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# Block additive functions on the Gaussian integers

by

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**1. Introduction.** Let $q = -a + i \in \mathbb{Z}[i]$ for a positive integer $a$ and

$$\mathcal{N} = \{0, 1, \ldots, a^2\}.$$

Then every Gaussian integer $z \in \mathbb{Z}[i]$ can be uniquely represented as

$$z = \sum_{j \geq 0} \varepsilon_j(z) q^j$$

with $\varepsilon_j(z) \in \mathcal{N}$. Formally we set $\varepsilon_j(z) = 0$ for all negative integers $j < 0$. It will be convenient sometimes to use infinite or even doubly infinite sequences (filled with zeros) for the representation of Gaussian integers. We denote the length of the expansion by

$$\mathrm{length}_q(z) = \max\{j \in \mathbb{N}_0 : \varepsilon_j(z) \neq 0\} + 1$$

and $\mathrm{length}_q(0) = 0$. (We denote the positive integers by $\mathbb{N}$ and use $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ for the non-negative integers.) Throughout the paper we will use the notation $\log_b$ for the logarithm to base $b$. The following lemma was proved in [13].

Lemma 1. *There exists a positive constant $c$ such that for all $z \in \mathbb{Z}[i]$,*

$$|\,\mathrm{length}_q(z) - \log_{|q|} |z|\,| \leq c.$$

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The fundamental domain of the base $q$ representation on $\mathbb{Z}[i]$ is defined by

$$\mathcal{F}_q = \left\{ \sum_{l=1}^{\infty} \frac{\varepsilon_l}{q^l} : \varepsilon_l \in \mathcal{N} \ \forall l \right\}.$$

This subset of $\mathbb{C}$ plays the same rôle for $q$-adic numeration as the unit interval does for classical number systems on the integers (cf. [8, 9, 13]). More generally, radix representations of elements of the ring of integers $\mathbb{Z}_K$ of a number field $K$ can be considered. A base $\alpha \in \mathbb{Z}_K$ together with the digit set $D = \{0, 1, \ldots, |\mathrm{N}_{K|\mathbb{Q}}(\alpha)| - 1\}$ is called a *canonical number system* (cf. [17, 18]) if every $\zeta \in \mathbb{Z}_K$ has a representation of the form

$$\zeta = \sum_{l=0}^{n} \varepsilon_l \alpha^l$$

with $\varepsilon_l \in D$ for $0 \le l \le n$. The point 0 is an inner point of $\mathcal{F}_q$. This follows from the general fact that $(\alpha, D)$ is a canonical number system if and only if the corresponding fundamental domain contains 0 as an inner point (cf. [1]).

Let $F : \mathcal{N}^{L+1} \to \mathbb{R}$ be any given function (for some $L \ge 0$) with $F(0, \ldots, 0) = 0$. Furthermore, set

$$s_F(z) = \sum_{j=-L}^{\infty} F(\varepsilon_j(z), \varepsilon_{j+1}(z), \ldots, \varepsilon_{j+L}(z)).$$

This means that we consider a weighted sum over all subsequent digital patterns of length $L+1$ of the digital expansion of $z$. The function $s_F$ is called a *block additive function* of rank $L+1$. This generalises the block additive digital functions studied in [4] for digital expansions on the rational integers. This definition readily extends to functions taking values in an arbitrary abelian group $A$. We will use this in the general considerations in Section 5.

For example, for $L = 0$ we obtain completely additive functions such as those studied in [16, Section 5], for instance for $F(\varepsilon) = \varepsilon$ we just have the sum-of-digits function studied in [10, 13, 14], or if $L = 1$ and $F(\varepsilon, \eta) = 1 - \delta_{\varepsilon,\eta}$ ($\delta_{x,y}$ denoting the Kronecker symbol) then $s_F(n)$ just counts the number of times that a digit is different from the preceding one etc.

**2. Overview of the results.** Our main objective is to get information on sums

$$S_N(x) = \sum_{|z|^2 < N} x^{s_F(z)}, \tag{2.1}$$

where $x$ is a complex variable. It is clear that $S_N(x)$ encodes the distribution of $s_F(z)$. For example, if we assume that $s_F(z)$ has only non-negative integer values then

$$S_N(x) = \sum_{k\ge 0} \#\{z \in \mathbb{Z}[i] : |z|^2 < N,\ s_F(z) = k\} x^k.$$

More generally, let $Y_N$ denote the random variable induced by the distribution of $s_F(z)$ for $|z|^2 < N$, that is, the distribution function of $Y_N$ is given by

$$\mathbb{P}\{Y_N \le y\} = \frac{1}{S_N(1)}\, \#\{|z|^2 < N : s_F(z) \le y\}. \tag{2.2}$$

Then

$$\mathbb{E}\, x^{Y_N} = \frac{1}{S_N(1)} \sum_{|z|^2<N} x^{s_F(z)} = \frac{1}{S_N(1)}\, S_N(x). \tag{2.3}$$

In particular, the moment generating function $\mathbb{E}\, e^{\lambda Y_N}$ and the characteristic function $\mathbb{E}\, e^{itY_N}$ of $Y_N$ can be expressed with the help of $S_N(x)$. (Note that $S_N(1) = \pi N + \mathcal{O}(N^{1/3})$.)

In what follows we will present three different methods to obtain asymptotic information for $S_N(x)$. In Section 3 we use a measure-theoretic approach to show that for real numbers $x$ sufficiently close to 1 we have

$$S_N(x) = \Phi(x, \log_{|q|^2} N)\, N^{\log_{|q|^2} \lambda(x)} (1 + \mathcal{O}(N^{-\kappa})), \tag{2.4}$$

where $\Phi(x,t)$ is a function that is analytic in $x$ and periodic (with period 1) and Hölder continuous in $t$, and $\lambda(x)$ is the dominant eigenvalue of a certain matrix $\mathbf{A}(x)$ defined in (3.1). This representation directly implies that the random variable

$$X_N = \frac{Y_N - \mu \log_{|q|^2} N}{\sqrt{\sigma^2 \log_{|q|^2} N}}$$

with

$$\mu = \frac{\lambda'(1)}{\lambda(1)} \quad \text{and} \quad \sigma^2 = \frac{\lambda''(1)}{\lambda(1)} + \frac{\lambda'(1)}{\lambda(1)} - \frac{\lambda'(1)^2}{\lambda(1)^2}$$

satisfies a central limit theorem and we have convergence of all moments. More precisely, we get (uniformly in $y$)

$$\frac{1}{\pi N}\, \#\Big\{|z|^2 < N : s_F(z) \le \mu \log_{|q|^2} N + y\sqrt{\sigma^2 \log_{|q|^2} N}\Big\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac12 u^2}\, du + o(1),$$

and (for every $L \ge 0$)

$$\frac{1}{\pi N} \sum_{|z|^2<N} (s_F(z) - \mu \log_{|q|^2} N)^L = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u^L e^{-\frac12 u^2}\, du + o(1).$$

The drawback of the method given in Section 3 is that it only works for real numbers $x$. In Section 4 we present a method that is based on Dirichlet series that extends (2.4) to a complex neighbourhood of $x = 1$. Furthermore, we provide upper bounds for $S_N(x)$ for complex $x$ with modulus close to 1. With the help of this extension we are able to provide more precise distributional results. Besides the central limit theorem we also get a local limit theorem, that is, asymptotic expansions for the numbers

$$\#\{z \in \mathbb{Z}[i] : |z|^2 < N,\ s_F(z) = k\}$$

if $k$ is close to the mean $\mu \log_{|q|^2} N$ and if $s_F(z)$ is integer-valued. Furthermore, we obtain very precise asymptotic expansions of the moments.

Next we consider the sequence $s_F(z)$ taking values in a compact abelian group $A$. Then the closure of the set $\{s_F(z) : z \in \mathbb{Z}[i]\}$ is a subgroup of $A$ denoted by $A(F)$. The results on exponential sums obtained in Section 4 are used to prove uniform distribution of $(s_F(z))_{z\in\mathbb{Z}[i]}$ in the groups $\mathbb{R}/\mathbb{Z}$ and $\mathbb{Z}/M\mathbb{Z}$ with respect to the Haar measure $\lambda_A$ under natural conditions. The method gives results on uniform distribution of the values of $s_F$ in large circles, i.e.

$$\lim_{N\to\infty} \frac{1}{\pi N}\, \#\{z \in \mathbb{Z}[i] : |z|^2 < N,\ s_F(z) \in B\} = \lambda_A(B)$$

for all $B \subseteq A$ with $\lambda_A(\partial B) = 0$.

In Section 5 we use an approach based on ergodic $\mathbb{Z}[i]$-actions and skew products to extend the distribution results for group-valued $s_F$ to well uniform distribution with respect to Følner sequences $(Q_n)_{n\in\mathbb{N}}$, i.e.

$$\lim_{n\to\infty} \frac{1}{\#Q_n}\, \#\{z \in Q_n : s_F(z+y) \in B\} = \lambda_A(B)$$

uniformly in $y \in \mathbb{Z}[i]$. This generalises the results on uniform distribution of $(s_F(z))_{z\in\mathbb{Z}[i]}$ obtained in Section 4. On the other hand, methods from ergodic theory do not allow one to obtain error terms, which come as a natural by-product of the method used in Section 4.

**3. A measure-theoretic method.** In the following we use ideas developed in [11, 12]. The measure-theoretic approach to asymptotic questions about digital functions gives a smooth proof for a real version of the asymptotic representation (2.4) for $S_N(x)$.

In order to formulate our results we have to introduce some notation.

For every block $B = (\eta_0, \eta_1, \dots, \eta_L) \in \mathcal{N}^{L+1}$ we set $B' = (\eta_1, \dots, \eta_L) \in \mathcal{N}^L$, that is, the block without the first digit, and $\eta(B) = \eta_0$, the first digit of $B$. Furthermore, set

$$g_F(B) = \sum_{i=0}^{L} (F(0,\dots,0,\eta_0,\eta_1,\dots,\eta_i) - F(0,\dots,0,0,\eta_1,\dots,\eta_i)).$$

Note that $g_F(B) = 0$ if $\eta_0 = 0$.

By the definition of block additive function we directly get the following property.

LEMMA 2. *For $z \in \mathbb{Z}[i]$ let $B = B(z) = (\varepsilon_0(z), \dots, \varepsilon_L(z))$ be the block of the first $L+1$ digits of the $q$-ary digital expansion of $z$. Then*

$$s_F(z) = g_F(B) + s_F(v),$$

*where $z = \varepsilon_0(z) + qv$.*

Now define a matrix $\mathbf{A}(x) = (A_{B,C}(x))_{B,C\in\mathcal{N}^{L+1}}$ by

$$A_{B,C}(x) = \begin{cases} x^{g_F(B)} & \text{if } C = (B', l) \text{ for some } l \in \mathcal{N}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

Finally, let $\lambda(x)$ be the dominant eigenvalue of the matrix $\mathbf{A}(x)$ that surely exists if $x$ is close to the positive real axis, in particular, if $x$ is close to 1 (cf. Lemma 4). Note that $\lambda(1) = |q|^2$.

THEOREM 1. *The following asymptotic relation holds uniformly for $x$ in some interval $I$ containing* 1:

$$\sum_{|z|^2<N} x^{s_F(z)} = \Phi(x, \log_{|q|^2} N) N^{\log_{|q|^2} \lambda(x)} (1 + \mathcal{O}(N^{-\kappa})) \tag{3.2}$$

*with some $\kappa > 0$, where $\Phi(x,t)$ is 1-periodic and Hölder continuous in $t$ and continuous in $x$.*

Before we present the proof of Theorem 1 we derive some direct corollaries.

COROLLARY 1. *Set*

$$\mu = \frac{\lambda'(1)}{\lambda(1)} \quad \text{and} \quad \sigma^2 = \frac{\lambda''(1)}{\lambda(1)} + \mu - \mu^2.$$

*If $\sigma^2 > 0$ then uniformly for real $y$,*

$$\frac{1}{\pi N} \#\left\{|z|^2 < N : s_F(z) \le \mu \log_{|q|^2} N + y\sqrt{\sigma^2 \log_{|q|^2} N}\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{1}{2}u^2}\, du + o(1), \tag{3.3}$$

*and for every $L \ge 0$,*

$$\frac{1}{\pi N} \sum_{|z|^2<N} (s_F(z) - \mu \log_{|q|^2} N)^L = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u^L e^{-\frac{1}{2}u^2}\, du + o(1). \tag{3.4}$$

*Furthermore, we have exponential tail estimates of the form*

$$\frac{1}{\pi N}\,\#\Big\{|z|^2 < N : |s_F(z) - \mu \log_{|q|^2} N| \ge \eta\sqrt{\log_{|q|^2} N}\Big\} \ll \min(e^{-c\eta}, e^{-c\eta^2 + \mathcal{O}(\eta^3/\sqrt{\log N})}) \tag{3.5}$$

*for some constant* $c > 0$.

Remark 1. The above result suggests that the distribution of $s_F(z)$ for $|z|^2 < N$ can be approximated by a sum of (weakly dependent) random variables. This is in fact a possible approach to this problem. Observe that the constant $\mu$ can be explicitly calculated from

$$\mu = \frac{\lambda'(1)}{\lambda(1)} = \frac{1}{|q|^{L+1}} \sum_{B\in\mathcal{N}^{L+1}} s_F(B).$$

Of course, this *mean value* corresponds to the contribution of one block of length $L+1$ in the digital expansion of $z$ that has approximately $\log_{|q|^2} N$ digits. It is also possible to represent $\sigma^2$ similarly, but this is much more involved.

