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On differential independence of the Riemann zeta function and the Euler gamma function

by

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It is well-known ([2] and [4]) that Γ, as well as ζ, is not a solution of any algebraic differential equation with coefficients in $\mathbb{C}$. In other words, if $P(u_0, u_1, \ldots, u_m)$ is any polynomial in $u_0, u_1, \ldots, u_m$ over $\mathbb{C}$, and

$$P(\Gamma, \Gamma', \ldots, \Gamma^{(m)})(z) \equiv 0 \quad \text{or} \quad P(\zeta, \zeta', \ldots, \zeta^{(m)})(z) \equiv 0,$$

for all $z \in \mathbb{C}$, then the polynomial P is identically zero. This answers Hilbert's conjecture [1] in his 18th problem. For a detailed discussion of this subject and other related topics, we refer the reader to [5].

Let $h(z) = \zeta(\sin(2\pi z))$. Recently, Markus [3] proved that if

$$P(h, h', \ldots, h^{(m)}; \Gamma, \Gamma', \ldots, \Gamma^{(n)})(z) \equiv 0 \quad \text{for } z \in \mathbb{C},$$

then the polynomial $P(u_0, u_1, \ldots, u_m; v_0, v_1, \ldots, v_n)$ is identically zero. Thus, in the terminology of differential algebraic theory, Γ and h are differentially independent over $\mathbb{C}$ (hence, over $\mathbb{C}(z)$). Furthermore, Markus [3] conjectured that if

$$P(\zeta, \zeta', \ldots, \zeta^{(m)}; \Gamma, \Gamma', \ldots, \Gamma^{(n)})(z) \equiv 0 \quad \text{for } z \in \mathbb{C},$$

then the polynomial $P(u_0, u_1, \ldots, u_m; v_0, v_1, \ldots, v_n)$ is identically zero, i.e. Γ and ζ are differentially independent. In this short note, we prove that ζ and Γ cannot satisfy a class of algebraic differential equations.

Let $P(u_0, u_1, \ldots, u_m; v_0, v_1, \ldots, v_n)$ be any polynomial with coefficients in $\mathbb{C}$. For a non-negative integer μ, we let

$$\Lambda = \Lambda(\mu) = \{(\lambda_0, \lambda_1, \ldots, \lambda_\mu) : \lambda_j \text{ is a non-negative integer and } 0 \le j \le \mu < \infty\}$$

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be a finite index set. Define

$$|\lambda| = \sum_{j=0}^{\mu} \lambda_j \quad \text{and} \quad \Lambda_k = \{\lambda \in \Lambda : |\lambda| = k\};$$
$$|\lambda|_* = \sum_{j=0}^{\mu} j\lambda_j \quad \text{and} \quad \Lambda_k^* = \{\lambda \in \Lambda : |\lambda|_* = k\}.$$

Thus, there is a non-negative integer N such that

$$P(u_0, u_1, \ldots, u_m; v_0, v_1, \ldots, v_n) = \sum_{j=0}^{N} \sum_{\lambda \in \Lambda_j} a_\lambda(u_0, \ldots, u_m) v_0^{\lambda_0} v_1^{\lambda_1} \cdots v_n^{\lambda_n},$$

where $a_\lambda(u_0, u_1, \ldots, u_m)$ is a polynomial in $u_0, u_1, \ldots, u_m$ with coefficients in $\mathbb{C}$. Set, for $j = 0, 1, \ldots, N$,

$$P_j(u_0, u_1, \ldots, u_m; v_0, v_1, \ldots, v_n) = \sum_{\lambda \in \Lambda_j} a_\lambda(u_0, \ldots, u_m) v_0^{\lambda_0} v_1^{\lambda_1} \cdots v_n^{\lambda_n}.$$

