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Seven octonary quadratic forms

by

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1. Introduction. Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$ and $\mathbb{C}$ denote the sets of positive integers, nonnegative integers, integers and complex numbers respectively. For $k, a_1, \ldots, a_k \in \mathbb{N}$ and $n \in \mathbb{N}_0$, we define

$$N(a_1, \ldots, a_k; n) := \operatorname{card}\{(x_1, \ldots, x_k) \in \mathbb{Z}^k \mid n = a_1 x_1^2 + \cdots + a_k x_k^2\}. \tag{1.1}$$

As $N(a_1, \ldots, a_k; n)$ remains invariant under a permutation of $a_1, \ldots, a_k$, we may suppose that

$$a_1 \le \cdots \le a_k. \tag{1.2}$$

Clearly,

$$N(a_1, \ldots, a_k; 0) = 1. \tag{1.3}$$

If l of $a_1, \ldots, a_k$ are equal, say

$$a_i = a_{i+1} = \cdots = a_{i+l-1} = a,$$

we indicate this in $N(a_1, \ldots, a_k; n)$ by writing a^l for $a_i, a_{i+1}, \ldots, a_{i+l-1}$. For $k \in \mathbb{N}$ the sum of divisors function $\sigma_k(n)$ is defined by

$$\sigma_k(n) := \begin{cases} \displaystyle\sum_{\substack{d \in \mathbb{N} \\ d \mid n}} d^k & \text{if } n \in \mathbb{N}, \\ 0 & \text{if } n \notin \mathbb{N}. \end{cases}$$

We write $\sigma(n)$ for $\sigma_1(n)$.

The authors and M. F. Lemire have recently proved formulae for $N(1^i, 4^{4-i}; n)$ for $i \in \{1, 2, 3, 4\}$ and all $n \in \mathbb{N}$ in terms of $\sigma(n), \sigma(n/2)$, $\sigma(n/4), \sigma(n/8)$ and $\sigma(n/16)$ [1, Theorems 1.6, 1.7, 1.11 and 1.18]. The origins of these formulae are given in [1, pp. 284–286].

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PROPOSITION 1.1. *Let $n \in \mathbb{N}$. Then*

(i) $$N(1^4; n) = 8\sigma(n) - 32\sigma(n/4),$$

(ii) $$N(1^3, 4; n) = \left(4 + 2\left(\frac{-4}{n}\right)\right)\sigma(n) - 20\sigma(n/4) + 24\sigma(n/8) - 32\sigma(n/16),$$

(iii) $$N(1^2, 4^2; n) = \left(2 + 2\left(\frac{-4}{n}\right)\right)\sigma(n) - 2\sigma(n/2) + 8\sigma(n/8) - 32\sigma(n/16),$$

(iv) $$N(1, 4^3; n) = \left(1 + \left(\frac{-4}{n}\right)\right)\sigma(n) - 3\sigma(n/2) + 10\sigma(n/4) - 32\sigma(n/16),$$

where $\left(\frac{-4}{n}\right)$ is the Legendre–Jacobi–Kronecker symbol for discriminant -4, that is,

$$\left(\frac{-4}{n}\right) = \begin{cases} +1 & \text{if } n \equiv 1 \pmod 4, \\ -1 & \text{if } n \equiv 3 \pmod 4, \\ 0 & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

Proof. See [1, pp. 296, 297, 298, 303]. ∎

DEFINITION 1.1. For $k, n \in \mathbb{N}$ we define

$$W_k(n) := \sum_{\substack{m \in \mathbb{N} \\ m < n/k}} \sigma(m)\sigma(n - km).$$

In recent years the convolution sums $W_k(n)$ have been evaluated explicitly for certain values of k and all $n \in \mathbb{N}$. We require the evaluations for $k = 1$ [6, eq. (3.10), p. 236], $k = 2$ [6, Theorem 2, p. 247], $k = 4$ [6, Theorem 4, p. 249], $k = 8$ [7, Theorem 1, p. 388], and $k = 16$ [2, Theorem 1.1, p. 4].