*Proof of Corollary 1.* Let $Y_N$ denote the random variable that is induced by the distribution of $s_F(z)$ for $|z|^2 < N$ given by (2.2). Then (by (2.3)) the moment generating function of $Y_N$ is given by

$$\mathbb{E}\, e^{tY_N} = \frac{1}{S_N(1)}\, S_N(e^t) = \frac{1}{\pi}\, \Phi(e^t, \log_{|q|^2} N) N^{\log_{|q|^2} \lambda(e^t) - 1}(1 + \mathcal{O}(N^{-\eta})).$$

Hence, by using the local expansion (recall that $\lambda(1) = |q|^2$)

$$\log \lambda(e^t) = \log |q|^2 + \mu t + \frac{\sigma^2}{2}\, t^2 + \mathcal{O}(t^3)$$

we directly see that the moment generating function of the normalised random variable

$$Z_N = \frac{Y_N - \mu \log_{|q|^2} N}{\sqrt{\sigma^2 \log_{|q|^2} N}}$$

is given by

$$\mathbb{E}\, e^{tZ_N} = e^{-t(\mu/\sigma)\sqrt{\log_{|q|^2} N}}\, \mathbb{E}\, e^{(t/\sqrt{\sigma^2 \log_{|q|^2} N}) Y_N} = e^{\frac{1}{2}t^2 + \mathcal{O}(t^3/\sqrt{\log N})}.$$

Of course, this directly translates to (3.3).

Furthermore, convergence of the moment generating function also implies convergence of all moments, that is, we get (3.4). Finally, the tail estimates (3.5) are a direct consequence of Chernov type inequalities. ∎

The proof of Theorem 1 runs along the lines of [12, Sections 4 and 5] and is organised in four steps.

*Step 1* defines a sequence of discrete measures, which are obtained by suitably rescaling point masses $x^{s_F(z)}$. Let $\delta_z$ denote the Dirac measure supported at $\{z\}$. Then we define a family of measures (depending on $n$ and $x$) by setting

$$\mu_{n,x} = \frac{\sum_{z\in\mathcal{B}_n} x^{s_F(z)}\delta_{z/q^n}}{\sum_{z\in\mathcal{B}_n} x^{s_F(z)}}, \tag{3.6}$$

where

$$\mathcal{B}_n = \{z \in \mathbb{Z}[i] : \text{length}(z) \le n\}.$$

Using the matrix $\mathbf{A}(x)$ introduced in (3.1), we can write the denominator in (3.6) as

$$(x^{g_F(B)})_B \cdot \mathbf{A}(x)^n \cdot (\delta_{\mathbf{0},C})_C^T,$$

$\delta_{\mathbf{0},C}$ denoting the Kronecker symbol, and $^T$ the transposition.

*Step 2* uses characteristic functions to show that the sequence $\mu_{n,x}$ has a weak limit. The fact that the values $x^{s_F(z)}$ are formed from the digital expansion of $z$ can be used to express the Fourier transforms $\widehat{\mu}_{n,x}$ of the measures $\mu_{n,x}$,

$$\widehat{\mu}_{n,x}(t) = \frac{\sum_{z\in\mathcal{B}_n} x^{s_F(z)} e(\Re(tz/q^n))}{\sum_{z\in\mathcal{B}_n} x^{s_F(z)}}, \tag{3.7}$$

in terms of matrix products. Here $t \in \mathbb{C}$ and as usual $e(\cdot) = e^{2\pi i(\cdot)}$. We define the matrix $\mathbf{H}(x,t)$ by setting

$$H_{B,C}(x,t) = A_{B,C}(x)e(\Re(tB_0)).$$

This allows us to write

$$\widehat{\mu}_{n,x}(t) = \frac{\mathbf{v}_1(x,tq^{-n}) \cdot \mathbf{H}(x,t/q^{n-1}) \cdots \mathbf{H}(x,t/q) \cdot \mathbf{v}_2}{\mathbf{v}_1(x,0) \cdot \mathbf{A}(x)^n \cdot \mathbf{v}_2} \tag{3.8}$$

with

$$\mathbf{v}_1(x,t) = (x^{s_F(B)} e(\Re(tB_0)))_B \quad \text{and} \quad \mathbf{v}_2 = (\delta_{\mathbf{0},C})_C^T.$$

The matrices $(1/\lambda(x))\mathbf{H}(x,t)$ satisfy the conditions of [12, Lemma 5] (*mutatis mutandis*) and therefore, the sequence of matrices

$$\mathbf{P}_n(x,t) = \lambda(x)^{-n}\mathbf{H}(x,t/q^{n-1}) \cdots \mathbf{H}(x,t/q)$$

converges to a limit $\mathbf{P}(x,t)$ and

$$\begin{aligned} \|\mathbf{P}_n(x,t) - \mathbf{P}_n(x,0)\| &\ll |t| && \text{for } |t| \le 1, \\ \|\mathbf{P}_n(x,t) - \mathbf{P}(x,t)\| &\ll (1+|t|)^{\eta(x)}|q|^{-\eta(x)n} && \text{for all } t, \end{aligned} \tag{3.9}$$

where

$$\eta(x) = \frac{\log\lambda(x) - \log|\lambda_1(x)|}{\log|q| + \log\lambda(x) - \log|\lambda_1(x)|},$$

with $\lambda_1(x)$ denoting the second largest eigenvalue of $\mathbf{A}(x)$. These relations hold uniformly for $x$ in compact subsets of $(0,\infty)$.

For $|t|\ge 1$, (3.9) together with (3.8) implies

$$|\widehat{\mu}_{n,x}(t)-\widehat{\mu}_x(t)|\ll |t|^{\eta(x)}q^{-n\eta(x)}. \tag{3.10}$$

For $|t|\le 1$ and $L>K>l$ we estimate using (3.8):

$$\begin{aligned}
&|\widehat{\mu}_{K,x}(t)-\widehat{\mu}_{L,x}(t)|\\
&=\Bigg|\lambda(x)^K\,\frac{\mathbf{v}_1(x,tq^{-K})\cdot\mathbf{P}_{K-l}(q^{-l}t)\mathbf{P}_l(t)\mathbf{v}_2}{\mathbf{v}_1(x,0)\cdot\mathbf{A}(x)^K\cdot\mathbf{v}_2}\\
&\qquad\qquad -\lambda(x)^L\,\frac{\mathbf{v}_1(x,tq^{-L})\cdot\mathbf{P}_{L-l}(q^{-l}t)\mathbf{P}_l(t)\mathbf{v}_2}{\mathbf{v}_1(x,0)\cdot\mathbf{A}(x)^L\cdot\mathbf{v}_2}\Bigg|\\
&\ll\Bigg|\lambda(x)^K\,\frac{\mathbf{v}_1(x,tq^{-K})\cdot\mathbf{P}_{K-l}(0)\mathbf{P}_l(t)\mathbf{v}_2}{\mathbf{v}_1(x,0)\cdot\mathbf{A}(x)^K\cdot\mathbf{v}_2}\\
&\qquad\qquad -\lambda(x)^L\,\frac{\mathbf{v}_1(x,tq^{-L})\cdot\mathbf{P}_{L-l}(0)\mathbf{P}_l(t)\mathbf{v}_2}{\mathbf{v}_1(x,0)\cdot\mathbf{A}(x)^L\cdot\mathbf{v}_2}\Bigg|+|q|^{-l}|t|\\
&=\Bigg|\lambda(x)^K\,\frac{\mathbf{v}_1(x,tq^{-K})\cdot\mathbf{P}_{K-l}(0)(\mathbf{P}_l(t)-\mathbf{P}_l(0))\mathbf{v}_2}{\mathbf{v}_1(x,0)\cdot\mathbf{A}(x)^K\cdot\mathbf{v}_2}\\
&\qquad\qquad -\lambda(x)^L\,\frac{\mathbf{v}_1(x,tq^{-L})\cdot\mathbf{P}_{L-l}(0)(\mathbf{P}_l(t)-\mathbf{P}_l(0))\mathbf{v}_2}{\mathbf{v}_1(x,0)\cdot\mathbf{A}(x)^L\cdot\mathbf{v}_2}\Bigg|+|q|^{-l}|t|\\
&\ll |t|\left(\left(\frac{\lambda_1(x)}{\lambda(x)}\right)^{K-l}+|q|^{-l}\right)\ll |t|\,|q|^{-\eta(x)K},
\end{aligned}$$

where we have chosen $l=\lceil\eta K\rceil$. Letting $L$ tend to infinity yields

$$|\widehat{\mu}_{n,x}(t)-\widehat{\mu}_x(t)|\ll |t|q^{-n\eta(x)} \tag{3.11}$$

for $|t|\le 1$. In particular, (3.10) and (3.11) establish the existence of a (weak) limiting measure $\mu_x$.

Remark 2. What we have proved up to now is enough to have the asymptotic relation (3.2) without error term for all $x>0$.

*Step 3* establishes estimates for the measure dimension of $\mu_x$, which will be needed in Step 4. We define the matrices $\mathbf{I}_\varepsilon$ by setting

$$(\mathbf{I}_\varepsilon)_{B,C}=\begin{cases}\delta_{B,C} & \text{if the block } B \text{ starts with the digit } \varepsilon,\\ 0 & \text{otherwise.}\end{cases}$$

Clearly $\mathbf{I}_0+\mathbf{I}_1+\cdots+\mathbf{I}_{a^2}$ is the identity matrix. Furthermore, we have

$$\mu_x\left(\frac{\varepsilon_1}{q}+\frac{\varepsilon_2}{q^2}+\cdots+\frac{\varepsilon_k}{q^k}+q^{-k}\mathcal{F}\right)
=\lim_{n\to\infty}\frac{\mathbf{v}_1(x,0)\cdot\mathbf{I}_{\varepsilon_1}\mathbf{A}(x)\mathbf{I}_{\varepsilon_2}\mathbf{A}(x)\cdots\mathbf{I}_{\varepsilon_k}\mathbf{A}(x)\mathbf{A}(x)^{n-k}\cdot\mathbf{v}_2}{\mathbf{v}_1(x,0)\cdot\mathbf{A}(x)^n\cdot\mathbf{v}_2}.$$

The limit can be computed by the Perron–Frobenius theorem and equals

$$\lambda(x)^{-k}\mathbf{v}_1(x,0)\cdot\mathbf{I}_{\varepsilon_1}\mathbf{A}(x)\mathbf{I}_{\varepsilon_2}\mathbf{A}(x)\cdots\mathbf{I}_{\varepsilon_k}\mathbf{v}(x),$$

where $\mathbf{v}(x)$ denotes the (Perron–Frobenius) eigenvector of $\mathbf{A}(x)$ associated to the eigenvalue $\lambda(x)$ normalised so that

$$\mathbf{v}_1(x,0)\cdot\mathbf{v}(x)=1.$$

Now we define

$$\xi(x)=\max_{\varepsilon}\max_{B}\frac{(\mathbf{A}(x)\mathbf{I}_\varepsilon\mathbf{v})_B}{(\mathbf{v}(x))_B}$$

(this is always finite, since all coordinates of $\mathbf{v}(x)$ are strictly positive). Clearly, $\xi(x)<\lambda(x)$ and $\xi(1)=1$. By definition of $\xi(x)$ we have the componentwise inequality

$$\mathbf{A}(x)\mathbf{I}_\varepsilon\mathbf{v}(x)\le\xi(x)\mathbf{v}(x),$$

from which we conclude that

$$\mathbf{v}_1(x,0)\cdot\mathbf{I}_{\varepsilon_1}\mathbf{A}(x)\mathbf{I}_{\varepsilon_2}\mathbf{A}(x)\cdots\mathbf{I}_{\varepsilon_k}\mathbf{v}(x)\le\xi(x)^k\mathbf{v}_1(x,0)\cdot\mathbf{v}(x)=\xi(x)^k$$

and

$$\mu_x\left(\frac{\varepsilon_1}{q}+\frac{\varepsilon_2}{q^2}+\cdots+\frac{\varepsilon_k}{q^k}+q^{-k}\mathcal{F}\right)\ll\left(\frac{\xi(x)}{\lambda(x)}\right)^k. \tag{3.12}$$

Since $(q,\{0,\dots,a^2\})$ is a canonical number system, every ball $B(z,r)$ can be covered by an absolutely bounded number of sets of the form

$$\frac{\varepsilon_1}{q}+\frac{\varepsilon_2}{q^2}+\cdots+\frac{\varepsilon_k}{q^k}+q^{-k}\mathcal{F}$$

for $k=\lfloor-\log_{|q|}r\rfloor$ and $r<1$. This together with (3.12) implies

$$\mu_x(B(z,r))\ll r^{\beta(x)} \tag{3.13}$$

with

$$\beta(x)=\frac{\log\lambda(x)-\log\xi(x)}{\log|q|}.$$

Notice that $\beta(1)=2$, which is no surprise, since $\mu_1$ is Lebesgue measure restricted to $\mathcal{F}$.

Furthermore, we need at most $\mathcal{O}(|q|^{2n})$ times the area of the annulus $B(0,r+\varepsilon+|q|^{-n})\setminus B(0,r-|q|^{-n})$ copies of $q^{-n}\mathcal{F}$ to cover the annulus $B(0,r+\varepsilon)\setminus B(0,r)$. This together with (3.12) implies

$$\mu_x(B(0,r+\varepsilon)\setminus B(0,r))\ll|q|^{-n\beta(x)}|q|^{2n}(2r+\varepsilon)(\varepsilon+|q|^{-n})$$

for all $n$. Setting $n=-\lceil\log_{|q|}\varepsilon\rceil$ gives

$$\mu_x(B(0,r+\varepsilon)\setminus B(0,r))\ll(r+\varepsilon)\varepsilon^{\beta(x)-1}. \tag{3.14}$$

This gives a reasonable estimate if $\beta(x)>1$ or equivalently $\log\xi(x)<\log\lambda(x)-\log|q|$. Since this inequality is satisfied for $x=1$ and $\beta(x)$ depends continuously on $x$, there exists an interval $I$ around $x = 1$ such that $\beta(x) \geq \beta_0 > 1$ for some $\beta_0 < 2$.

