suggests asking the following question.

PROBLEM 1. *Let d be a positive integer. Let α be a real number that is not algebraic of degree less than or equal to d. For every prime number p and every positive real number ε, does there exist a non-zero real algebraic number ξ of degree at most d satisfying*

$$|\alpha - \xi| \cdot \min\{|\mathrm{Norm}(\xi)|_p, 1\} < \varepsilon H(\xi)^{-d-1}?$$

The answer to Problem 1 is clearly positive, unless (perhaps) when α is an element of the set $\boldsymbol{Bad}_d$ of real numbers that are badly approximable by algebraic numbers of degree at most d, where

$$\boldsymbol{Bad}_d = \{\alpha \in \mathbb{R} : \text{there exists } c > 0 \text{ such that } |\alpha - \xi| > cH(\xi)^{-d-1} \\ \text{for all algebraic numbers } \xi \text{ of degree at most } d\}.$$

For $d \geq 1$, the set $\boldsymbol{Bad}_d$ contains the set of algebraic numbers of degree $d+1$, but it remains an open problem to decide whether this inclusion is strict for $d \geq 2$; see the monograph [2] for more information. The purpose of the present note is to give a positive answer to Problem 1 for every positive integer d and every real algebraic number α of degree $d+1$. This extends the result from [7], which deals with the case $d = 1$.

2. Results. Throughout this paper, for a prime number p, a number field $\mathbb{K}$, and a non-Archimedean place v on $\mathbb{K}$ lying above p, we normalize the absolute value $|\cdot|_v$ in such a way that $|\cdot|_v$ and $|\cdot|_p$ coincide on $\mathbb{Q}$.

Our main result includes a positive answer to Problem 1 when α is a real algebraic number of degree $d+1$.

THEOREM 1. *Let d be a positive integer. Let α be a real algebraic number of degree $d+1$ and denote by r the unit rank of $\mathbb{Q}(\alpha)$. Let p be a prime number. There exist positive constants c_1, c_2, c_3, depending on α and p, and infinitely many real algebraic numbers ξ of degree d such that*

$$|\alpha - \xi| < c_1 H(\xi)^{-d-1}, \tag{2.1}$$

$$|\xi|_v < c_2(\log 3H(\xi))^{-1/(rd)} \tag{2.2}$$

for every absolute value $|\cdot|_v$ on $\mathbb{Q}(\xi)$ above the prime p, and

$$|\alpha - \xi| \cdot \min\{|\mathrm{Norm}(\xi)|_p, 1\} < c_3 H(\xi)^{-d-1}(\log 3H(\xi))^{-1/r}. \tag{2.3}$$

Theorem 1 extends Théorème 2.1 of [7], which only concerns the case $d = 1$.

Under the assumptions of Theorem 1, Wirsing [9] established that there are infinitely many real algebraic numbers ξ satisfying (2.1).

The proof of Theorem 1 is very much inspired by a paper of Peck [8] on simultaneous rational approximation to real algebraic numbers. Roughly speaking, we use a method dual to Peck's to construct integer polynomials $P(X)$ that take small values at α, and we need an extra argument to ensure that our polynomials have a root ξ very close to α.

De Mathan [6] used the theory of linear forms in non-Archimedean logarithms to prove that Theorem 1 for $d = 1$ is best possible, in the sense that the absolute value of the exponent of $\log 3H(\xi)$ in (2.2) cannot be too large. The next theorem extends this result to all values of d.

THEOREM 2. *Let p be a prime number, d a positive integer and α a real algebraic number of degree $d+1$. Let λ be a positive real number. There exists a positive real number $\kappa = \kappa(\lambda)$ such that for every non-zero real algebraic number ξ of degree d satisfying*

$$|\alpha - \xi| \leq \lambda H(\xi)^{-d-1} \tag{2.4}$$

we have

$$|\xi|_v \geq (\log 3H(\xi))^{-\kappa}$$

for at least one absolute value $|\cdot|_v$ on $\mathbb{Q}(\xi)$ above the prime p.

As in [6], the proof of Theorem 2 rests on the theory of linear forms in non-Archimedean logarithms.