PROPOSITION 1.2. *Let $n \in \mathbb{N}$. Then*

(i) $$W_1(n) = \frac{5}{12}\sigma_3(n) + \left(\frac{1}{12} - \frac{n}{2}\right)\sigma(n).$$

(ii) $$W_2(n) = \frac{1}{12}\sigma_3(n) + \frac{1}{3}\sigma_3(n/2) + \left(\frac{1}{24} - \frac{n}{8}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma(n/2).$$

(iii) $$W_4(n) = \frac{1}{48}\sigma_3(n) + \frac{1}{16}\sigma_3(n/2) + \frac{1}{3}\sigma_3(n/4) + \left(\frac{1}{24} - \frac{n}{16}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma(n/4).$$

(iv) $$W_8(n) = \frac{1}{192}\sigma_3(n) + \frac{1}{64}\sigma_3(n/2) + \frac{1}{16}\sigma_3(n/4) + \frac{1}{3}\sigma_3(n/8) + \left(\frac{1}{24} - \frac{n}{32}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma(n/8) - \frac{1}{64}c_8(n),$$

where the integers $c_8(n)$ $(n \in \mathbb{N})$ *are given by*

$$\sum_{n=1}^{\infty} c_8(n)q^n = q\prod_{n=1}^{\infty}(1-q^{2n})^4(1-q^{4n})^4. \tag{1.4}$$

(v) $$W_{16}(n) = \frac{1}{768}\sigma_3(n) + \frac{1}{256}\sigma_3(n/2) + \frac{1}{64}\sigma_3(n/4) + \frac{1}{16}\sigma_3(n/8) + \frac{1}{3}\sigma_3(n/16) + \left(\frac{1}{24} - \frac{n}{64}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma(n/16) - \frac{7}{256}c_{16}(n),$$

where the rational numbers $c_{16}(n)$ $(n \in \mathbb{N})$ *are given by*

$$\sum_{n=1}^{\infty} c_{16}(n)q^n = \frac{1}{32}A_1(q) + \frac{3}{112}A_2(q) + \frac{1}{224}A_3(q) - \frac{1}{32}A_5(q) - \frac{3}{112}A_6(q) - \frac{1}{224}A_7(q) \tag{1.5}$$

with

$$A_k(q) := \prod_{n=1}^{\infty}(1+q^n)^{24-4k}(1-q^n)^8(1-q^{4n-2})^{16-2k}. \tag{1.6}$$

We remark that for $n \in \mathbb{N}$,

$$c_8(n) = 0 \quad \text{if } n \equiv 0 \pmod 2 \tag{1.7}$$

(see [7, p. 388]), and

$$7c_{16}(n) \in \mathbb{Z} \tag{1.8}$$

(see [2, eq. (1.6), p. 4]).

Definition 1.2. For $a \in \mathbb{Z}$ and $m, n \in \mathbb{N}$ we define

$$S_{a,m}(n) := \sum_{\substack{l=1 \\ l \equiv a \ (\mathrm{mod}\ m)}}^{n-1} \sigma(l)\sigma(n-l).$$

Clearly,

$$S_{a,m}(n) = S_{b,m}(n) \quad \text{if } a \equiv b \pmod m,$$

and

$$\sum_{a=0}^{m-1} S_{a,m}(n) = W_1(n).$$

These sums were introduced in [6, p. 255]. We require the evaluation of $S_{a,4}(n)$ given in [3, Theorem 1.1].

PROPOSITION 1.3. *Let* $n \in \mathbb{N}$.

(i) *If* $n \equiv 0 \pmod 4$ *then*

$$S_{0,4}(n) = \frac{29}{192}\,\sigma_3(n) + \frac{17}{64}\,\sigma_3(n/2) + \left(\frac{1}{12} - \frac{n}{2}\right)\sigma(n),$$

$$S_{1,4}(n) = \frac{1}{16}\,\sigma_3(n) - \frac{1}{16}\,\sigma_3(n/2),$$

$$S_{2,4}(n) = \frac{9}{64}\,\sigma_3(n) - \frac{9}{64}\,\sigma_3(n/2),$$

$$S_{3,4}(n) = \frac{1}{16}\,\sigma_3(n) - \frac{1}{16}\,\sigma_3(n/2).$$

(ii) *If* $n \equiv 1 \pmod 4$ *then*

$$S_{0,4}(n) = \frac{11}{96}\,\sigma_3(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma(n) + \frac{3}{32}\,c_8(n),$$

$$S_{1,4}(n) = \frac{11}{96}\,\sigma_3(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma(n) + \frac{3}{32}\,c_8(n),$$

$$S_{2,4}(n) = \frac{3}{32}\,\sigma_3(n) - \frac{3}{32}\,c_8(n),$$

$$S_{3,4}(n) = \frac{3}{32}\,\sigma_3(n) - \frac{3}{32}\,c_8(n).$$

(iii) *If* $n \equiv 2 \pmod 4$ *then*

$$S_{0,4}(n) = \frac{11}{72}\,\sigma_3(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma(n),$$