*Step 4* uses the estimates for the measure dimension of $\mu_x$ and a suitable version of the Berry–Esseen inequality to provide bounds for $|\mu_{n,x}(B(0,r)) - \mu_x(B(0,r))|$. Since $\mu_{n,x}(B(0,r))$ can be easily related to the sum occurring in (3.2), this gives the error term in (3.2).

We recall the following result obtained in [12]. The statement uses the notation $\mathbf{c}(\phi) = (\cos\phi, \sin\phi)^T$.

PROPOSITION 1 ([12, Proposition 1]). *Let $\nu_1$ and $\nu_2$ be two probability measures in $\mathbb{R}^2$ with their Fourier transforms defined by*

$$\widehat{\nu}_k(\mathbf{t}) = \int_{\mathbb{R}^2} e(\langle \mathbf{x}, \mathbf{t}\rangle)\, d\nu_k(\mathbf{x}).$$

*Suppose that*

$$\nu_2(B(\mathbf{0}, r+\varepsilon) \setminus B(\mathbf{0}, r)) \ll \varepsilon^{\theta} \tag{3.15}$$

*for some $0 < \theta < 1$ and all $r \geq 0$. Then for all $r \geq 0$ and $T > 0$,*

$$\begin{aligned} &|\nu_1(B(\mathbf{0}, r)) - \nu_2(B(\mathbf{0}, r))| \\ &\quad \ll \int_0^T \int_0^{2\pi} K_r(t,T) |\widehat{\nu}_1(t\mathbf{c}(\phi)) - \widehat{\nu}_2(t\mathbf{c}(\phi))| t\, d\phi\, dt + T^{-2\theta/(\theta+2)}, \end{aligned} \tag{3.16}$$

*where the kernel function $K_r(t,T)$ satisfies*

$$K_r(t,T) \ll \frac{1}{T^2} + \min\biggl(r^2, \frac{r^{1/2}}{t^{3/2}}\biggr).$$

*The implied constant in* (3.16) *depends only on the implied constant in* (3.15).

Inserting (3.10) and (3.11) into (3.16) with $\theta = \beta(x) - 1$ yields

$$\begin{aligned} &|\mu_{n,x}(B(0,r)) - \mu_x(B(0,r))| \\ &\quad \ll \int_0^1 K_r(t,T) t |q|^{-\eta(x)n} t\, dt + \int_1^T K_r(t,T) t^{\eta(x)} |q|^{-\eta(x)n} t\, dt + T^{-2\frac{\beta(x)-1}{\beta(x)+1}}. \end{aligned} \tag{3.17}$$

Using the bounds for $K_r(t,T)$ and setting

$$\log T = \frac{\eta(x)}{\eta(x) + \frac{1}{2} + 2\frac{\beta(x)-1}{\beta(x)+1}}\, n \log|q|$$

yields

$$|\mu_{n,x}(B(0,r)) - \mu_x(B(0,r))| \ll |q|^{-2\kappa(x)n}$$

uniformly in $r$ with

$$\kappa(x)=\frac{\eta(x)(\beta(x)-1)}{(\eta(x)+1/2)(\beta(x)+1)+2\beta(x)-2}.$$

Choosing $\kappa$ to be the minimum attained by $\kappa(x)$ on a compact interval $I$, where $\beta(x)\ge\beta_0>1$ for some $\beta_0<2$, gives

$$|\mu_{n,x}(B(0,r))-\mu_x(B(0,r))|\ll|q|^{-2\kappa n} \tag{3.18}$$

for all $x\in I$.

Now, by definition of $\mu_{k,x}$, we have

$$\sum_{|z|^2<N}x^{s_F(z)}=\mathbf{v}_1(x,0)\cdot\mathbf{A}(x)^k\cdot\mathbf{v}_2\cdot\mu_{k,x}(B(0,|q|^{-k}\sqrt{N}))$$

for $k=\lfloor\log_{|q|^2}N\rfloor+M$ and some integer constant $M>0$, which is chosen so that $B(0,|q|^{1-M})\subset\mathcal{F}$. Inserting (3.18) and $\mathbf{v}_1(x,0)\cdot\mathbf{A}(x)^k\cdot\mathbf{v}_2=C(x)\lambda(x)^k+\mathcal{O}(\lambda_1(x)^k)$ yields

$$\begin{aligned}\sum_{|z|^2<N}x^{s_F(z)}&=C(x)\lambda(x)^k\mu_x(B(0,|q|^{-k}\sqrt{N}))+\mathcal{O}(\lambda_1(x)^k)+\mathcal{O}(\lambda(x)^k|q|^{-2\kappa k})\\&=N^{\log_{|q|^2}\lambda(x)}C(x)\lambda(x)^{\{\log_{|q|^2}N\}+M}\mu_x(B(0,q^{\{\log_{|q|^2}N\}-M}))(1+\mathcal{O}(N^{-\kappa})).\end{aligned}$$

We observe that the measure $\mu_x$ satisfies the self-similarity relation

$$\mu_x(B(0,|q|r))=\lambda(x)\mu_x(B(0,r))$$

for $r$ sufficiently small. Setting

$$\Phi(x,t)=C(x)\lambda(x)^{t+M}\mu_x(B(0,q^{t-M}))\quad\text{for } t<1$$

and noting that (3.13) implies the Hölder continuity of $\Phi$ as a function of $t$ completes the proof of Theorem 1. ■

REMARK 3. For complex values of $x$ this method breaks down, because the weak limits $\mu_x$ have infinite total variation and are therefore not complex measures.

**4. A Dirichlet series method.** The goal of this section is to generalise Theorem 1 to complex $x$. The proof relies on Dirichlet series and Mellin–Perron techniques.

THEOREM 2. *There exists a complex neighbourhood of* $x=1$ (*that is*, $|x-1|\le\delta$ *for some* $\delta>0$) *such that uniformly*

$$\sum_{|z|^2<N}x^{s_F(z)}=\Phi(x,\log_{|q|^2}N)N^{\log_{|q|^2}\lambda(x)}(1+\mathcal{O}(N^{-\kappa})) \tag{4.1}$$

*with some* $\kappa>0$, *where* $\Phi(x,t)$ *is analytic in* $x$ *and* 1-*periodic and Hölder continuous in* $t$.

*Furthermore, if* $F$ *is integer-valued with the property that*

$$d = \gcd\{g_F(B) : B \in \mathcal{N}^{L+1}\} = 1, \tag{4.2}$$

*then uniformly for* $|x-1| \ge \delta$ *and* $|\Re(x) - 1| \le \delta_2$,

$$\sum_{|z|^2<N} x^{s_F(z)} \ll N^{\log_{|q|^2}\lambda(|x|)-\kappa} \tag{4.3}$$

*with some* $\kappa > 0$ *and some* $\delta_2$ *with* $0 < \delta_2 < \delta$.

Remark 4. We will also show that $\Phi(x,t)$ has an explicit representation (see (4.21)). For example, for the sum-of-digits function $s_q(z)$ we have

$$\Phi(x,t) = \frac{X^{-t}}{1-X^{-1}} \sum_{l=1}^{a^2} x^l X^{\lfloor t-\log_{|q|^2} l^2\rfloor} + \frac{X^{-t}}{1-X^{-1}} \sum_{l=1}^{a^2} x^l \sum_{z\ne 0} x^{s_q(z)} (X^{\lfloor t-\log_{|q|^2}|qz+l|^2\rfloor} - X^{\lfloor t-\log_{|q|^2}|qz|^2\rfloor}),$$

where $X$ abbreviates

$$X = \frac{x^{|q|^2}-1}{x-1}.$$

The asymptotic representations (4.1) and (4.3) can be used in various ways (cf. also [5] and [6]). We directly derive asymptotic expansions for moments (Corollary 2) and a refinement of the central limit theorem stated in Corollary 1, further a local limit theorem (Corollary 3), uniform distribution in residue classes (Corollary 4) and uniform distribution modulo 1 (Corollary 5).

Corollary 2. *For every integer* $r \ge 1$ *we have*

$$\frac{1}{\pi N} \sum_{|z|^2<N} s_F(z)^r = \mu^r (\log_{|q|^2} N)^r + \sum_{l=0}^{r-1} G_{r,l}(\log_{|q|^2} N)(\log_{|q|^2} N)^l + \mathcal{O}(N^{-\kappa}), \tag{4.4}$$

*where the functions* $G_{r,l}(t)$ $(0 \le l < r)$ *are continuous and* 1*-periodic.*

*Proof.* Since (4.1) is uniform in a neighbourhood of 1 and $\Phi(x,t)$ is analytic in $x$ one can take derivatives at $x=1$ of arbitrary order by using the formula

$$G^{(r)}(1) = \frac{r!}{2\pi i} \int_{|x-1|=\delta/2} \frac{G(x)}{(x-1)^{r+1}}\, dx.$$

Furthermore, note that $\Phi(1,t) = \pi$. Hence, the asymptotic leading term is given by $(\lambda'(1)/\lambda(1))^r (\log_{|q|^2} N)^r$ and has no periodic fluctuations. ∎

Note that if we combine Corollaries 1 and 2 then we also get error terms for the central moments of the form

$$\frac{1}{\pi N}\sum_{|z|^2<N}(s_F(z)-\mu\log_{|q|^2}N)^L=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u^L e^{-\frac{1}{2}u^2}\,du+\mathcal{O}(N^{-\kappa}),$$

for every integer $L\ge 0$. Furthermore, if we use the characteristic function $\mathbb{E}\,e^{itY_N}=S_N(e^{it})/S_N(1)$ instead of the moment generating function $\mathbb{E}\,e^{tY_N}$, that is, if we set $x=e^{it}$ in Theorem 2, combined with Berry–Esseen techniques we also get a central limit theorem with error terms:

$$\frac{1}{\pi N}\#\Big\{|z|^2<N : s_F(z)\le\mu\log_{|q|^2}N+y\sqrt{\sigma^2\log_{|q|^2}N}\Big\}$$
$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y}e^{-\frac{1}{2}u^2}\,du+\mathcal{O}((\log N)^{-1/2}).$$

Corollary 3. *Suppose that $F$ is integer-valued and that (4.2) holds. Set*

$$\mu(x)=\frac{x\lambda'(x)}{\lambda(x)}\quad\text{and}\quad\sigma^2(x)=\frac{x^2\lambda''(x)}{\lambda(x)}+\mu(x)-\mu(x)^2.$$

*Furthermore, for $k\in K(N)=\mathbb{Z}\cap[\mu(1-\delta_2)\log_{|q|^2}N,\mu(1+\delta_2)\log_{|q|^2}N]$ we define $x_{k,N}$ by $\mu(x_{k,N})=k/\log_{|q|^2}N$, where $\delta$ and $\delta_2$ are from Theorem 2. Then uniformly for $k\in K(N)$,*

$$\#\{z\in\mathbb{Z}[i]:|z|^2<N,\ s_F(z)=k\}$$
$$=\frac{\Phi(x_{k,N},\log_{|q|^2}N)}{\sqrt{2\pi\sigma^2(x_{k,N})\log_{|q|^2}N}}N^{\log_{|q|^2}\lambda(x_{k,N})}x_{k,N}^{-k}\Big(1+\mathcal{O}\Big(\frac{1}{\log N}\Big)\Big).\tag{4.5}$$

*Furthermore, if $|k-\mu\log_{|q|^2}N|\le C\sqrt{\log_{|q|^2}N}$ (for some $C>0$) then also*

$$\#\{z\in\mathbb{Z}[i]:|z|^2<N,\ s_F(z)=k\}$$
$$=\frac{\pi N}{\sqrt{2\pi\sigma^2\log_{|q|^2}N}}\exp\Big(-\frac{(k-\mu\log_{|q|^2}N)^2}{2\sigma^2\log_{|q|^2}N}\Big)\Big(1+\mathcal{O}\Big(\frac{1}{\sqrt{\log N}}\Big)\Big).\tag{4.6}$$

Note that $\mu=\mu(1)$ and $\sigma^2=\sigma^2(1)$.

*Proof.* We apply (4.1) and (4.3) and use Cauchy's formula:

$$\#\{z\in\mathbb{Z}[i]:|z|^2<N,\ s_F(z)=k\}=\frac{1}{2\pi i}\int_{|x|=x_{k,N}}\Big(\sum_{|z|^2<N}x^{s_F(z)}\Big)x^{-k-1}\,dx,$$

where $x_{k,N}$ is the saddle point of the asymptotic leading term of the integrand:

$$N^{\log_{|q|^2}\lambda(x)}x^{-k}=e^{\log\lambda(x)\cdot\log_{|q|^2}N-k\log x}.$$

We do not work out the details of standard saddle point techniques. We just refer to [6], where problems of almost the same kind have been discussed. ∎

Corollary 4. *Suppose that $F$ is integer-valued and that* (4.2) *holds. Then for every integer $M \ge 1$ and all $m \in \{0,1,\dots,M-1\}$ we have*

$$\frac{1}{\pi N}\#\{|z|^2 < N : s_F(z) \equiv m \bmod M\} = \frac{1}{M} + \mathcal{O}(N^{-\eta})$$

*for some $\eta > 0$.*

Remark 5. Alternatively to condition (4.2) we can assume that $s_F$ attains a value that is relatively prime to $M$. Then the same assertion holds (cf. Corollary 8).