Let d be a positive integer. We recall that it follows from the p-adic version of the Schmidt Subspace Theorem that for every algebraic number α of degree $d+1$ and for every positive real number ε, there are only finitely many non-zero integer polynomials $P(X) = a_0 + a_1X + \cdots + a_dX^d$ of degree at most d, with $a_0 \neq 0$, that satisfy

$$|P(\alpha)| \cdot |a_0|_p < H(P)^{-d-\varepsilon}.$$

Let ξ be a real algebraic number of degree at most d, and denote by $P(X) = a_0 + a_1X + \cdots + a_dX^d$ its minimal polynomial over $\mathbb{Z}$. Then

$$\min\{|\mathrm{Norm}(\xi)|_p, 1\} \geq |a_0|_p$$

and there exists a constant $c(\alpha)$, depending only on α, such that

$$|P(\alpha)| \leq c(\alpha)H(\xi) \cdot |\xi - \alpha|.$$

Let ε be a positive real number. Applying the above statement deduced from the p-adic version of the Schmidt Subspace Theorem to these polynomials $P(X)$, we deduce that

$$|\alpha - \xi| \cdot \min\{|\mathrm{Norm}(\xi)|_p, 1\} \geq H(P)^{-d-1-\varepsilon}$$

holds if $H(P)$ is sufficiently large. This implies that if ξ satisfies (2.4) and if $H(\xi)$ is sufficiently large, then

$$|\mathrm{Norm}(\xi)|_p \geq H(\xi)^{-\varepsilon},$$

accordingly

$$\max_{v|p} |\xi|_v \geq H(\xi)^{-\varepsilon/d}.$$

The result of Theorem 2 is more precise, but we cannot obtain a good lower bound for $|\mathrm{Norm}(\xi)|_p$.

We conclude this section by pointing out that Einsiedler and Kleinbock [4] showed that a slight modification of the de Mathan–Teulié conjecture easily follows from a theorem of Furstenberg [5, 1].

Theorem EK. *Let p_1 and p_2 be distinct prime numbers. Then*

$$\inf_{q \geq 1} q \cdot \|q\alpha\| \cdot |q|_{p_1} \cdot |q|_{p_2} = 0$$

for every real number α.

In view of Theorem EK, we formulate the following question, presumably easier to solve than Problem 1.

Problem 2. *Let d be a positive integer. Let α be a real number that is not algebraic of degree less than or equal to d. For any distinct prime numbers p_1, p_2 and every positive real number ε, does there exist a non-zero real algebraic number ξ of degree at most d satisfying*

$$|\alpha - \xi| \cdot \min\{|\mathrm{Norm}(\xi)|_{p_1}, 1\} \cdot \min\{|\mathrm{Norm}(\xi)|_{p_2}, 1\} < \varepsilon H(\xi)^{-d-1}?$$

Theorem EK gives a positive answer to Problem 2 when $d = 1$.

The remainder of the paper is organized as follows. We gather several auxiliary results in Section 3, and Theorems 1 and 2 are established in Sections 4 and 5, respectively.

In the next sections, we fix a real algebraic number field $\mathbb{K}$ of degree $d+1$. The notation $A \ll B$ means, unless specifically indicated otherwise, that the implicit constant depends on $\mathbb{K}$. Furthermore, we write $A \asymp B$ if $A \ll B$ and $B \ll A$ simultaneously.

3. Auxiliary lemmas. Let $\mathbb{K}$ be a real algebraic number field of degree $d+1$. Let $\mathcal{O}$ denote its ring of integers, and let $\alpha_0 = 1, \alpha_1, \dots, \alpha_d$ be a basis of $\mathbb{K}$. Let D be a positive integer satisfying

$$D(\mathbb{Z} + \alpha_1\mathbb{Z} + \dots + \alpha_d\mathbb{Z}) \subset \mathcal{O} \subset \frac{1}{D}(\mathbb{Z} + \alpha_1\mathbb{Z} + \dots + \alpha_d\mathbb{Z})$$

and the corresponding inequalities for the dual basis $\beta_0, \dots, \beta_d$ defined by

$$\mathrm{Tr}(\alpha_i\beta_j) = \delta_{i,j},$$

where Tr is the trace and $\delta_{i,j}$ is the Kronecker symbol.