$$S_{1,4}(n) = \frac{1}{18}\,\sigma_3(n) + \frac{1}{2}\,c_8(n/2),$$

$$S_{2,4}(n) = \frac{11}{72}\,\sigma_3(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma(n),$$

$$S_{3,4}(n) = \frac{1}{18}\,\sigma_3(n) - \frac{1}{2}\,c_8(n/2).$$

(iv) *If* $n \equiv 3 \pmod 4$ *then*

$$S_{0,4}(n) = \frac{11}{96}\,\sigma_3(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma(n) + \frac{3}{32}\,c_8(n),$$

$$S_{1,4}(n) = \frac{3}{32}\,\sigma_3(n) - \frac{3}{32}\,c_8(n),$$
$$S_{2,4}(n) = \frac{3}{32}\,\sigma_3(n) - \frac{3}{32}\,c_8(n),$$
$$S_{3,4}(n) = \frac{11}{96}\,\sigma_3(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma(n) + \frac{3}{32}\,c_8(n).$$

In this paper we use Propositions 1.1, 1.2 and 1.3 to determine $N(1^i, 4^{8-i}; n)$ for $i \in \{1,2,3,4,5,6,7\}$ and all $n \in \mathbb{N}$. We prove

THEOREM 1.1. *Let $n \in \mathbb{N}$. Then*

(i)
$$\begin{aligned} N(1^7,4;n) = {} & \left(8 - \left(\frac{-4}{n}\right)\right)\sigma_3(n) - 16\sigma_3(n/2) + 136\sigma_3(n/4) \\ & - 144\sigma_3(n/8) + 256\sigma_3(n/16) \\ & + 7\left(\frac{-4}{n}\right)c_8(n) + 28c_8(n/2). \end{aligned}$$

(ii)
$$\begin{aligned} N(1^6,4^2;n) = {} & \left(4 - \left(\frac{-4}{n}\right)\right)\sigma_3(n) - 4\sigma_3(n/2) - 16\sigma_3(n/8) \\ & + 256\sigma_3(n/16) + \left(2 + 7\left(\frac{-4}{n}\right)\right)c_8(n) + 28c_8(n/2). \end{aligned}$$

(iii)
$$\begin{aligned} N(1^5,4^3;n) = {} & \left(2 - \frac{1}{2}\left(\frac{-4}{n}\right)\right)\sigma_3(n) + 2\sigma_3(n/2) - 68\sigma_3(n/4) \\ & + 48\sigma_3(n/8) + 256\sigma_3(n/16) \\ & + \left(3 + \frac{11}{2}\left(\frac{-4}{n}\right)\right)c_8(n) + 20c_8(n/2). \end{aligned}$$

(iv)
$$\begin{aligned} N(1^4,4^4;n) = {} & \sigma_3(n) + 3\sigma_3(n/2) - 68\sigma_3(n/4) + 48\sigma_3(n/8) \\ & + 256\sigma_3(n/16) + \left(3 + 4\left(\frac{-4}{n}\right)\right)c_8(n) + 12c_8(n/2). \end{aligned}$$

(v)
$$\begin{aligned} N(1^3,4^5;n) = {} & \left(\frac{1}{2} + \frac{1}{4}\left(\frac{-4}{n}\right)\right)\sigma_3(n) + \frac{3}{2}\,\sigma_3(n/2) - 34\sigma_3(n/4) \\ & + 16\sigma_3(n/8) + 256\sigma_3(n/16) \\ & + \left(\frac{5}{2} + \frac{11}{4}\left(\frac{-4}{n}\right)\right)c_8(n) + 6c_8(n/2). \end{aligned}$$

(vi)
$$\begin{aligned} N(1^2,4^6;n) = {} & \left(\frac{1}{4} + \frac{1}{4}\left(\frac{-4}{n}\right)\right)\sigma_3(n) - \frac{1}{4}\,\sigma_3(n/2) - 16\sigma_3(n/8) \\ & + 256\sigma_3(n/16) + \left(\frac{7}{4} + \frac{7}{4}\left(\frac{-4}{n}\right)\right)c_8(n) + 2c_8(n/2). \end{aligned}$$

(vii) $$N(1,4^7;n) = \left(\frac{1}{8}+\frac{1}{8}\left(\frac{-4}{n}\right)\right)\sigma_3(n) - \frac{9}{8}\,\sigma_3(n/2) + 17\sigma_3(n/4) - 32\sigma_3(n/8) + 256\sigma_3(n/16) + \left(\frac{7}{8}+\frac{7}{8}\left(\frac{-4}{n}\right)\right)c_8(n).$$

Part (iv) of Theorem 1.1 was proved in [2, Theorem 1.2, p. 4] in terms of c_{16} rather than c_8.