*Proof of Corollary 4.* We use (4.3) for all $M$th roots of unity $x = e^{2\pi i m/M}$ and apply simple discrete Fourier techniques. ∎

Corollary 5. *Let $s_F$ be a block-additive function which attains one irrational value. Then the sequence $(s_F(z))_{z\in\mathbb{Z}[i]}$ is uniformly distributed modulo* 1.

Remark 6. Note that Corollary 5 in particular applies to sequences of the kind $(\alpha s_F(z))_{z\in\mathbb{Z}[i]}$ if $s_F$ is integer-valued and $\alpha$ is irrational.

*Proof of Corollary 5.* We only have to prove that there exists a block $B$ of length $L+1$ such that $g_F(B)$ is irrational. For this purpose we find a $z_0 \in \mathbb{Z}[i]$ with $s_F(z_0)$ irrational and with base $q$ representation of minimal length. Then by Lemma 2 we write $z_0 = \varepsilon_0 + qv$ and $g_F(B) = s_F(z_0) - s_F(v)$. Since the base $q$ representation of $v$ has one digit less than the representation of $z_0$, $s_F(v)$ is rational, and therefore $g_F(B)$ is irrational.

Choosing $x^{g_F(B)} = e(hg_F(B))$ for $h \in \mathbb{Z}\setminus\{0\}$ gives a matrix $\mathbf{A}(x)$ with eigenvalues strictly less than $|q|^2$. By Weyl's criterion this implies the assertion. ∎

We now turn to the proof of Theorem 2. For this purpose we will consider the Dirichlet series

$$G_B(x,s) = \sum_{z\in\mathbb{Z}[i]\setminus\{0\},\,(\varepsilon_0(z),\dots,\varepsilon_L(z))=B} \frac{x^{s_F(z)}}{|z|^{2s}}$$

for $B \in \mathcal{N}^{L+1}$. It is easy to see that these series are well defined in a certain range. Set $A_1 = \max_{B\in\mathcal{N}^{L+1}} F(B)$ and $A_2 = \min_{B\in\mathcal{N}^{L+1}} F(B)$. Then we have $A_2 \log_{|q|^2}|z| - \mathcal{O}(1) \le s_F(z) \le A_2 \log_{|q|^2}|z| + \mathcal{O}(1)$. Hence, if $|x| \ge 1$ then $G_B(x,s)$ is surely absolutely convergent for $\Re(s) > 1 + \frac{1}{2}A_1 \log_{|q|^2}|x|$. Similarly, if $|x| \le 1$ then $G_B(x,s)$ is absolutely convergent for $\Re(s) > 1 - \frac{1}{2}A_2 \log_{|q|^2}(1/|x|)$.

Next we provide a representation for $G_B(x,s)$ that can be used for analytic continuation.

LEMMA 3. *Define the vectors* $\mathbf{G}(x,s) = (G_B(x,s))_{B\in\mathcal{N}^{L+1}}$ *and* $\mathbf{H}(x,s) = (H_B(x,s))_{B\in\mathcal{N}^{L+1}}$, *where*

$$H_B(x,s)=\begin{cases} 0 & \text{if } \eta(B)=0,\\ \dfrac{x^{s_F(\eta_0)}}{|\eta_0|^{2s}}+\dfrac{x^{g_F(B)}}{|q|^{2s}}\displaystyle\sum_{\substack{v\in\mathbb{Z}[i]\setminus\{0\}\\ (\varepsilon_0(v),\dots,\varepsilon_{L-1}(v))=(0,\dots,0)}} x^{s_F(v)}\left(\frac{1}{|v+\eta_0/q|^{2s}}-\frac{1}{|v|^{2s}}\right) & \\ & \text{if } \eta_0=\eta(B)\neq 0 \text{ and } B'=(0,\dots,0),\\ \dfrac{x^{g_F(B)}}{|q|^{2s}}\displaystyle\sum_{\substack{v\in\mathbb{Z}[i]\setminus\{0\}\\ (\varepsilon_0(v),\dots,\varepsilon_{L-1}(v))=B'}} x^{s_F(v)}\left(\frac{1}{|v+\eta_0/q|^{2s}}-\frac{1}{|v|^{2s}}\right) & \\ & \text{if } \eta_0=\eta(B)\neq 0 \text{ and } B'\neq(0,\dots,0).\end{cases}$$

*Then* $H_B(x,s)$ *is absolutely convergent for* $\Re(s) > \frac{1}{2}+\frac{1}{2}A_1\log_{|q|^2}|x|$ *if* $|x|\geq 1$ *and for* $\Re(s) > \frac{1}{2}-\frac{1}{2}A_2\log_{|q|^2}(1/|x|)$ *if* $|x|\leq 1$. *More precisely, in that range*

$$H(x,\sigma+it) \ll \begin{cases} (1+|t|)^{2(1-\sigma)+A_1\log_{|q|^2}|x|} & \text{if } |x|\geq 1,\\ (1+|t|)^{2(1-\sigma)-A_2\log_{|q|^2}(1/|x|)} & \text{if } |x|\leq 1,\end{cases} \tag{4.7}$$

*and a meromorphic continuation of* $\mathbf{G}(x,s) = (G_B(x,s))_{B\in\mathcal{N}^{L+1}}$ *is given by*

$$\mathbf{G}(x,s) = \left(\mathbf{I} - \frac{1}{|q|^{2s}}\,\mathbf{A}(x)\right)^{-1}\mathbf{H}(x,s), \tag{4.8}$$

*where* $\mathbf{A}(x)$ *is defined in* (3.1).

*Proof.* We use the substitution $z=\eta_0+qv$. If $\varepsilon_0(z)=\eta_0=0$ we have $s_F(z)=s_F(q)$ and consequently

$$G_B(x,s) = \frac{1}{|q|^{2s}}\sum_{v\in\mathbb{Z}[i]\setminus\{0\},\,(\varepsilon_0(v),\dots,\varepsilon_L(v))=B'}\frac{x^{s_F(v)}}{|v|^{2s}} = \frac{1}{|q|^{2s}}\sum_{l=0}^{a^2} G_{(B',l)}(x,s).$$

Similarly, if $\eta_0>0$ and $B'=(0,\dots,0)$ we get

$$\begin{aligned} G_B(x,s) &= \frac{x^{s_F(\eta_0)}}{|\eta_0|^{2s}}+\frac{x^{g_F(B)}}{|q|^{2s}}\sum_{v\in\mathbb{Z}[i]\setminus\{0\},\,(\varepsilon_0(v),\dots,\varepsilon_{L-1}(v))=(0,\dots,0)}\frac{x^{s_F(v)}}{|v+\eta_0/q|^{2s}}\\ &= \frac{x^{g_F(B)}}{|q|^{2s}}\sum_{v\in\mathbb{Z}[i]\setminus\{0\},\,(\varepsilon_0(v),\dots,\varepsilon_{L-1}(v))=(0,\dots,0)}\frac{x^{s_F(v)}}{|v|^{2s}} + H_B(x,s)\\ &= \frac{x^{g_F(B)}}{|q|^{2s}}\sum_{l=0}^{a^2} G_{(0,\dots,0,l)}(x,s) + H_B(x,s). \end{aligned}$$

Finally, if $\eta_0>0$ and $B'\neq(0,\dots,0)$ then the case $v=0$ cannot appear and we also get

$$G_B(x,s) = \frac{x^{g_F(B)}}{|q|^{2s}} \sum_{v\in\mathbb{Z}[i]\setminus\{0\},\,(\varepsilon_0(v),\dots,\varepsilon_{L-1}(v))=B'} \frac{x^{s_F(v)}}{|v+\eta_0/q|^{2s}}$$

$$= \frac{x^{g_F(B)}}{|q|^{2s}} \sum_{l=0}^{a^2} G_{(B',l)}(x,s) + H_B(x,s).$$

Now with $\mathbf{A}(x) = (A_{B,C}(x))_{B,C\in\mathcal{N}^{L+1}}$ this directly translates to

$$\mathbf{G}(x,s) = \frac{1}{|q|^{2s}}\,\mathbf{A}(x)\mathbf{G}(x,s) + \mathbf{H}(x,s),$$

which implies (4.8).

Set $s = \sigma + it$. Since

$$|\,|v + l/\eta_0|^{2s} - |v|^{2s}| \ll |v|^{2\sigma}\min\biggl(1, \frac{1+|t|}{|v|}\biggr)$$

it easily follows that $H_B(x,s)$ is absolutely convergent for $\Re(s) > \frac{1}{2} + \frac{1}{2}A_1\log_{|q|^2}|x|$ if $|x| \ge 1$ and for $\Re(s) > \frac{1}{2} - \frac{1}{2}A_2\log_{|q|^2}(1/|x|)$ if $|x| \le 1$, and that $H(x,s)$ is bounded by (4.7). ∎

If we set $a_n = \sum_{|z|^2=n} x^{s_F(z)}$ then $G(s,x) = \sum_{n\ge1} a_n n^{-s}$ and Mellin–Perron's formula gives (for non-integral $N$)

$$\sum_{n<N} a_n = \sum_{0\ne|z|^2<N} x^{s_q(z)} = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT} G(x,s)\,\frac{N^s}{s}\,ds \tag{4.9}$$

for any sufficiently large $c$ such that the line $\Re(s) = c$ is contained in the half-plane of convergence of $G(x,s)$.

We will first use this representation to get upper bounds for the sum $\sum_{0\ne|z|^2<N} x^{s_q(z)}$. For this purpose we have to know something on the dominant eigenvalue $\lambda(x)$ of $\mathbf{A}(x)$.

LEMMA 4. *If $x$ is sufficiently close to the positive real axis then $\lambda(x)$ is a simple eigenvalue of $\mathbf{A}(x)$ and all other eigenvalues have smaller modulus. Furthermore, if $F$ is integer-valued such that* (4.2) *holds and if $x \ne 0$ is not a positive real number then all eigenvalues $\beta$ of $\mathbf{A}(x)$ satisfy*

$$|\beta| < \lambda(|x|). \tag{4.10}$$

*Proof.* Suppose first that $x$ is a positive real number. Then it easily follows that $\mathbf{A}(x)$ is a primitive irreducible non-negative matrix. We just have to observe that for every pair of blocks $B, C \in \mathcal{N}_{L+1}$ there exists a Gaussian integer $z$ such that both $B$ and $C$ occur in the $q$-ary digital expansion of $z$. Hence, all elements of $\mathbf{A}(x)^{L+1}$ are positive and consequently by [22, Theorem 2.1, p. 49], $\mathbf{A}(x)$ is primitive and irreducible. Thus, $\lambda(x) > 0$

is simple and all other eigenvalues have smaller modulus. By continuity, this property remains true if $x$ is sufficiently close to the positive real axis.

Next, suppose that $x = |x|e^{i\varphi}$ with $0 < \varphi < 2\pi$. Since $|x^{g_F(B)}| = |x|^{g_F(B)}$, [22, Theorem 2.1, p. 36] implies that all eigenvalues $\beta$ of $\mathbf{A}(x)$ satisfy $|\beta| \le \lambda(|x|)$. Furthermore, the equality $|\beta| = \lambda(|x|)$ holds if and only if there exists a complex number $\mu$ with $|\mu| = 1$ and a diagonal matrix $D = \mathrm{diag}(\mu_B)_{B\in\mathcal{N}_{L+1}}$ with complex numbers $\mu_B$ of modulus $|\mu_B| = 1$ such that

$$\mathbf{A}(x) = \lambda D\mathbf{A}(|x|)D^{-1}.$$

Without loss of generality we may assume that $\mu_{0\cdots 0} = 1$.

We now show that in this case $\mu = 1$ and $\mu_B = 1$ for all $B \in \mathcal{N}_{L+1}$, resp. $\mathbf{A}(x) = \mathbf{A}(|x|)$. First observe that $A_{0\cdots 0,\,0\cdots 0}(x) = 1$ (for all $x$). Thus, $\mu = 1$. Furthermore, observe that $A_{B,C}(x) = A_{B,C}(|x|) \ne 0$ implies $\mu_B = \mu_C$. Obviously, we have $A_{B,C}(x) = A_{B,C}(|x|) \ne 0$ if $C = (B', l)$ (for some $l$) and $\eta_B = 0$. Thus, if $B = (\eta_1, \ldots, \eta_L)$ is any block in $\mathcal{N}_{L+1}$ then we can consider the sequence of blocks

$$B_0 = (0,\ldots,0),\; B_1 = (0,\ldots,0,\eta_1),\; B_2 = (0,\ldots,0,\eta_1,\eta_2),\; \ldots,\; B_L = B$$

and conclude inductively that

$$1 = \mu_{B_0} = \mu_{B_1} = \cdots = \mu_B.$$

However, if (4.2) holds then for every $0 < \varphi < 2\pi$ there exists $B \in \mathcal{N}_{L+1}$ with $e^{i\varphi g_F(B)} \ne 1$, and thus $x^{g_F(B)} \ne |x|^{g_F(B)}$. Consequently, all eigenvalues $\beta$ of $\mathbf{A}(|x|e^{i\varphi})$ are strictly bounded by $|\beta| < \lambda(|x|)$. ∎

Next note that the inverse matrix $(\mathbf{I} - u\mathbf{A}(x))^{-1}$ can be written as

$$(\mathbf{I} - u\mathbf{A}(x))^{-1} = \frac{1}{\det(\mathbf{I} - u\mathbf{A}(x))}\,(P_{BC}(u,x))_{B,C\in\mathcal{N}^{L+1}}$$

with polynomials $P_{BC}(u,x)$ having degree in $u$ smaller than $D := |\mathcal{N}^{L+1}| = |q|^{2L+2}$. As above let $\lambda(x)$ be the dominating eigenvalue of $\mathbf{A}(x)$ and $\lambda_2(x)$, $\ldots, \lambda_D(x)$ the remaining ones (where we assume that $x$ is sufficiently close to the real axis and that all roots are simple). Then by the partial fraction decomposition we have

$$\frac{P_{BC}(u,x)}{\det(\mathbf{I} - u\mathbf{A}(x))} = \frac{C_{BC}(x)}{1 - u\lambda(x)} + \sum_{j=2}^{D} \frac{C_{j,BC}(x)}{1 - u\lambda_j(x)} \tag{4.11}$$

for certain (analytic) functions $C_{BC}(x)$ and $C_{j,BC}(x)$. This also shows that $G(x,s)$ can be represented as

$$G(x,s) = \frac{K(x,s)}{1 - \frac{1}{|q|^{2s}}\lambda(x)} + \sum_{j=2}^{D} \frac{K_j(x,s)}{1 - \frac{1}{|q|^{2s}}\lambda_j(x)}, \tag{4.12}$$

where $K(x,s)$ and $K_j(x,s)$ are linear combinations of the functions $H_B(x,s)$ with coefficients that are analytic in $x$ (cf. also (4.20)).