For simplicity, we write $P(u_0, u_1, \ldots, u_m; v_0, v_1, \ldots, v_n)$ as $P(u; v)$ if it does not cause confusion, and similarly for $P_j(u; v)$. Further, we write

$$P_j(u; v) = \sum_{p=0}^{M_j} \sum_{\lambda \in \Lambda_j \cap \Lambda_p^*} a_\lambda(u_0, \ldots, u_m) v_0^{\lambda_0} v_1^{\lambda_1} \cdots v_n^{\lambda_n}, \tag{1}$$

where M_j is a non-negative integer and, in the same manner,

$$P_{j,p}(u; v) = \sum_{\lambda \in \Lambda_j \cap \Lambda_p^*} a_\lambda(u_0, \ldots, u_m) v_0^{\lambda_0} v_1^{\lambda_1} \cdots v_n^{\lambda_n}. \tag{2}$$

Consequently, we obtain

$$\begin{aligned} P_j(u; v) &= \sum_{p=0}^{M_j} P_{j,p}(u; v), \\ P(u; v) &= \sum_{j=0}^{N} P_j(u; v) = \sum_{j=0}^{N} \sum_{p=0}^{M_j} P_{j,p}(u; v). \end{aligned} \tag{3}$$

Theorem. *Let $P(u; v)$ be a non-trivial polynomial defined as in* (3). *If*

$$\sum_{\lambda \in \Lambda_j \cap \Lambda_p^*} a_\lambda(u_0, \ldots, u_m) \not\equiv 0 \quad \textit{whenever} \quad P_{j,p}(u; v) \not\equiv 0,$$

for all possible j's and p's, then

$$P(\zeta, \zeta', \ldots, \zeta^{(m)}; \Gamma, \Gamma', \ldots, \Gamma^{(n)})(z) \not\equiv 0.$$

To prove our theorem, we need the following celebrated theorem.

LEMMA (Voronin, [7]). *For any* $1/2 < \sigma < 1$, *the curve* $(\zeta(\sigma+it), \zeta'(\sigma+it), \ldots, \zeta^{(m)}(\sigma+it))$, $-\infty < t < \infty$, *is everywhere dense in* $\mathbb{C}^{m+1}$.

Proof of Theorem. If $N=0$, then $P(u;v)$ is a polynomial in $u_0, u_1, \ldots, u_m$ only; denote it by $P_0(u)$. If $P_0(u) = P_{0,0}(u) \neq 0$ at a point $w_* = (w_0, w_1, \ldots, w_m) \in \mathbb{C}^{m+1}$, then there is a neighborhood U of w_* such that $P_0(u) \neq 0$ for $u \in U \subset \mathbb{C}^{m+1}$. By Voronin's lemma, there is a sequence of positive real numbers t_q such that

$$(\zeta, \zeta', \ldots, \zeta^{(m)})(3/4 + it_q) \in U$$

for all large q. Therefore,

$$P_0(\zeta, \zeta', \ldots, \zeta^{(m)})(3/4 + it_q) \neq 0$$

for all large q and the theorem is proved in this case.

We now assume that N in (3) is greater than zero. Since $P(u;v)$ is non-trivial, we can assume that $P_{j_0}(u;v)$ is the first one that is not identically zero in the sequence

$$P_0(u;v), P_1(u;v), \ldots, P_N(u;v),$$

and that $P_{j_0,p_0}(u;v)$ is the first one that is not identically zero in the sequence

$$P_{j_0,M_{j_0}}(u;v), P_{j_0,M_{j_0}-1}(u;v), \ldots, P_{j_0,0}(u;v).$$

Without loss of generality, we may assume that $c = j_0$ and $d = M_{j_0}$ for simplicity. It is clear from (3) that

$$P(\zeta, \zeta', \ldots, \zeta^{(m)}; \Gamma, \Gamma', \ldots, \Gamma^{(n)})(z) \not\equiv 0$$