We denote by $\sigma_0 = \mathrm{Id}, \dots, \sigma_d$ the complex embeddings of $\mathbb{K}$, numbered in such a way that $\sigma_0, \dots, \sigma_{r_1-1}$ are real, $\sigma_{r_1}, \dots, \sigma_d$ are imaginary and $\sigma_{r_1+r_2+j} = \overline{\sigma}_{r_1+j}$ for $0 \le j < r_2$. Set also $r = r_1 + r_2 - 1$, and let $\varepsilon_1, \dots, \varepsilon_r$ be multiplicatively independent units in $\mathbb{K}$.

Lemma 1. *Let η be a unit in $\mathcal{O}$ such that $-1 < \eta < 1$ and define the real number N by $|\eta| = N^{-1}$. The conditions*

$$|\sigma_j(\eta)| \asymp N^{1/d}, \qquad 0 < j \le d, \tag{3.1}$$

and

$$|\sigma_i(\eta)| \asymp |\sigma_j(\eta)|, \quad 0 < i < j \le d, \tag{3.2}$$

are equivalent. Let $\gamma \neq 0$ be in $\mathbb{K}$ and let Δ be a positive integer such that $\Delta\gamma \in \mathcal{O}$. If η satisfies (3.1) *or* (3.2), *write*

$$\gamma\eta = a_0 + \dots + a_d\alpha_d$$

with $a_0, \dots, a_d$ in $\mathbb{Q}$. Then $D\Delta a_k \in \mathbb{Z}$ for $k = 0, \dots, d$ and

$$\max_{k=0,\dots,d} |a_k| \asymp N^{1/d},$$

where the implicit constants depend on γ.

Proof. Since η is a unit, we have $\prod_{0 \le j \le d} \sigma_j(\eta) = \pm 1$, and (3.1) and (3.2) are clearly equivalent. The formula

$$a_k = \mathrm{Tr}(\gamma\eta\beta_k) = \gamma\eta\beta_k + \sum_{j=1}^{d} \sigma_j(\eta)\sigma_j(\gamma\beta_k)$$

implies that if η satisfies (3.1), then

$$|a_k| \ll N^{1/d}, \qquad 0 \le k \le d.$$

Combined with

$$\sigma_1(\gamma)\sigma_1(\eta) = a_0 + \dots + a_d\sigma_1(\alpha_d),$$

this shows that $N^{1/d} \asymp |\sigma_1(\eta)| \ll \max_{k=0,\dots,d} |a_k|$. ∎

Let α be a real algebraic number of degree $d + 1$. We keep the above notation with the field $\mathbb{K} = \mathbb{Q}(\alpha)$ and the basis $1, \alpha, \dots, \alpha^d$ of $\mathbb{K}$ over $\mathbb{Q}$, and we display an immediate consequence of Lemma 1.

Corollary 1. *Let η be a unit in $\mathcal{O}$ such that $-1 < \eta < 1$ and set $N = |\eta|^{-1}$. Then*

$$D\Delta\gamma\eta = P(\alpha),$$

where $P(X)$ is an integral polynomial of degree at most d satisfying

$$H(P) \asymp N^{1/d}, \qquad |P(\alpha)| \asymp N^{-1},$$

and thus $|P(\alpha)| \asymp H(P)^{-d}$.

Denote by τ_j, $j = 0, \ldots, d$ the embeddings of $\mathbb{K}$ into $\mathbb{C}_p$. Recall that the absolute value $|\cdot|_p$ on $\mathbb{Q}$ has an extension to $\mathbb{C}_p$, that we also denote by $|\cdot|_p$. In Lemmata 2 to 4 below we work in $\mathbb{C}_p$. Let $P(X)$ be an irreducible integer polynomial of degree $n \geq 1$. Let ξ be a complex root of $P(X)$ and $\xi_1, \ldots, \xi_n$ be the roots of $P(X)$ in $\mathbb{C}_p$. We point out that the sets

$$\{|\xi|_v : v \text{ is above } p \text{ on } \mathbb{Q}(\xi)\} \quad \text{and} \quad \{|\xi_i|_p : 1 \leq i \leq n\}$$

coincide, since all the absolute values above p over $\mathbb{Q}(\xi)$ are obtained by starting from $|\cdot|_p$ over $\mathbb{C}_p$, after embedding $\mathbb{Q}(\xi)$ in $\mathbb{C}_p$.