This shows that (4.8) provides an analytic continuation of $G(s,x)$ to the range $\Re(s) > \log_{|q|^2} |\lambda(x)|$ if $x$ is sufficiently close to 1, say $|x-1| \le \delta$. Furthermore, if $|x-1| \ge \delta$ and $|\Re(x)-1| \le \delta_2$ then Lemma 4 shows that all eigenvalues $\beta$ of $\mathbf{A}(x)$ satisfy $|\beta| \le \lambda(|x|) - \eta'$ for some $\eta'$. Consequently, for all $x$ in that range the function $G(x,s)$ is analytic in the half-plane $\Re(s) > \log_{|q|^2}(\lambda(|x|) - \eta')$.

With this knowledge we are now ready to prove the second part of Theorem 2. The argument is close to that of [14].

LEMMA 5. *Suppose that $F$ is integer-valued and that* (4.2) *holds. Then there exist $\delta, \kappa > 0$ such that*

$$\sum_{|z|^2<N} x^{s_F(z)} \ll N^{\log_{|q|^2} \lambda(|x|) - \kappa} \tag{4.13}$$

*uniformly for $|x-1| \ge \delta$ and $|\Re(x)-1| \le \delta_2$.*

*Proof.* Our starting point is formula (4.9). Observe that the integral there is not absolutely convergent. However, a slight variation of the Mellin–Perron formula gives

$$S_N^{(2)}(x) = \sum_{0\ne|z|^2<N} x^{s_q(z)}\left(1 - \frac{|z|^2}{N}\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} G(x,s)\,\frac{N^s}{s(s+1)}\,ds \tag{4.14}$$

with an integral that will be absolutely convergent in the range of interest.

Suppose now that $|x-1| \ge \delta$ and $|\Re(x)-1| \le \delta_2$. Then we already know that $G(x,s)$ is analytic for $\Re(s) > \log_{|q|^2}(\lambda(|x|) - \eta')$ and that

$$|G(x,s)| \ll (1+|t|)^{2(1-\sigma)+\eta''}$$

if $\sigma = \Re(s) \ge \log_{|q|^2}(\lambda(|x|) - \eta'/2) > \log_{|q|^2}\lambda(|x|) - \eta'''$. It follows that

$$S_N^{(2)}(x) \ll N^{\log_{|q|^2} \lambda(|x|) - \eta'''}.$$

It is now easy to derive proper upper bounds for

$$S_N(x) = \sum_{0\ne|z|^2<N} x^{s_q(z)}.$$

Observe that for every factor $\varrho > 1$ we have

$$S_N(x) = \frac{\varrho S_{\varrho N}^{(2)}(x) - S_N^{(2)}(x)}{\varrho - 1} + \frac{1}{\varrho-1}\sum_{N\le|z|^2<\varrho N} x^{s_F(z)}\left(1 - \frac{|z|^2}{N}\right).$$

Set $c = \log_{|q|^2}\lambda(|x|) - \eta'''$. By adjusting $\delta_2$ we can assume that $c < 1$. Finally, with

$$\varrho = 1 + N^{-(1-c)/2}$$

it follows that

$$S_N(x) \ll N^{(1+c)/2} N^{\max(A_1 \log_{|q|^2}(1+\delta_2), A_2 \log_{|q|^2}(1-\delta_2))}.$$

Since $\delta_2$ can be chosen arbitrarily small it finally follows that

$$S_N(x) \ll N^{\log_{|q|^2} \lambda(|x|) - \eta}$$

for some $\eta > 0$. ∎

In order to prove the asymptotic expansion (4.1) for complex $x$ (close to 1) we will use the following properties (see also [2, p. 243]).

Lemma 6. *Suppose that $a$ and $c$ are positive real numbers. Then*

$$\left| \frac{1}{2\pi i} \int_{c-iT}^{c+iT} a^s \frac{ds}{s} - 1 \right| \le \frac{a^c}{\pi T \log a} \qquad (a > 1), \tag{4.15}$$

$$\left| \frac{1}{2\pi i} \int_{c-iT}^{c+iT} a^s \frac{ds}{s} \right| \le \frac{a^c}{\pi T \log(1/a)} \qquad (0 < a < 1), \tag{4.16}$$

$$\left| \frac{1}{2\pi i} \int_{c-iT}^{c+iT} a^s \frac{ds}{s} - \frac{1}{2} \right| \le \frac{C}{T} \qquad (a = 1). \tag{4.17}$$

*Proof.* Suppose first that $a > 1$. By considering the contour integral of the function $F(s) = a^s/s$ around the rectangle with vertices $-A - iT, c - iT, c + iT, -A + iT$ and passing $A$ to infinity one directly gets the representation

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} a^s \frac{ds}{s} = \operatorname{Res}(a^s/s; s = 0) + \frac{1}{2\pi i} \int_{-\infty}^{c} \frac{a^{x+iT}}{x + iT}\, dx + \frac{1}{2\pi i} \int_{-\infty}^{c} \frac{a^{x-iT}}{x - iT}\, dx.$$

Since

$$\left| \frac{1}{2\pi i} \int_{-\infty}^{c} \frac{a^{x\pm iT}}{x \pm iT}\, dx \right| \le \frac{a^c}{\pi T \log a}$$

we directly obtain the bound in the case $a > 1$.

The case $0 < a < 1$ can be handled in the same way. Finally, in the case $a = 1$ the integral can be explicitly calculated (and estimated). ∎

For the formulation of the next lemma we use Iverson's notation $[\![p]\!]$ which is 1 if $p$ is a true proposition and 0 otherwise.

Lemma 7. *Suppose that $l$ is a positive real number, $\lambda$ a non-zero complex number, $c$ a real number with $c > \log_{|q|^2} |\lambda|$. Then for all real $N > l^2$,*

$$\text{(4.18)} \qquad \frac{1}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT}\frac{\frac{1}{l^{2s}}}{1-\frac{1}{|q|^{2s}}\lambda}\,\frac{N^s}{s}\,ds$$
$$= \frac{\lambda^{\lfloor \log_{|q|^2}(N/l^2)\rfloor+1}-1}{\lambda-1} - \frac{1}{2}\,\lambda^{\lfloor \log_{|q|^2}(N/l^2)\rfloor}[\![\log_{|q|^2}(N/l^2)\in\mathbb{Z}]\!].$$

*Furthermore, if* $c > \max\{1, \log_{|q|^2}|\lambda|\}$ *and* $x$ *is sufficiently close to* 1 *then for every set of* $S$ *of Gaussian integers with* $0 \notin S$ *and all irrational numbers* $N > 1$,

$$\text{(4.19)} \qquad \frac{1}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT}\frac{\sum_{z\in S} x^{s_F(z)}\left(\frac{1}{|qz+l|^{2s}}-\frac{1}{|qz|^{2s}}\right)}{1-\frac{1}{|q|^{2s}}\lambda}\,\frac{N^s}{s}\,ds$$
$$= \frac{1}{1-\lambda^{-1}}\sum_{z\in S} x^{s_F(z)}\left(\lambda^{\lfloor \log_{|q|^2}(N/|qz+l|^2)\rfloor} - \lambda^{\lfloor \log_{|q|^2}(N/|qz|^2)\rfloor}\right)$$
$$-\frac{1}{2}\sum_{z\in S} x^{s_F(z)}\lambda^{\lfloor \log_{|q|^2}(N/|qz+l|^2)\rfloor}[\![\log_{|q|^2}(N/|qz+l|^2)\in\mathbb{Z}]\!]$$
$$+\frac{1}{2}\sum_{z\in S} x^{s_F(z)}\lambda^{\lfloor \log_{|q|^2}(N/|qz|^2)\rfloor}[\![\log_{|q|^2}(N/|qz|^2)\in\mathbb{Z}]\!] + \mathcal{O}(1).$$

*Proof.* By assumption we have $|\lambda/|q|^{2s}| < 1$. Thus, by using a geometric series expansion and Lemma 6, for all $N > 1$ such that $\log_{|q|^2}(N/l^2)$ is not an integer we get

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{\frac{1}{l^{2s}}}{1-\frac{1}{|q|^{2s}}\lambda}\,\frac{N^s}{s}\,ds = \sum_{k\ge 0}\lambda^k\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\left(\frac{N}{|q|^{2k}l^2}\right)^s\frac{ds}{s}$$
$$= \sum_{k\le \log_{|q|^2}(N/l^2)}\lambda^k + \mathcal{O}\left(\frac{1}{T}\sum_{k\ge 0}\frac{|\lambda|^k\left(\frac{N}{|q|^{2k}l^2}\right)^c}{\left|\log\left(\frac{N}{|q|^{2k}l^2}\right)\right|}\right)$$
$$= \frac{\lambda^{\lfloor \log_{|q|^2}(N/l^2)\rfloor+1}-1}{\lambda-1} + \mathcal{O}\left(\frac{1}{T}\,\frac{(N/l^2)^c}{1-\frac{1}{|q|^{2c}}|\lambda|}\right).$$

If $\log_{|q|^2}(N/l^2)$ is an integer, we can proceed similarly. Of course, this implies (4.18).

Next assume that neither $\log_{|q|^2}(N/|qz+l|^2)$ nor $\log_{|q|^2}(N/|qz|^2)$ are integers for all $z \in S$. Hence, if $N > |qz+l|^2$ then

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{\frac{1}{|qz+l|^{2s}}}{1-\frac{1}{|q|^{2s}}\lambda}\,\frac{N^s}{s}\,ds$$
$$= \frac{\lambda^{\lfloor \log_{|q|^2}(N/|qz+l|^2)\rfloor+1}-1}{\lambda-1} + \mathcal{O}\left(\frac{1}{T}\sum_{k\ge 0}\frac{|\lambda|^k\left(\frac{N}{|q|^{2k}|qz+l|^2}\right)^c}{\left|\log\left(\frac{N}{|q|^{2k}|qz+l|^2}\right)\right|}\right)$$

and if $N < |qz+l|^2$ then we just have

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{\frac{1}{|qz+l|^{2s}}}{1-\frac{1}{|q|^{2s}}\lambda}\,\frac{N^s}{s}\,ds = \mathcal{O}\biggl(\frac{1}{T}\sum_{k\ge 0}\frac{|\lambda|^k\bigl(\frac{N}{|q|^{2k}|qz+l|^2}\bigr)^c}{\bigl|\log\bigl(\frac{N}{|q|^{2k}|qz+l|^2}\bigr)\bigr|}\biggr).$$

Furthermore, for given $N$ there are only finitely many pairs $(k,z)$ with

$$\left|\frac{N}{|q|^{2k}|qz+l|^2}-1\right| < \frac{1}{2}.$$

Hence, the series

$$\sum_{z\in S}\sum_{k\ge 0}\frac{|\lambda|^k\bigl(\frac{N}{|q|^{2k}|qz+l|^2}\bigr)^c}{\bigl|\log\bigl(\frac{N}{|q|^{2k}|qz+l|^2}\bigr)\bigr|}$$

is convergent if $x$ is sufficiently close to 1. Consequently, we get

$$\frac{1}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT}\frac{\sum_{z\in S}x^{s_F(z)}\bigl(\frac{1}{|qz+l|^{2s}}-\frac{1}{|qz|^{2s}}\bigr)}{1-\frac{1}{|q|^{2s}}\lambda}\,\frac{N^s}{s}\,ds$$
$$= \frac{1}{1-\lambda^{-1}}\sum_{z\in S,\,|z|^2<N}x^{s_F(z)}\bigl(\lambda^{\lfloor\log_{|q|^2}(N/|qz+l|^2)\rfloor}-\lambda^{\lfloor\log_{|q|^2}(N/|qz|^2)\rfloor}\bigr)+\mathcal{O}(1).$$

Finally, since $|qz+l|^2 = |qz|^2(1+\mathcal{O}(1/|z|))$ it follows that for $x$ sufficiently close to 1 we have

$$\sum_{z\in S,\,|z|^2\ge N}x^{s_F(z)}\bigl(\lambda^{\lfloor\log_{|q|^2}(N/|qz+l|^2)\rfloor}-\lambda^{\lfloor\log_{|q|^2}(N/|qz|^2)\rfloor}\bigr)=\mathcal{O}(1).$$