2. Some preliminary results. The following sums will be needed in the proof of Theorem 1.1.

DEFINITION 2.1. For $r,s \in \mathbb{N}_0$ and $n \in \mathbb{N}$, we define

$$X(2^r,2^s;n) := \sum_{m=1}^{n-1} \sigma(m/2^r)\sigma((n-m)/2^s). \tag{2.1}$$

Clearly,

$$X(2^r,2^s;n) = X(2^s,2^r;n). \tag{2.2}$$

PROPOSITION 2.1. *Let $r,s \in \mathbb{N}_0$ and $n \in \mathbb{N}$. Then*

$$X(2^r,2^s;n) = \begin{cases} W_{2^{r-s}}(n/2^s) & \text{if } r \ge s, \\ W_{2^{s-r}}(n/2^r) & \text{if } r \le s. \end{cases}$$

Proof. If $r \ge s$ then

$$X(2^r,2^s;n) = \sum_{\substack{m\in\mathbb{N}\\ m<n/2^r}} \sigma(m)\sigma\left(\frac{n}{2^s} - 2^{r-s}m\right) = W_{2^{r-s}}(n/2^s).$$

If $r \le s$ then

$$X(2^r,2^s;n) = X(2^s,2^r;n) = W_{2^{s-r}}(n/2^r). \ ■$$

PROPOSITION 2.2. *For $n \in \mathbb{N}$,*

$$\sum_{m=1}^{n-1}\left(\frac{-4}{m}\right)\sigma(m)\sigma(n-m) = \left(\frac{-4}{n}\right)\left(\frac{1}{48}\,\sigma_3(n) + \left(\frac{1}{24}-\frac{n}{4}\right)\sigma(n) + \frac{3}{16}\,c_8(n)\right) + c_8(n/2).$$

Proof. We have

$$\begin{aligned}\sum_{m=1}^{n-1}\left(\frac{-4}{m}\right)\sigma(m)\sigma(n-m) &= \sum_{\substack{m=1\\ m\equiv 1 \,(\mathrm{mod}\,4)}}^{n-1}\sigma(m)\sigma(n-m) - \sum_{\substack{m=1\\ m\equiv 3 \,(\mathrm{mod}\,4)}}^{n-1}\sigma(m)\sigma(n-m)\\ &= S_{1,4}(n) - S_{3,4}(n)\end{aligned}$$

by Definition 1.2, and the asserted result follows from Proposition 1.3. ■

PROPOSITION 2.3. *For* $n \in \mathbb{N}$,

$$\sum_{m=1}^{n-1}\left(\frac{-4}{m}\right)\sigma(m)\sigma((n-m)/2) = \left(\frac{-4}{n}\right)(-W_2(n)+6W_4(n)-4W_8(n)).$$

Proof. We have

$$\begin{aligned}\sum_{m=1}^{n-1}\left(\frac{-4}{m}\right)\sigma(m)\sigma((n-m)/2) &= \sum_{\substack{l\in\mathbb{N}\\ l<n/2}}\left(\frac{-4}{n-2l}\right)\sigma(n-2l)\sigma(l)\\ &= \sum_{\substack{l\in\mathbb{N}\\ l<n/4}}\left(\frac{-4}{n-4l}\right)\sigma(2l)\sigma(n-4l)\\ &\quad + \sum_{\substack{l\in\mathbb{N}\\ l<(n+2)/4}}\left(\frac{-4}{n-2(2l-1)}\right)\sigma(2l-1)\sigma(n-2(2l-1))\\ &= \left(\frac{-4}{n}\right)\sum_{\substack{l\in\mathbb{N}\\ l<n/4}}\sigma(2l)\sigma(n-4l)\\ &\quad + \left(\frac{-4}{n+2}\right)\sum_{\substack{l\in\mathbb{N}\\ l<(n+2)/4}}\sigma(2l-1)\sigma(n-2(2l-1))\\ &= \left(\frac{-4}{n}\right)\sum_{\substack{l\in\mathbb{N}\\ l<n/4}}\sigma(2l)\sigma(n-4l)\\ &\quad - \left(\frac{-4}{n}\right)\Big(\sum_{\substack{l\in\mathbb{N}\\ l<n/2}}\sigma(l)\sigma(n-2l) - \sum_{\substack{l\in\mathbb{N}\\ l<n/4}}\sigma(2l)\sigma(n-4l)\Big)\end{aligned}$$