Keeping the notation of Lemma 1, we have the following auxiliary result.

LEMMA 2. *Assume that $\gamma = \alpha_d$. Then*

$$|a_k|_p \ll \max_{0\leq j\leq d} |\tau_j(\eta) - 1|_p, \qquad 0 \leq k < d,$$

and

$$|a_d - 1|_p \ll \max_{0\leq j\leq d} |\tau_j(\eta) - 1|_p.$$

Proof. Since $\mathrm{Tr}(\alpha_d\beta_k) = 0$ for $k = 0, \ldots, d-1$, we get

$$a_k = \mathrm{Tr}(\gamma\eta\beta_k) = \mathrm{Tr}(\alpha_d(\eta - 1)\beta_k) = \sum_{j=0}^{d} (\tau_j(\eta) - 1)\tau_j(\alpha_d\beta_k),$$

and deduce that

$$|a_k|_p \ll \max_{0\leq j\leq d} |\tau_j(\eta) - 1|_p, \qquad 0 \leq k < d.$$

It follows from $\mathrm{Tr}(\alpha_d\beta_d) = 1$ that

$$a_d = 1 + \mathrm{Tr}(\alpha_d\beta_d(\eta - 1)) = 1 + \sum_{j=0}^{d} (\tau_j(\eta) - 1)\tau_j(\alpha_d\beta_d),$$

and we derive the last conclusion of the lemma. ∎

LEMMA 3. *Let $0 < \delta < 1$. There exist arbitrarily large positive real numbers H and units η satisfying $\eta = H^{-d}$,*

$$\left|\frac{\sigma_j(\eta)}{\sigma_1(\eta)} - 1\right| \leq \delta, \qquad 2 \leq j \leq d, \tag{3.3}$$

and

$$|\tau_j(\eta) - 1|_p \ll (\log H)^{-1/r}, \qquad 0 \leq j \leq d.$$

Proof. By replacing ε_i by $\varepsilon_i^{2p^{m_i}(p-1)(p^2-1)\cdots(p^{d+1}-1)}$ with a suitable positive integer m_i, we can assume that ε_i is positive, together with its real conjugates, and that $|\tau_j(\varepsilon_i) - 1|_p < p^{-1/(p-1)}$ for $i = 1, \ldots, r$ and $j = 0, \ldots, d$. This is possible since $|\tau_j(\varepsilon_i)|_p = 1$ for $i = 1, \ldots, r$ and $j = 0, \ldots, d$. This allows us to consider the p-adic logarithms of each $\tau_j(\varepsilon_i)$. Our aim is to construct a suitable unit η of the form

$$\eta = \varepsilon_1^{\mu_1 p^s} \cdots \varepsilon_r^{\mu_r p^s},$$

where $\mu_i \in \mathbb{Z}$. The conditions for (3.3) are then

$$p^s \left| \mu_1 \log \frac{|\sigma_j(\varepsilon_1)|}{|\sigma_1(\varepsilon_1)|} + \cdots + \mu_r \log \frac{|\sigma_j(\varepsilon_r)|}{|\sigma_r(\varepsilon_r)|} \right| \le C_1, \quad 2 \le j \le r,$$

where $C_1 = C_1(\delta) > 0$ is a constant, and

$$\left\| \frac{p^s}{2\pi} (\mu_1 \arg \sigma_j(\varepsilon_1) + \cdots + \mu_r \arg \sigma_j(\varepsilon_r)) \right\| \le C_2, \quad r_1 \le j \le r,$$

with $C_2 = C_2(\delta) > 0$. Set

$$Y_j = p^s \left(\mu_1 \log \frac{|\sigma_1(\varepsilon_1)|}{|\sigma_j(\varepsilon_1)|} + \cdots + \mu_r \log \frac{|\sigma_1(\varepsilon_r)|}{|\sigma_j(\varepsilon_r)|} \right), \qquad 2 \le j \le r,$$

$$Z_k = \frac{p^s}{2\pi} (\mu_1 \arg \sigma_k(\varepsilon_1) + \cdots + \mu_r \arg \sigma_k(\varepsilon_r)) \in \mathbb{R}/\mathbb{Z}, \quad r_1 \le k \le r.$$