This proves (4.19) if neither $\log_{|q|^2}(N/|qz+l|^2)$ nor $\log_{|q|^2}(N/|qz|^2)$ are integers. It is, however, easy to adapt the above calculation in the general case. ∎

We now come back to the representation (4.12) for $G(s,x)$. We already mentioned that $K(s,x)$ and $K_j(s,x)$ are linear combinations of the functions $H_B(x,y)$ with coefficients that are analytic in $x$. We make this explicit for $K(s,x)$ in the following form:

$$K(s,x) = \sum_{l=1}^{a^2}\frac{c_l'(x)}{l^{2s}} + \sum_{l=1}^{a^2}\sum_{B'\in\mathcal{L}^L}c_{l,B'}''(x)\sum_{\substack{z\in\mathbb{Z}[i]\setminus\{0\}\\(\varepsilon_0(z),\dots,\varepsilon_{L-1}(z))=B'}}x^{s_F(z)}\biggl(\frac{1}{|qz+l|^{2s}}-\frac{1}{|qz|^{2s}}\biggr). \tag{4.20}$$

Hence, for $N>1$ we obtain

$$\frac{1}{2\pi i}\lim_{T\to\infty}\int\limits_{c-iT}^{c+iT}\frac{K(x,s)}{1-\frac{1}{|q|^{2s}}\lambda(x)}\,\frac{N^s}{s}\,ds=\frac{1}{1-\lambda(x)^{-1}}\sum_{l=1}^{a^2}c_l'(x)\lambda(x)^{\lfloor\log_{|q|^2}\frac{N}{l^2}\rfloor}$$
$$+\sum_{l=1}^{a^2}\sum_{B'\in\mathcal{L}^L}\frac{c''_{l,B'}(x)}{1-\lambda(x)^{-1}}\sum_{z\neq0}x^{s_F(z)}\Big(\lambda(x)^{\lfloor\log_{|q|^2}\frac{N}{|qz+l|^2}\rfloor}-\lambda(x)^{\lfloor\log_{|q|^2}\frac{N}{|qz|^2}\rfloor}\Big)$$
$$-\frac{1}{2}\sum_{l=1}^{a^2}c_l'(x)\lambda(x)^{\lfloor\log_{|q|^2}\frac{N}{l^2}\rfloor}\left[\!\!\left[\log_{|q|^2}\frac{N}{l^2}\in\mathbb{Z}\right]\!\!\right]$$
$$-\frac{1}{2}\sum_{l=1}^{a^2}\sum_{B'\in\mathcal{L}^L}c''_{l,B'}(x)\sum_{z\neq0}x^{s_F(z)}\lambda(x)^{\lfloor\log_{|q|^2}\frac{N}{|qz+l|^2}\rfloor}\left[\!\!\left[\log_{|q|^2}\frac{N}{|qz+l|^2}\in\mathbb{Z}\right]\!\!\right]$$
$$+\frac{1}{2}\sum_{l=1}^{a^2}\sum_{B'\in\mathcal{L}^L}c''_{l,B'}(x)\sum_{z\neq0}x^{s_F(z)}\lambda(x)^{\lfloor\log_{|q|^2}\frac{N}{|qz|^2}\rfloor}\left[\!\!\left[\log_{|q|^2}\frac{N}{|qz|^2}\in\mathbb{Z}\right]\!\!\right]+\mathcal{O}(1),$$

where the $\mathcal{O}(1)$-term is uniform for $N>1$ and for $x$ in a complex neighbourhood of $x=1$. Note that the *correction terms* vanish if $N$ is, for example, irrational. Actually, we will prove in Lemma 8 that these correction terms can always be neglected since they sum up to zero in all cases.

Furthermore, note that the right hand side of this representation is of order $\mathcal{O}(N^{\log_{|q|^2}\Re(\lambda(x))})$. Thus, if we do corresponding calculations for $K_j(x,s)$ and $\lambda_j(x)$ we also get

$$\frac{1}{2\pi i}\lim_{T\to\infty}\int\limits_{c-iT}^{c+iT}\frac{K_j(x,s)}{1-\frac{1}{|q|^{2s}}\lambda_j(x)}\,\frac{N^s}{s}\,ds=\mathcal{O}(N^{\log_{|q|^2}\Re(\lambda_j(x))}).$$

Hence, setting

$$(4.21)\qquad \overline{\Phi}(x,t)=\frac{\lambda(x)^{-t}}{1-\lambda(x)^{-1}}\sum_{l=1}^{a^2}c_l'(x)\lambda(x)^{\lfloor t-\log_{|q|^2}l^2\rfloor}$$
$$+\sum_{l=1}^{a^2}\sum_{B'\in\mathcal{L}^L}\frac{\lambda(x)^{-t}c''_{l,B'}(x)}{1-\lambda(x)^{-1}}\sum_{z\neq0}x^{s_F(z)}(\lambda(x)^{t-\lfloor\log_{|q|^2}|qz+l|^2\rfloor}-\lambda(x)^{\lfloor t-\log_{|q|^2}|qz|^2\rfloor})$$

and

$$(4.22)\qquad \overline{\overline{\Phi}}(x,t)=-\frac{\lambda(x)^{-t}}{2}\sum_{l=1}^{a^2}c_l'(x)\lambda(x)^{\lfloor t-\log_{|q|^2}l^2\rfloor}[\![t-\log_{|q|^2}l^2\in\mathbb{Z}]\!]$$
$$-\frac{\lambda(x)^{-t}}{2}\sum_{l=1}^{a^2}\sum_{B'\in\mathcal{L}^L}c''_{l,B'}(x)\sum_{z\neq0}x^{s_F(z)}\lambda(x)^{\lfloor t-\log_{|q|^2}|qz+l|^2\rfloor}[\![t-\log_{|q|^2}|qz+l|^2\in\mathbb{Z}]\!]$$
$$+\frac{\lambda(x)^{-t}}{2}\sum_{l=1}^{a^2}\sum_{B'\in\mathcal{L}^L}c''_{l,B'}(x)\sum_{z\neq0}x^{s_F(z)}\lambda(x)^{\lfloor t-\log_{|q|^2}|qz|^2\rfloor}[\![t-\log_{|q|^2}|qz|^2\in\mathbb{Z}]\!]$$

we end up with the representation

$$S_N(x) = (\overline{\Phi}(x, \log_{|q|^2} N) + \overline{\overline{\Phi}}(x, \log_{|q|^2} N)) \times N^{\log_{|q|^2} \lambda(x)}(1 + \mathcal{O}(N^{-\kappa})), \tag{4.23}$$

where $\kappa > 0$ is just the minimal difference between $\Re(\lambda(x))$ and $\Re(\lambda_j(x))$ $(j \geq 2)$ when $x$ varies in a sufficiently small neighbourhood of $x = 1$. By definition it is clear that $\overline{\Phi}(x,t) = \overline{\Phi}(x,t+1)$, $\overline{\overline{\Phi}}(x,t) = \overline{\overline{\Phi}}(x,t+1)$ and that $\overline{\Phi}(x,t)$ and $\overline{\overline{\Phi}}(x,t)$ represent analytic functions in $x$ if $t$ is fixed. However, $\overline{\overline{\Phi}}(x, \log_{|q|^2} N) = 0$ if $N$ is irrational. Thus, it is natural to expect that $\overline{\overline{\Phi}}(x,t) = 0$ for all $t$ which is in fact true. The next lemma provides this fact and also the continuity of $\overline{\Phi}(x,t)$, thus completing the proof of Theorem 2.

Lemma 8. *The function $\overline{\Phi}(x,t)$ is Hölder continuous in $t$ and analytic for $x$ in a complex neighbourhood of $x = 1$. Furthermore, $\overline{\overline{\Phi}}(x,t) = 0$ for all $t$.*

Remark 7. In particular this shows that $\Phi(x,t)$ from Theorem 1 equals $\overline{\Phi}(x,t)$ for real $x$.

*Proof of Lemma 8.* First assume that $x$ is real. By considering $N = |q|^{2(n+t)}$ for $n = 0, 1, 2, \ldots$ it follows from Theorem 1 and (4.23) that $\Phi(x,t) = \overline{\Phi}(x,t) + \overline{\overline{\Phi}}(x,t)$. Furthermore, we have $\overline{\overline{\Phi}}(x,t) = 0$ if $t$ is not of the form $t = \log_{|q|^2} m - k$ for some positive integers $m$ and $k$. (This occurs, for example, if $t = \log_{|q|^2} T$ for some irrational number $T$.) Since the numbers $t$ with this property are dense in $[0,1)$ it follows that $\overline{\Phi}(x,t)$ is continuous in $t$ if and only if $\overline{\overline{\Phi}}(x,t) = 0$ for all $t$. This observation can also be deduced from the inequality (4.24) below which is also true for complex $x$. Hence, continuity of the mapping $t \mapsto \overline{\Phi}(x,t)$ follows from $\overline{\overline{\Phi}}(x,t) = 0$ even if $x$ is a complex number.

We now suppose that $t \in [0,1)$ is of the form $t = \log_{|q|^2} m - k$ for some positive integers $m$ and $k$ where we assume that $k$ is chosen to be minimal. If $s \neq t$ is also of that form, that is, $s = \log_{|q|^2} n - j \in [0,1)$ for positive integers $n$ and $j$, then (for a properly chosen constant $c > 0$) we have

$$|s - t| \geq c|\,|q|^{2s} - |q|^{2t}| \geq \frac{1}{|q|^{2(k+j)}}.$$

In particular,

$$|q|^{2j} \geq c\,\frac{1}{|q|^{2k}|s-t|}.$$

Observe that only terms of the form $\lambda(x)^{-j}$ contribute to $\overline{\overline{\Phi}}(x,s)$; notice that for these values of $t$ we have $n = |qz + l|^2$ for some $z \in \mathbb{Z}[i]$ and $l \in \{0, \ldots, a^2\}$. Thus, if we fix some $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|\overline{\overline{\Phi}}(x,s)| < \varepsilon \quad \text{for all } s \text{ with } |s-t| < \delta.$$

Next observe that if $t = \log_{|q|^2} m - k$ then for $0 < \theta < 1$,

$$\frac{\lambda(x)^{\lfloor (t+\theta)-\log_{|q|^2} l^2\rfloor} - \lambda(x)^{\lfloor (t-\theta)-\log_{|q|^2} l^2\rfloor}}{1-1/\lambda(x)} = \frac{\lambda^{-k}-\lambda^{-k-1}}{1-1/\lambda(x)}$$
$$= \lambda^{-k} = \lambda(x)^{\lfloor t-\log_{|q|^2} l^2\rfloor}.$$

Thus, by a similar reasoning we also get

$$\left|\overline{\Phi}(x,t+\theta) - \overline{\Phi}(x,t-\theta) + \frac{1}{2}\overline{\overline{\Phi}}(x,t)\right| < \varepsilon \tag{4.24}$$

if $0 < \theta < \delta$. Furthermore, by continuity of $\Phi(x,t) = \overline{\Phi}(x,t) + \overline{\overline{\Phi}}(x,t)$,

$$|\Phi(x,t+\theta) - \Phi(x,t-\theta)|$$
$$= |\overline{\Phi}(x,t+\theta) + \overline{\overline{\Phi}}(x,t+\theta) - \overline{\Phi}(x,t-\theta) + \overline{\overline{\Phi}}(x,t-\theta)| < \varepsilon$$

in that range. Consequently,

$$|\overline{\overline{\Phi}}(x,t)| \le 2|\overline{\Phi}(x,t+\theta) - \overline{\Phi}(x,t-\theta)| + 2\varepsilon$$
$$\le 2|\Phi(x,t+\theta) - \Phi(x,t-\theta)| + 6\varepsilon < 7\varepsilon.$$

Since $\varepsilon > 0$ can be chosen arbitrarily small it follows that $\overline{\overline{\Phi}}(x,t) = 0$.

Thus, we have shown that $\overline{\overline{\Phi}}(x,t) = 0$ for all $t$ if $x$ is a real number close to 1. Since $\overline{\overline{\Phi}}(x,t)$ is an analytic function in $x$ we also obtain $\overline{\overline{\Phi}}(x,t) = 0$ for complex $x$ close to 1. As mentioned above, this implies that $\overline{\Phi}(x,t)$ is continuous in $t$ even if $x$ is a complex number close to 1.

Similarly we show that $\Phi(x,t)$ is Hölder continuous in $t$. Here we just have to use a quantified version of (4.24). We leave the details to the reader. ∎

**5. A method based on ergodic $\mathbb{Z}[i]$-actions and skew products.** In this section we will consider block additive functions $s_F$ taking values in an abelian group $A$, hence $F : \mathcal{A}^{L+1} \to A$. The neutral element will be denoted by $0_A$. We assume that $A$ is compact metrisable, equipped with its Haar measure $\lambda_A$, and we introduce the metrisable compact space $\Omega := A^{\mathbb{Z}[i]}$. The shift $\mathbb{Z}[i]$-action $\Sigma : \zeta \mapsto \Sigma_\zeta$ on $\Omega$ is defined by setting, for all $\omega : z \mapsto \omega_z$ and all $\zeta \in \mathbb{Z}[i]$,

$$(\Sigma_\zeta(\omega))_z := \omega_{\zeta+z}\,.$$

For any $\omega \in \Omega$, consider its orbit closure $K_\omega$ which is the topological closure of its orbit

$$\mathcal{O}_\omega := \{\Sigma_\zeta(\omega) : \zeta \in \mathbb{Z}[i]\}$$

under the shift. Clearly $K_\omega$ is a compact subspace of $\Omega$ and $\Sigma_\zeta(K_\omega) = K_\omega$ for all $\zeta \in \mathbb{Z}[i]$. The restriction of $\Sigma_\zeta$ on $K_\omega$, still denoted by $\Sigma_\zeta$, is a homeomorphism of $K_\omega$, defining the shift $\mathbb{Z}[i]$-action $\Sigma : \zeta \mapsto \Sigma_\zeta$ on $K_\omega$. By definition, the couple $\mathcal{K}_\omega := (\Sigma, K_\omega)$ is the flow associated to $\omega$.