if and only if

$$\begin{aligned}(4)\quad & \frac{P(\zeta, \zeta', \ldots, \zeta^{(m)}; \Gamma, \Gamma', \ldots, \Gamma^{(n)})(z)}{\Gamma^N(z)} \\ &= \sum_{j=0}^{N} \frac{1}{\Gamma^{N-j}(z)} \sum_{p=0}^{M_j} P_{j,p}(\zeta, \zeta', \ldots, \zeta^{(m)}; 1, \Gamma'/\Gamma, \ldots, \Gamma^{(n)}/\Gamma)(z) \\ &= \sum_{j=0}^{N} \frac{1}{\Gamma^{N-j}(z)} P_j(\zeta, \zeta', \ldots, \zeta^{(m)}; 1, \Gamma'/\Gamma, \ldots, \Gamma^{(n)}/\Gamma)(z) \not\equiv 0.\end{aligned}$$

Now we estimate the term $P_{c,d}(\zeta, \zeta', \ldots, \zeta^{(m)}; 1, \Gamma'/\Gamma, \ldots, \Gamma^{(n)}/\Gamma)(z)$. It is known (e.g. [6, p. 151]) that

$$\log \Gamma(z) = (z - 1/2)\log z - z + \frac{1}{2}\log(2\pi) + \int_0^\infty \frac{[u] - u + 1/2}{u + z}\, du.$$

It follows that there is a $\delta > 0$ such that

$$\Gamma'(z) = (1 + o(1))\Gamma(z)\log z$$

uniformly for all $z \in \mathbb{C} \setminus \{z : |\arg z - \pi| \leq \delta\}$, where $o(1)$ stands for a quantity that goes to zero as $|z| \to \infty$. Similarly, for any positive integer q, there exists a $\delta > 0$ such that

$$\Gamma^{(q)}(z) = (1 + o(1))(\log z)^q \Gamma(z)$$

uniformly for all $z \in \mathbb{C} \setminus \{z : |\arg z - \pi| \le \delta\}$. Thus, from (2) we find that

$$\begin{aligned}(5)\qquad & P_{c,d}(\zeta, \zeta', \ldots, \zeta^{(m)}; 1, \Gamma'/\Gamma, \ldots, \Gamma^{(n)}/\Gamma)(z) \\ &= (\log z)^d \sum_{\lambda \in \Lambda_c \cap \Lambda_d^*} a_\lambda(\zeta, \zeta', \ldots, \zeta^{(m)})(z)(1 + o(1))^{\lambda_1} \cdots (1 + o(1))^{\lambda_n}\end{aligned}$$

uniformly for all $z \in \mathbb{C} \setminus \{z : |\arg z - \pi| \le \delta\}$.

Since $\sum_{\lambda \in \Lambda_c \cap \Lambda_d^*} a_\lambda(u_0, \ldots, u_m)$ is not identically zero, there are a $\delta_0 > 0$ and a bounded neighborhood U of $w_* = (w_0, w_1, \ldots, w_m)$ such that

$$\Big| \sum_{\lambda \in \Lambda_c \cap \Lambda_d^*} a_\lambda(u_0, \ldots, u_m) \Big| \ge \delta_0 \quad \text{for } u = (u_0, u_1, \ldots, u_m) \in U \subset \mathbb{C}^{m+1}.$$

By Voronin's lemma, there is a sequance $\{t_q\}_{q=1}^\infty$ of positive real numbers converging to ∞ such that

$$(\zeta(3/4 + it_q), \zeta'(3/4 + it_q), \ldots, \zeta^{(m)}(3/4 + it_q)) \in U$$

for all $q = 1, 2, \ldots$. It follows from (5) that there are an $\varepsilon_0 > 0$ and q_0 such that, for $z_q = 3/4 + it_q$,

$$|P_{c,d}(\zeta, \zeta', \ldots, \zeta^{(m)}; 1, \Gamma'/\Gamma, \ldots, \Gamma^{(n)}/\Gamma)(z_q)| \ge \varepsilon_0 |\log z_q|^d$$