Taking $0 \le \mu_i < M$, we have M^r points $(\mu_i)_{1 \le i \le r}$. The $(Y_j, Z_k)_{2 \le j \le r,\, r_1 \le k \le r}$ are in the product of intervals I_j, $2 \le j \le r$, of lengths $O(Mp^s)$ and of r_2 factors identical to $\mathbb{R}/\mathbb{Z}$. This set can be covered by $C_3(Mp^s)^{r-1}$ sets of diameter at most $\max\{C_1, C_2\}$, where C_3 is a constant depending on δ. By Dirichlet's *Schubfachprinzip*, choosing M such that

$$C_3(Mp^s)^{r-1} < M^r,$$

which can be done with $M \asymp p^{(r-1)s}$, we deduce that there is $(\mu_1, \ldots, \mu_r) \in \mathbb{Z}^r \setminus \{0\}$ such that

$$\max_{1 \le i \le r} |\mu_i| \ll M,$$

$$|Y_j| \le C_1, \quad 2 \le j \le r,$$

$$\|Z_k\| \le C_2, \quad r_1 \le k \le r.$$

Set then $\eta = (\varepsilon_1^{\mu_1} \cdots \varepsilon_r^{\mu_r})^{p^s}$ in such a way that $0 < \eta < 1$ (if needed, just consider $1/\eta$). This choice implies that

$$|\tau_i(\eta) - 1|_p = |\log_p \tau_i(\eta)|_p \le p^{-s}, \quad 0 \le i \le d,$$

and

$$|\log \eta| \ll p^s M \ll p^{rs}. \ \blacksquare$$

Lemma 4. *Let $P(X) = a_0 + \cdots + a_d X^d \in \mathbb{C}_p[X]$ be a polynomial of degree d. Let ξ_i $(1 \le i \le d)$ be the roots of $P(X)$ in $\mathbb{C}_p$. Let c be a real number satisfying $0 \le c \le 1$. If*

$$|\xi_i|_p \le c, \quad 1 \le i \le d,$$

then

$$|a_k|_p \le c|a_d|_p, \quad 0 \le k < d. \tag{3.4}$$

Conversely, *if* (3.4) *holds*, *then*

$$|\xi_i|_p \le c^{1/d}, \quad 1 \le i \le d.$$

Proof. Since $P(X) = a_d \prod_{1\le i\le d}(X - \xi_i)$, if $|\xi_i|_p \le c \le 1$ for $i = 1, \dots, d$ then

$$|a_k|_p \le c|a_d|_p \quad \text{for } k = 0, \dots, d-1.$$

Conversely, if

$$|a_k|_p \le c|a_d|_p, \quad 0 \le k < d,$$

and if $\xi \in \mathbb{C}_p$ is such that $a_d\xi^d + \dots + a_0 = 0$, then there exists k with $0 \le k < d$ and

$$|a_k\xi^k|_p \ge |a_d\xi^d|_p, \quad \text{thus} \quad |\xi|_p^d \le |\xi|_p^{d-k} \le c. \ \blacksquare$$

We conclude this section with two lemmas used in the proof of Theorem 2. The first of them was proved by Peck [8].

LEMMA 5. *There exists a sequence* $(\eta_m)_{m\ge1}$ *of positive units in* $\mathcal{O}$ *such that*

$$\eta_m \asymp e^{-dm}, \quad |\sigma_j(\eta_m)| \asymp e^m, \quad 1 \le j \le d.$$

Proof. Let us search for the unit η_m in the form $\eta_m = \varepsilon_1^{\mu_1}\cdots\varepsilon_r^{\mu_r}$ with $\mu_i \in \mathbb{Z}$. We construct *real* numbers $\nu_1, \dots, \nu_r$ such that

$$\nu_1 \log\varepsilon_1 + \dots + \nu_r \log\varepsilon_r = -dm \tag{3.5}$$

and

$$\nu_1 \log|\sigma_j(\varepsilon_1)| + \dots + \nu_r \log|\sigma_j(\varepsilon_r)| = m, \quad 1 \le j \le d. \tag{3.6}$$