The function $s_F$ can be viewed as an element of the compact space $\Omega := A^{\mathbb{Z}[i]}$. For short we write $K(F)$ (resp. $\mathcal{K}(F)$) for $K_{s_F}$ (resp. $\mathcal{K}_{s_F}$) and we set $I(F) := \{s_F(z) : z \in \mathbb{Z}[i]\}$.

Lemma 9. *Assume that $A$ is a compact metrisable group. Then the closure $A(F)$ of the set $I(F)$ is a subgroup of $A$.*

*Proof.* It is clear that the neutral element $0_A$ of $A$ belongs to $I(F)$ so that, due to compactness, it is enough to prove that $a + a' \in A(F)$ for any $a$ and $a'$ in $A(F)$. Let $U$ be any neighbourhood of $0_A$ and let $V$ be another neighbourhood of $0_A$ such that $V + V \subset U$. By assumption there exist Gaussian integers $z$ and $z'$ such that $s_F(z) - a \in V$ and $s_F(z') - a' \in V$. Setting $z'' = z + q^{\mathrm{length}_q(z)+L+1} z'$ one gets $s_F(z'') - (a + a') = s_F(z) - a + s_F(z') - a' \in V + V$. Hence $s_F(z'') - (a + a') \in U$, proving that $a + a' \in A[F]$. ∎

In the next theorem we make use of the following simple result:

Lemma 10. *For any neighbourhood $V$ of $0_A$ in $A$ there exists a finite set $B = B(V)$ of $\mathbb{Z}[i]$ such that for all $r \in \mathbb{Z}[i]$ there exists $b \in B$ such that $s_F(r + b) \in V$.*

*Proof.* We may assume that $V = -V$ otherwise replace $V$ by $V \cap (-V)$. Since $I(F)$ is dense in $A(F)$ and $A(F)$ is compact there exists an integer $N = N(V)$ such that

$$A(F) \subseteq \bigcup_{z,\, \mathrm{length}_q(z) \le N} s_F(z) + V.$$

Given any Gaussian integer $r$, we use the $q$-adic expansion of $r$ to write the decomposition $r = r' + q^{N+L+1} t$ with $\mathrm{length}_q(r') \le N + L + 1$ and choose $r''$ with $\mathrm{length}_q(r'') \le N$ such that $-s_F(t) \in V + s_F(r'')$. With $b = -r' + r''$ we get

$$s_F(r + b) = s_F(r'' + q^{N+L+1} t) = s_F(r'') + s_F(t) \in V.$$

In addition, from Lemma 1,

$$\begin{aligned}\mathrm{length}(b) &\le c + \frac{\log(|r'| + |r''|)}{\log|q|} \le c + \frac{\log 2 + \log|q|(c + N + L + 1)}{\log|q|}\\ &\le c' + N + L + 1.\end{aligned}$$

The proof ends by taking $B := \{z \in \mathbb{Z}[i] : \mathrm{length}_q(z) \le c' + N + L + 1\}$. ∎

We are ready to prove the main result on the topological structure of $\mathcal{K}(F)$.

THEOREM 3. *The flow $\mathcal{K}(F)$ is minimal, that is, if $M$ is a non-empty compact subspace of $K(F)$ such that $\Sigma_\zeta(M) \subset M$ for all $\zeta \in \mathbb{Z}[i]$ then $M = K(F)$.*

*Proof.* Since $K(F)$ is the orbit closure of $s_F$, it is enough to prove that $s_F$ is uniformly recurrent (see [7, Section 4]). To this end we have to show that for any neighbourhood $W$ of $0_\Omega$, the neutral element of $\Omega$, the set $S(W) := \{u \in \mathbb{Z}[i] : \Sigma_u(s_F) - s_F \in W\}$ is *syndetic*, that is, there is a finite set $E$ such that $\mathbb{Z}[i] = S(W) + E$. We may restrict ourselves to fundamental neighbourhoods of the form

$$W(M,U) = \bigcap_{\mathrm{length}_q(z)\le M} \{\omega \in \Omega : \omega_z \in U\}$$

where $U$ is any neighbourhood of $0_A$. Choose a neighbourhood $V$ of $0_A$ such that $V + V \subset U$ and a finite subset $B = B(V)$ of $\mathbb{Z}[i]$ as in Lemma 10 and let $h = \max\{\mathrm{length}_q(b) : b \in B\}$. Fix any Gaussian integer $z$ and decompose it as $z = z' + q^{M+L+1} r$ with $\mathrm{length}_q(z') \le M+L+1$. By Lemma 1, $\mathrm{length}(-z') \le 2c+M+L+1$ and there exists $r' \in B$ such that $s_F(r+r') \in V$. Now set $\zeta = -z' + q^{M+L+1} r'$. By construction $z+\zeta = q^{M+L+1}(r+r')$, which implies $s_F(z+\zeta+t) - s_F(t) \in V$ for all Gaussian integers $t$ of length at most $M$. This means that $z+\zeta \in S(W)$ with

$$\mathrm{length}_q(\zeta) \le c + \frac{\log(|z'| + |r'|\,|q|^{M+L+1})}{\log|q|} \le c'' + M + L + 1 + h$$

where $c''$ is an absolute constant. Therefore $\zeta$ belongs to a finite subset of $\mathbb{Z}[i]$ and consequently $S(W)$ is syndetic. ∎

Now we introduce tools from ergodic theory to prove rather general distribution results on block-additive functions. We will use ideas discussed in more detail in [14] and refer to that paper for a detailed exposition of the method.

The general idea of the approach motivated by ergodic theory is to build a dynamical system $(X, T, \mu)$ from the underlying digital expansion. The space $X$ is then a suitably chosen compactification of $\mathbb{Z}[i]$, the action $T : \mathbb{Z}[i] \to \mathrm{Aut}(X)$ is simply addition by elements of $\mathbb{Z}[i]$. Since the compactification $X$ carries a natural group structure in our case, $\mu$ is chosen as the Haar measure on this group. Since no non-trivial block additive function can be extended to a continuous or even measurable function on $X$ (see Remark 9 below), we use a trick developed by T. Kamae [15], which overcomes this problem by constructing a suitable cocycle (we will introduce this notion below). The fact that the additive function has no extension to $X$ is then reflected by the non-triviality of the cocycle.

Consider the infinite product space

$$\mathcal{K}_q = \{0, 1, \dots, a^2\}^{\mathbb{N}_0}$$

and embed $\mathbb{Z}[i]$ by $q$-adic digital expansion

$$\iota : \mathbb{Z}[i] \to \mathcal{K}_q, \qquad z \mapsto (\varepsilon_0(z), \varepsilon_1(z), \dots, \varepsilon_L(z), 0, 0, \dots).$$

Then it was proved in [14] that addition in $\mathbb{Z}[i]$ can be extended continuously to $\mathcal{K}_q$. By this construction $\mathcal{K}_q$ inherits a group structure by

$$\mathcal{K}_q = \operatorname{proj}\lim_{n\to\infty} \mathbb{Z}[i]/q^n\mathbb{Z}[i].$$

The corresponding Haar measure $\mu$ is the infinite product measure of uniform distribution on the digits. The cylinder set of base $(x_0, \dots, x_n) \in \{0, \dots, a^2\}^{n+1}$ is given by

$$\begin{aligned}[x_0, \dots, x_n] &:= x_0 + x_1 q + \dots + x_n q^n + q^{n+1}\mathcal{K}_q \\ &= \{z \in \mathcal{K}_q : \varepsilon_0(z) = x_0, \dots, \varepsilon_n(z) = x_n\}.\end{aligned}$$

The Haar measure of such sets is given by $\mu([x_0, \dots, x_n]) = |q|^{-n-1}$. The Gaussian integers $\mathbb{Z}[i]$ act on $\mathcal{K}_q$ by addition

$$T : \mathbb{Z}[i] \to \operatorname{Aut}(\mathcal{K}_q), \qquad z \mapsto (x \mapsto x + z).$$

This continuous action is uniquely ergodic.

Definition 1. A sequence $(Q_n)_{n\in\mathbb{N}}$ of finite subsets of $\mathbb{Z}[i]$ is called a *Følner sequence* if it has the following properties:

(1) $Q_n \subset Q_{n+1}$ for all $n$;
(2) There exists a constant $K$ such that $\#(Q_n - Q_n) \le K\#Q_n$ for all $n$;
(3) $\displaystyle\lim_{n\to\infty} \frac{\#(Q_n \triangle (g + Q_n))}{\#Q_n} = 0$ for all $g \in \mathbb{Z}[i]$.

($\triangle$ denotes symmetric difference.)

Classical examples of such sequences are the sequence of balls of radius $\sqrt{n}$, $Q_n = \{z \in \mathbb{Z}[i] : |z|^2 < n\}$, or the squares $Q_n = \{z \in \mathbb{Z}[i] : |\Re(z)| < n, |\Im(z)| < n\}$. Another example more connected to digital expansions is the "discrete $q$-adic dragons" $Q_n = \{z \in \mathbb{Z}[i] : \operatorname{length}_q(z) \le n\}$.

We recall that a point $x \in X$ is called $(T, \mu)$-*generic* (or simply *generic*, if the underlying action is clear) if

$$\forall f \in C(X) : \quad \lim_{n\to\infty} \frac{1}{\#Q_n} \sum_{z\in Q_n} f \circ T_z(x) = \int_X f\, d\mu \tag{5.1}$$

for a Følner sequence $(Q_n)_{n\in\mathbb{N}}$. By Tempel'man's ergodic theorem (cf. [19, Chapter 6, Theorem 4.4]) $\mu$-almost all points are generic. Clearly, for a uniquely ergodic continuous action every point is generic, and even more: the convergence in (5.1) is uniform in $x$.

For uniquely ergodic non-continuous actions we need additional conditions, which will be developed below, to have the same conclusion. To this end we introduce the following definition.

DEFINITION 2. Let $X$ be a compact metrisable space and $T : \mathbb{Z}[i] \times X \to X$ a Borel measurable $\mathbb{Z}[i]$-action. A subset $A \subset X$ is called *uniformly $T$-negligible* if

$$\forall \varepsilon > 0\ \exists g \in C(X), g \geq \mathbb{1}_A : \quad \limsup_{n\to\infty} \left\| \frac{1}{\#Q_n} \sum_{z\in Q_n} g \circ T_z \right\|_\infty < \varepsilon$$

for a Følner sequence $(Q_n)_{n\in\mathbb{N}}$.

DEFINITION 3. Let $X$ be a compact metrisable space and $T : \mathbb{Z}[i] \times X \to X$ a Borel measurable $\mathbb{Z}[i]$-action. The action $T$ is called *uniformly quasi-continuous* if for every $z \in \mathbb{Z}[i]$ the set of discontinuity points of $T_z$ is uniformly $T$-negligible.

REMARK 8. If $T$ is uniformly quasi-continuous and $\mu$ is a $T$-invariant Borel probability measure on $X$, then $T$ is $\mu$-continuous.

The following theorem is an adapted version of [21, Annexe, Théorème]. The proof is slightly simplified by the fact that the action is invertible.

THEOREM 4. *Let $T$ be a uniformly quasi-continuous $\mathbb{Z}[i]$-action on the compact metric space $X$ and assume that $T$ is uniquely ergodic with invariant measure $\lambda$. Then for any $\lambda$-continuous function $f$ we have*

$$\lim_{n\to\infty} \frac{1}{\#Q_n} \sum_{z\in Q_n} f \circ T_z(x) = \int_X f\, d\lambda \tag{5.2}$$

*uniformly in $x$.*

*Proof.* Let $\mathcal{R}_\lambda$ denote the Banach space of real-valued $\lambda$-continuous functions on $X$ equipped with the uniform norm and let

$$E = \overline{\langle \{g - g\circ T_z : g \in \mathcal{R}_\lambda,\ z \in \mathbb{Z}[i]\}\rangle}.$$

Then $\lambda$ defines a linear form on $\mathcal{R}_\lambda$ with $\ker(\lambda) \subseteq E$. We will show that we have equality in fact.

Let $L : \mathcal{R}_\lambda \to \mathbb{R}$ be a continuous linear form with $E \subseteq \ker(L)$ and $L(1) = 1$. For $f \geq 0$ define

$$|L|(f) = \sup\{L(g) : g \in \mathcal{R}_\lambda,\ |g| \leq f\}.$$

Then $|L|$ can be extended to a continuous positive linear form on $\mathcal{R}_\lambda$. Thus $|L|$ determines a measure $\ell$ on $X$.

We will now prove that $|L|$ and therefore $\ell$ is $T$-invariant. By definition we have, for $f \geq 0$,

$$\begin{aligned}|L|(f\circ T_z) &= \sup\{L(g): g\in\mathcal{R}_\lambda,\ |g|\le f\circ T_z\}\\ &\ge \sup\{L(g\circ T_z): g\circ T_z\in\mathcal{R}_\lambda,\ |g\circ T_z|\le f\circ T_z\}\ge |L|(f),\end{aligned}$$

where we have used $L(g)=L(g\circ T_z)$ since $E\subseteq\ker(L)$. Applying the same inequality to $f\circ T_{-z}$ shows the $T$-invariance.