for all large $q \ge q_0$. If $d = 0$, then (1) gives

$$\begin{aligned}&|P_c(\zeta, \zeta', \ldots, \zeta^{(m)}; 1, \Gamma'/\Gamma, \ldots, \Gamma^{(n)}/\Gamma)(z_q)| \\ &\qquad = |P_{c,0}(\zeta, \zeta', \ldots, \zeta^{(m)}; 1, \Gamma'/\Gamma, \ldots, \Gamma^{(n)}/\Gamma)(z_q)| \ge \varepsilon_0\end{aligned}$$

for all large q. If $d \ge 1$, then noting that $\zeta^{(p)}(z_q)$ is bounded for any p and all large q, we see from (1) that

$$\begin{aligned}&|P_c(\zeta, \zeta', \ldots, \zeta^{(m)}; 1, \Gamma'/\Gamma, \ldots, \Gamma^{(n)}/\Gamma)(z_q)| \\ &\qquad \ge \varepsilon_0 |\log z_q|^d - C|\log z_q|^{d-1} \to \infty \quad \text{as } q \to \infty,\end{aligned}$$

where C is an absolute positive constant. Therefore, for any $d \ge 0$,

$$(6)\qquad |P_c(\zeta, \zeta', \ldots, \zeta^{(m)}; 1, \Gamma'/\Gamma, \ldots, \Gamma^{(n)}/\Gamma)(z_q)| \ge \varepsilon_0,$$

and, for any $d < p \le N$,

$$(7)\qquad |P_p(\zeta, \zeta', \ldots, \zeta^{(m)}; 1, \Gamma'/\Gamma, \ldots, \Gamma^{(n)}/\Gamma)(z_q)| \le C|\log z_q|^{M_p}$$

for all large q. It is also known that (e.g. [6, p. 151]),

$$|\Gamma(3/4 + iy)| \sim e^{-\pi|y|/2} |y|^{1/4} \sqrt{2\pi}$$

as $y \to \infty$. If $c < N$, we deduce from (4), (6) and (7) that

$$
\begin{aligned}
&\left|\frac{P(\zeta,\zeta',\ldots,\zeta^{(m)};\Gamma,\Gamma',\ldots,\Gamma^{(n)})(z_q)}{\Gamma^N(z_q)}\right| \\
&\qquad = \left|\sum_{j=c}^{N}\frac{1}{\Gamma^{N-j}(z_q)}\,P_j(\zeta,\zeta',\ldots,\zeta^{(m)};1,\Gamma'/\Gamma,\ldots,\Gamma^{(n)}/\Gamma)(z_q)\right| \\
&\qquad \ge \varepsilon_0\left(\frac{e^{\pi t_q/2}}{t_q^{1/4}\sqrt{2\pi}}\right)^{N-c} - C|\log z_q|^{M_N}\left(\frac{e^{\pi t_q/2}}{t_q^{1/4}}\right)^{N-c-1} \to \infty
\end{aligned}
$$

as $q\to\infty$, which completes the proof of the theorem in this case. If $c=N$, then

$$
\begin{aligned}
&P(\zeta,\zeta',\ldots,\zeta^{(m)};\Gamma,\Gamma',\ldots,\Gamma^{(n)})(z_q) \\
&\qquad = \Gamma^N(z_q)P_N(\zeta,\zeta',\ldots,\zeta^{(m)};1,\Gamma'/\Gamma,\ldots,\Gamma^{(n)}/\Gamma)(z_q) \ne 0
\end{aligned}
$$

for all large q, where we choose z_q as in (6). Thus, the theorem is proved in this case. Therefore, we have completely proved the theorem.

Corollary. *If a non-trivial polynomial has the form* $P(u_0,u_1,\ldots,u_m;v_0,v_1)$, *then*

$$P(\zeta,\zeta',\ldots,\zeta^{(m)};\Gamma,\Gamma')(z)\not\equiv 0.$$

Proof. For all possible j's and p's, the set $\Lambda_j\cap\Lambda_p^*$ only contains one element. So, the assumption in the theorem is satisfied. The corollary is proved.

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