Taking into account that, by complex conjugation, the equations (3.6) corresponding to an index j with $r_1 \le j < r_1 + r_2$ and to the index $j + r_2$ are identical, and that the sum of (3.5) and (3.6) is zero, we simply have to deal with a Cramer system, since the matrix $(\sigma_j(\varepsilon_i))_{1\le j\le r,\,1\le i\le r}$ is regular. We solve this system and then replace every ν_i by a rational integer μ_i such that $|\mu_i - \nu_i| \le 1/2$. $\blacksquare$

LEMMA 6. *Let* λ' *be a positive real number. Let* $(\eta_m)_{m\ge1}$ *be a sequence of positive units as in Lemma* 5. *There exists a finite set* $\Gamma = \Gamma(\lambda')$ *of non-zero elements of* $\mathbb{K}$ *such that for every integer polynomial* $P(X)$ *of degree at most* d *that satisfies*

$$|P(\alpha)| \le \lambda' H(P)^{-d}, \tag{3.7}$$

there exist a positive integer m *and* γ *in* Γ *for which*

$$P(\alpha) = \gamma\eta_m.$$

Proof. Below, all the constants implicit in $\ll$ depend on $\mathbb{K}$ and on λ'. Let m be a positive integer such that

$$H(P) \asymp e^m,$$

and set

$$\gamma = P(\alpha)\eta_m^{-1}.$$

Since $D\alpha^k$ is an algebraic integer for $k = 0, \ldots, d$, the algebraic number $D\gamma$ is an algebraic integer, and, by (3.7),

$$|\gamma| \ll 1.$$

Furthermore, for $j = 1, \ldots, d$, we have

$$|\sigma_j(\gamma)| = |P(\sigma_j(\alpha))| \cdot |\sigma_j(\eta_m^{-1})| \ll H(P)e^{-m} \ll 1.$$

The algebraic integers $D\gamma \in \mathcal{O}$ and all their complex conjugates being bounded, they form a finite set. ∎

4. Proof of Theorem 1. Let δ be in $(0, 1)$, to be selected later. Apply Lemma 3 with this δ to get a unit η and apply Lemma 1 with this unit and with $\gamma = \alpha^d$. Since $D^2\alpha^d\eta \in \mathbb{Z} + \cdots + \alpha^d\mathbb{Z}$, we obtain

$$D^2\eta\alpha^d = a_0 + a_1\alpha + \cdots + a_d\alpha^d = P(\alpha),$$

where, by Corollary 1, $P(X)$ is an integer polynomial of degree d and

$$|P(\alpha)| \asymp H(P)^{-d} \asymp H^{-d}.$$

By Lemmata 2 and 3, each coefficient of $P(X)$ has its p-adic absolute value $\ll (\log 3H(P))^{-1/r}$, except the leading coefficient, whose p-adic absolute value equals $|D|_p^2$.

We then infer from Lemma 4 that all the roots of $P(X)$ in $\mathbb{C}_p$ have their p-adic absolute value $\ll (\log 3H(P))^{-1/(dr)}$. This proves (2.2).

It now remains to guarantee that $P(X)$ has a root very close to α. To this end, we proceed to check that

$$|P'(\alpha)| \gg H(P).$$

Since

$$P'(\alpha) = a_1 + \cdots + da_d\alpha^{d-1},$$

we get

$$P'(\alpha) = D^2(\mathrm{Tr}(\eta\alpha^d\beta_1) + 2\alpha\mathrm{Tr}(\eta\alpha^d\beta_2) + \cdots + d\alpha^{d-1}\mathrm{Tr}(\eta\alpha^d\beta_d)),$$

hence,

$$P'(\alpha) = D^2 \sum_{i=0}^{d} \sum_{k=1}^{d} k\alpha^{k-1}\sigma_i(\eta\alpha^d\beta_k).$$