By unique ergodicity we have $\ell=\lambda$. On the other hand, $|L|-L$ is also a $T$-invariant positive linear form. Thus we have $|L|-L=a\lambda$ with $a\ge 0$. Hence $L=(1-a)\lambda$ and as $L(1)=1$ we get $a=0$, and we have $E=\ker(L)$ by the Hahn–Banach theorem.

Summing up, for every $f\in\mathcal{R}_\lambda$ and every $\varepsilon>0$ there exist $k\in\mathbb{N}$, $g_1,\dots,g_k\in\mathcal{R}_\lambda$, and $z_1,\dots,z_k\in\mathbb{Z}[i]$ such that

$$\Big\|f-\lambda(f)-\sum_{m=1}^{k}(g_m-g_m\circ T_{z_m})\Big\|_\infty<\varepsilon.$$

Applying the ergodic means to this inequality and using (3) Definition 1 finishes the proof. ∎

We recall the definition of a cocycle:

Definition 4. Let $(X,T,\mu)$ be a $\mathbb{Z}[i]$-action on $X$ and $A$ an abelian group. A $T$-*cocycle* (or simply a cocycle, if the underlying action $T$ is fixed) is a Borel map $a:\mathbb{Z}[i]\times X\to A$ such that

(i) $a(g+h,x)=a(g,T_hx)+a(h,x)$ $\mu$-a.e.,
(ii) $\mu(\bigcup_{g\in\mathbb{Z}[i]}(\{x:T_gx=x\}\cap\{x:a(g,x)\ne 0_A\}))=0$.

If we assume that $T$ is *aperiodic*, i.e. $\mu(\{x:\exists g\ne 0,\ T_gx=x\})=0$, then condition (ii) is always satisfied.

A cocycle $a$ is called a *coboundary* if there exists a Borel map $f:X\to A$ such that

$$\forall x\in X, g\in\mathbb{Z}[i]:\quad a(g,x)=f(T_gx)-f(x).$$

The skew product $(X\times A,T^a,\mu\otimes\lambda_A)$ corresponding to the cocycle $a$ is given by

$$T^a:\mathbb{Z}[i]\to\operatorname{Aut}(X\times A),\qquad z\mapsto((x,b)\mapsto(x+z,b+a(z,x))). \tag{5.3}$$

Definition 5. An element $\alpha\in A$ is said to be an *essential value* of the cocycle $a$ if for every neighbourhood $N(\alpha)$ of $\alpha$ in $A$ and for every $B\in\mathfrak{B}(X)$ (Borel sets) with $\mu(B)>0$,

$$\mu\Big(\bigcup_{g\in\mathbb{Z}[i]}(B\cap T_g^{-1}(B)\cap\{x:a(g,x)\in N(\alpha)\})\Big)>0. \tag{5.4}$$

Let

$$E(a)=\{\alpha\in A:\alpha\text{ is an essential value of }a\}.$$

This definition does not require ergodicity of $T$. We have the following proposition.

PROPOSITION 2 (cf. [23]). *Let $a : \mathbb{Z}[i] \times X \to A$ be a cocycle. Then the following properties hold*:

(1) *If $b : \mathbb{Z}[i] \times X \to A$ is a coboundary then $E(a+b) = E(a)$.*
(2) *$E(a)$ is a closed subgroup of $A$.*
(3) *$a$ is a coboundary $\Leftrightarrow E(a) = \{0_A\}$.*

Let $\mathcal{I}$ be the set of $T^a$-invariant elements in $\mathfrak{B} \otimes \mathfrak{B}_A$ and put

$$I(a) = \{\beta \in A : \mu \otimes h_A(\tau_\beta B \bigtriangleup B) = 0 \text{ for every } B \in \mathcal{I}\}$$

where $\tau_\beta : X \times A \to X \times A$ is given by

$$\tau_\beta(x, \alpha) = (x, \alpha + \beta).$$

The set of essential values is directly related to the ergodicity of the skew product action $T^a$ by the following theorem of K. Schmidt.

THEOREM 5 ([23, Theorem 5.2]). *Let $T$ be an ergodic action on $(X, \mathfrak{B}, \mu)$ which is assumed to be non-atomic. Then for any cocycle $a : G \times X \to A$,*

$$E(a) = I(a).$$

COROLLARY 6. *If $T$ is ergodic, then*

$$T^a \text{ is ergodic} \Leftrightarrow E(a) = A.$$

The cocycle suitable for our purposes is defined as

$$a_F(z, x) = \begin{cases} \lim\limits_{\substack{w \to x \\ w \in \mathbb{Z}[i]}} (s_F(w+z) - s_F(w)) & \text{if the limit exists,} \\ 0 & \text{otherwise.} \end{cases} \tag{5.5}$$

The limit exists if the carry propagation in the addition $x + z$ terminates after finitely many steps. It was proved in [14] that for almost all $x \in \mathcal{K}_q$ the addition $x + z$ produces only finitely many carries. Thus $a_F(z, x)$ is defined for $\mu$-almost all $x$. Furthermore, since $a_F(z, \cdot)$ is constant on cylinder sets defined by the different possible carries in the addition $x + z$ (cf. [14]), $a_F$ is also $\mu$-continuous. Moreover, the set of discontinuity points of $a_F(z, \cdot)$ is closed, hence it is also uniformly $T$-negligible by the unique ergodicity of the continuous action $T$. Thus we have proved

LEMMA 11. *The skew product action $T^{a_F}$ given by* (5.3) *is uniformly quasi-continuous.*

We naturally define

$$V(a_F) = \overline{\{a_F(z, x) : x \in \mathcal{K}_q,\ z \in \mathbb{Z}[i]\}},$$

the closed subgroup consisting of the values of $a_F$. Recalling the definition of the group $A(F) = \overline{\{s_F(z) : z \in \mathbb{Z}[i]\}}$, we readily have

PROPOSITION 3. *The groups generated by the values of $s_F$ and $a_F$ are equal*:

$$V(a_F) = A(F) = \overline{\{s_F(z) : z \in \mathbb{Z}[i]\}}\,.$$

PROPOSITION 4. *Let $s_F$ be a block additive function on $\mathbb{Z}[i]$ and $a_F$ be the corresponding cocycle on $\mathcal{K}_q$. Then the set of essential values of $a_F$ equals the closed subgroup $A(F)$ of $A$ generated by the values of $s_F$*:

$$E(a_F) = A(F).$$

*Proof.* We need the following lemma which is the analog of [3, Lemma 12] but in the case of cocycles for a $\mathbb{Z}[i]$-action.

LEMMA 12. *Let $\alpha \in A$ and assume that for any neighbourhood $V = V(\alpha)$ of $\alpha$ in $A$ there exists a constant $\kappa > 0$ such that for every non-empty cylinder set $C$ of $\mathcal{K}_q$ there exists $\zeta \in \mathbb{Z}[i]$ such that*

$$\mu(C \cap T_\zeta(C) \cap \{x \in \mathcal{K}_q : a_F(\zeta, x) \in V\}) \geq \kappa\mu(C).$$

*Then $\alpha \in E(a_F)$.*

*Proof of Lemma 12.* Set for short $W(V,\zeta) := \{x \in \mathcal{K}_q : a_F(\zeta,x) \in V\}$. If $B$ is a Borel subset of $\mathcal{K}_q$, then due to the regularity of the Haar measure, for any $\varepsilon > 0$ (and $\varepsilon < 1$), there exists a non-empty cylinder set $C$ such that $\mu(B \cap C) \geq (1-\varepsilon)\mu(C)$, hence $\mu(C \setminus (B \cap C)) \leq \varepsilon\mu(C)$, leading to

$$\begin{aligned}\mu(B \cap T_\zeta(B) \cap W(V,\zeta)) &\geq \mu((B \cap C) \cap T_\zeta(B \cap C) \cap W(V,\zeta)) \\ &\geq \mu(C \cap T_\zeta(C) \cap W(V,\zeta)) - 2\varepsilon\mu(C).\end{aligned}$$

Choose $\zeta$ such that $\mu(C \cap T_\zeta(C) \cap W(V,\zeta)) \geq \kappa\mu(C)$ and $\varepsilon < \kappa/2$. Then we get $\mu(B \cap T_\zeta(B) \cap W(V,\zeta)) > 0$. Hence $\zeta \in E(a_F)$ as expected. ∎

Going back to the proof of Proposition 4, it is enough to prove that $a_F(y, z_0) \in E(a_F)$ for all $y, z_0 \in \mathbb{Z}[i]$, where $y = (y_0, y_1, \ldots, y_t)_q$. Let $C$ be any non-empty cylinder set, say

$$C = [\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k].$$

Set $\zeta = q^{k+L+3}z_0$ and consider

$$C_0 = [\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k, \underbrace{0, \ldots, 0}_{L+2}, y_0, y_1, \ldots, y_t, \underbrace{0, \ldots, 0}_{M}]$$

with $M = 4 + \max(0, \mathrm{length}_q(z_0) - t)$. One has $\mu(C_0) = \kappa\mu(C)$ with $\kappa = 1/|q|^{M+t+L+2}$. The $M$ digits 0 at the end ensure that there is no carry propagation beyond the $k+L+t+M+4$ fixed digits. This means that for any $x \in C_0$,

$$a_F(\zeta, x) = a_F(z_0, y) \quad\text{and}\quad C_0 \subset C \cap T_\zeta^{-1}(C).$$

This implies that for any neighbourhood $V$ of $a_F(z_0,y)$,

$$\mu(C\cap T_\zeta^{-1}(C)\cap W(V,\zeta))\ge \kappa\mu(C),$$

and Lemma 12 gives $a_F(z_0,y)\in E(a_F)$. ∎

Remark 9. By considering both Proposition 3 and Proposition 2(3) one sees that if $s_F$ can be extended to a measurable map on $\mathcal{K}_q$, then the cocycle $a_F$ is a coboundary, hence $s_F$ is trivial, i.e., $s_F(z)=0_A$ for all $z\in\mathbb{Z}[i]$.

Putting together Proposition 4, Corollary 6, and Lemma 11 we obtain

Proposition 5. *Let $s_F$ be a block additive function taking its values in the compact abelian metrisable group $A$, let $a_F$ be the corresponding cocycle defined by* (5.5), *and assume that $A(F)=A$. Then the skew product $T^{a_F}$ is uniquely ergodic and more precisely, for all $\mu\otimes\lambda_A$-continuous maps $f: X\times A\to\mathbb{C}$,*

$$\lim_{n\to\infty}\frac{1}{\#Q_n}\sum_{z\in Q_n} f\circ T_z^{a_F}(x,g)=\int_{X\times A} f\,d(\mu\otimes\lambda_A)$$

*uniformly in $(x,g)\in\mathcal{K}_q$.*

Corollary 7. *Let $s_F$ be a real-valued block additive function which attains an irrational value. Then $(s_F(z))_{z\in\mathbb{Z}[i]}$ is well uniformly distributed modulo* 1 *with respect to any Følner sequence $(Q_n)_{n\in\mathbb{N}}$, i.e.*

$$\lim_{n\to\infty}\frac{1}{\#Q_n}\#\{z\in Q_n:\{s_F(z+y)\}\in I\}=\lambda(I)$$

*for every interval $I\subset[0,1]$* ($\{\cdot\}$ denotes the fractional part), *uniformly in $y\in\mathbb{Z}[i]$.*

*Proof.* The assumption that $s_F$ attains an irrational value clearly implies that $V(a_F \pmod 1)=\mathbb{R}/\mathbb{Z}$. By Weyl's criterion (cf. [20]) the assertion is equivalent to

$$\forall k\in\mathbb{Z}\setminus\{0\}:\quad \lim_{n\to\infty}\frac{1}{\#Q_n}\sum_{z\in Q_n} e(ks_F(z+y))=0$$

uniformly in $y\in\mathbb{Z}[i]$. The points $(y,0)$ are uniformly generic for $T^{a_F}$ by Proposition 5. Now, by definition of $T^{a_F}$ we have

$$\begin{aligned}T_z^{a_F}(y,0)&=(y+z,a_F(z,y))\\&=(y+z,s_F(y+z)-s_F(y)) \pmod 1.\end{aligned}$$

Genericity of $(y,0)$ implies

$$\lim_{n\to\infty}\frac{1}{\#Q_n}\Big|\sum_{z\in Q_n}\chi_0\otimes e_k(T_z^{a_F}(y,0))\Big|=\lim_{n\to\infty}\frac{1}{\#Q_n}\Big|\sum_{z\in Q_n} e(ks_F(y+z))\Big|=0,$$

where $\chi_0$ denotes the trivial character of $\mathcal{K}_q$ and $e_k(\cdot) = e(k\cdot)$. The convergence is uniform in $y \in \mathbb{Z}[i]$. ∎

COROLLARY 8. *Let $s_F$ be an integer-valued block additive function. Then for any integer $M \geq 2$ for which there exists a value $s_F(z)$ that is coprime to $M$ the sequence $(s_F(z))_{z\in\mathbb{Z}[i]}$ is well uniformly distributed in residue classes modulo $M$ with respect to any Følner sequence $(Q_n)_{n\in\mathbb{N}}$, i.e.*

$$\lim_{n\to\infty} \frac{1}{\#Q_n} \#\{z \in Q_n : s_F(z+y) \equiv m \pmod{M}\} = \frac{1}{M}$$

*for $m \in \{0, 1, \ldots, M-1\}$, uniformly in $y \in \mathbb{Z}[i]$.*

*Proof.* After observing that $V(a_F \pmod{M}) = \mathbb{Z}/M\mathbb{Z}$, the proof runs along the same lines as the proof of Corollary 7. ∎

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