Let us write

$$P'(\alpha) = D^2 \sum_{i=0}^{d} A_i\sigma_i(\eta)$$

with

$$A_i = \sigma_i(\alpha^d) \sum_{k=1}^{d} k\alpha^{k-1}\sigma_i(\beta_k), \qquad i = 0, \ldots, d.$$

Observe first that

$$\sum_{i=1}^{d} A_i \neq 0.$$

Indeed, if this is not the case, note that the above formulæ hold for any unit η in $\mathbb{K}$, thus we can in particular work with the unit $\eta = 1$, that is, with $P(X) = D^2X^d$ and $P'(\alpha) = dD^2\alpha^{d-1}$; we get

$$d\alpha^{d-1} = A_0 = \alpha^d \sum_{k=1}^{d} k\alpha^{k-1}\beta_k,$$

hence,

$$d = \sum_{k=1}^{d} k\alpha^k\beta_k.$$

Taking the trace, and recalling that $\mathrm{Tr}(\alpha^k\beta_k) = 1$, we get $d(d+1) = \sum_{k=1}^{d} k$, a contradiction.

Write

$$P'(\alpha) = D^2 \sum_{i=1}^{d} A_i\sigma_i(\eta) + O(H^{-d}) = D^2\sigma_1(\eta) \sum_{i=1}^{d} A_i + B$$

with

$$|B| \leq D^2 \sum_{2\leq i\leq d} |A_i| \cdot |\sigma_1(\eta)| \cdot \left|\frac{\sigma_i(\eta)}{\sigma_1(\eta)} - 1\right| + O(H^{-d}).$$

Selecting now δ such that

$$\delta \sum_{2\leq i\leq d} |A_i| \leq \frac{1}{3}\Big|\sum_{i=1}^{d} A_i\Big|,$$

we infer from Lemma 3 that

$$|P'(\alpha)| \geq \frac{1}{2}\, D^2\Big|\sigma_1(\eta) \sum_{i=1}^{d} A_i\Big|$$

when H is sufficiently large. This gives

$$|P'(\alpha)| \gg |\sigma_1(\eta)| \gg H.$$

Consequently, $P(X)$ has a root ξ such that

$$|\alpha - \xi| \ll H(P)^{-d-1} \ll H(\xi)^{-d-1}.$$

Classical arguments (see end of the proof of Theorem 2.11 in [2]) show that ξ must be real and of degree d if H is sufficiently large. This proves (2.1). Inequality (2.3) follows from (2.1) and (2.2) together with the fact that ξ is of degree d. ■

5. Proof of Theorem 2. The constants implicit in $\ll$ and $\gg$ below depend on $\mathbb{K}$, p and λ. By Rolle's theorem, there exists a positive real number λ', depending on λ and on d, such that the minimal polynomial $P(X)$ of any real number ξ of sufficiently large height and for which (2.4) holds is of degree d and satisfies

$$|P(\alpha)| \le \lambda' H(P)^{-d}.$$

Let $(\eta_m)_{m\ge 1}$ be as in Lemma 5. By Lemma 6, it is sufficient to prove Theorem 2 for the integer polynomials $P(X)$ as above such that

$$P(\alpha) = \gamma\eta_m = a_0 + a_1\alpha + \cdots + a_d\alpha^d.$$

Let ξ_i be the roots of $P(X)$ in $\mathbb{C}_p$ and set

$$u := \max_{1\le i\le d} |\xi_i|_p.$$

Assume that $u \le 1$. It follows from Lemma 4 that

$$|a_k|_p \le u|a_d|_p, \qquad 0 \le k < d,$$

thus, taking $|a_d|_p P(X)$, which is still an integer polynomial, in place of $P(X)$, we can assume that $|a_d|_p = 1$ and

$$|a_k|_p \le u, \qquad 0 \le k < d.$$

For $j = 1, \dots, d$, we then have

$$\gamma\eta_m\alpha^{-d} - \tau_j(\gamma\eta_m\alpha^{-d}) = \sum_{k=0}^{d-1} a_k(\alpha^{k-d} - \tau_j(\alpha^{k-d})),$$

hence,

$$|\gamma\eta_m\alpha^{-d} - \tau_j(\gamma\eta_m\alpha^{-d})|_p \ll u.$$

Since $|\eta_m|_p = 1$, we get

$$\left|\frac{\tau_j(\eta_m)}{\eta_m}\,\frac{\tau_j(\gamma)\alpha^d}{\gamma\tau_j(\alpha^d)} - 1\right|_p \ll u.$$

Upon writing

$$\eta_m = \varepsilon_1^{\mu_{1,m}} \cdots \varepsilon_r^{\mu_{r,m}},$$

we thus have

$$u \gg \left|\left(\frac{\tau_j(\varepsilon_1)}{\varepsilon_1}\right)^{-\mu_{1,m}} \cdots \left(\frac{\tau_j(\varepsilon_r)}{\varepsilon_r}\right)^{-\mu_{r,m}} \frac{\tau_j(\gamma)\alpha^d}{\gamma\tau_j(\alpha^d)} - 1\right|_p.$$

If

$$\frac{\tau_j(\eta_m)}{\eta_m} = \frac{\gamma\tau_j(\alpha^d)}{\tau_j(\gamma)\alpha^d}$$

for $j = 1, \ldots, d$, then the number $\gamma\eta_m\alpha^{-d}$ is equal to all its conjugates, hence is rational, and we have

$$P(\alpha) = b\alpha^d$$

with $b \in \mathbb{Q}$, hence $P(X) = bX^d$, a contradiction. For every m, there thus exists an index j such that $1 \le j \le d$ and

$$\left(\frac{\tau_j(\varepsilon_1)}{\varepsilon_1}\right)^{-\mu_{1,m}} \cdots \left(\frac{\tau_j(\varepsilon_r)}{\varepsilon_r}\right)^{-\mu_{r,m}} \frac{\tau_j(\gamma)\alpha^d}{\gamma\tau_j(\alpha^d)} \ne 1.$$

Consequently, by the theory of linear forms in non-Archimedean logarithms (see e.g. Yu [10]), there exists a positive constant κ such that

$$u \gg (\max_{1\le i\le r} |\mu_{i,m}|)^{-\kappa}. \tag{5.1}$$

As in the proof of Lemma 5, the matrix $(\log|\sigma_j(\varepsilon_i)|)_{1\le i\le r,\, 1\le j\le r}$ being regular, we have

$$|\log \eta_m| \asymp \max_{1\le i\le r} |\mu_{i,m}|.$$

Combined with $\eta_m \asymp H(P)^{-d}$ and (5.1), this gives

$$u \gg (\log 3H(\xi))^{-\kappa}. \blacksquare$$

References

[1] M. D. Boshernitzan, *Elementary proof of Furstenberg's Diophantine result*, Proc. Amer. Math. Soc. 122 (1994), 67–70.

[2] Y. Bugeaud, *Approximation by Algebraic Numbers*, Cambridge Tracts in Math. 160, Cambridge Univ. Press, Cambridge, 2004.

[3] Y. Bugeaud, M. Drmota, and B. de Mathan, *On a mixed Littlewood conjecture in Diophantine approximation*, Acta Arith. 128 (2007), 107–124.

[4] M. Einsiedler and D. Kleinbock, *Measure rigidity and p-adic Littlewood-type problems*, Compos. Math. 143 (2007), 689–702.

[5] H. Furstenberg, *Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation*, Math. Systems Theory 1 (1967), 1–49.

[6] B. de Mathan, *On a mixed Littlewood conjecture for quadratic numbers*, J. Théor. Nombres Bordeaux 17 (2005), 207–215.

[7] B. de Mathan et O. Teulié, *Problèmes diophantiens simultanés*, Monatsh. Math. 143 (2004), 229–245.

[8] L. G. Peck, *Simultaneous rational approximations to algebraic numbers*, Bull. Amer. Math. Soc. 67 (1961), 197–201.

[9] E. Wirsing, *Approximation mit algebraischen Zahlen beschränkten Grades*, J. Reine Angew. Math. 206 (1961), 67–77.

[10] K. Yu, *p-adic logarithmic forms and group varieties II*, Acta Arith. 89 (1999), 337–378.

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