$$= -\left(\frac{-4}{n}\right) \sum_{\substack{l \in \mathbb{N} \\ l < n/2}} \sigma(l)\sigma(n-2l) + 2\left(\frac{-4}{n}\right) \sum_{\substack{l \in \mathbb{N} \\ l < n/4}} \sigma(2l)\sigma(n-4l)$$

$$= -\left(\frac{-4}{n}\right) \sum_{\substack{l \in \mathbb{N} \\ l < n/2}} \sigma(l)\sigma(n-2l) + 2\left(\frac{-4}{n}\right) \sum_{\substack{l \in \mathbb{N} \\ l < n/4}} (3\sigma(l) - 2\sigma(l/2))\sigma(n-4l)$$

$$= \left(\frac{-4}{n}\right)(-W_2(n) + 6W_4(n) - 4W_8(n)),$$

as asserted. ■

PROPOSITION 2.4. *For $k, n \in \mathbb{N}$ with $k \geq 2$,*

$$\sum_{m=1}^{n-1} \left(\frac{-4}{m}\right)\sigma(m)\sigma((n-m)/2^k) = \left(\frac{-4}{n}\right)W_{2^k}(n).$$

Proof. As $k \geq 2$ we have

$$\left(\frac{-4}{n-2^k l}\right) = \left(\frac{-4}{n}\right),$$

and so

$$\sum_{m=1}^{n-1} \left(\frac{-4}{m}\right)\sigma(m)\sigma((n-m)/2^k) = \sum_{\substack{l \in \mathbb{N} \\ l < n/2^k}} \left(\frac{-4}{n-2^k l}\right)\sigma(n-2^k l)\sigma(l)$$

$$= \left(\frac{-4}{n}\right) \sum_{\substack{l \in \mathbb{N} \\ l < n/2^k}} \sigma(l)\sigma(n-2^k l) = \left(\frac{-4}{n}\right)W_{2^k}(n)$$

by Definition 1.1. ■

3. The relationship between $c_8(n)$ and $c_{16}(n)$. Let q be a complex variable with $|q| < 1$. As in [4, p. 6] we set

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}. \tag{3.1}$$

The infinite product representations of $\varphi(q)$ and $\varphi(-q)$ are due to Jacobi, namely

$$\varphi(q) = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^5}{(1-q^n)^2(1-q^{4n})^2}, \qquad \varphi(-q) = \prod_{n=1}^{\infty} \frac{(1-q^n)^2}{1-q^{2n}}. \tag{3.2}$$

DEFINITION 3.1. For $k \in \mathbb{N}$ and $q \in \mathbb{C}$ with $|q| < 1$, we define

$$E_k = E_k(q) := \prod_{n=1}^{\infty} (1-q^{kn}).$$

From (3.2) and Definition 3.1, we deduce

$$\varphi(q) = E_1^{-2}E_2^5E_4^{-2}, \tag{3.3}$$

$$\varphi(-q) = E_1^2E_2^{-1}. \tag{3.4}$$

LEMMA 3.1. *For $k \in \mathbb{N}$ and $q \in \mathbb{C}$ with $|q| < 1$,*

$$A_k(q) = \varphi^{8-k}(q)\varphi^k(-q).$$

Proof. We have

$$\prod_{n=1}^{\infty}(1+q^n) = \prod_{n=1}^{\infty}\frac{1-q^{2n}}{1-q^n} = E_1^{-1}E_2,$$

$$\prod_{n=1}^{\infty}(1-q^{4n-2}) = \prod_{n=1}^{\infty}\frac{(1-q^{4n-2})(1-q^{4n})}{1-q^{4n}} = \prod_{n=1}^{\infty}\frac{1-q^{2n}}{1-q^{4n}} = E_2E_4^{-1}.$$

Thus, by (1.6), we obtain

$$\begin{aligned} A_k(q) &= \prod_{n=1}^{\infty}(1+q^n)^{24-4k}(1-q^n)^8(1-q^{4n-2})^{16-2k} = E_1^{4k-16}E_2^{40-6k}E_4^{2k-16} \\ &= (E_1^{-2}E_2^5E_4^{-2})^{8-k}(E_1^2E_2^{-1})^k = \varphi^{8-k}(q)\varphi^k(-q). \ \blacksquare \end{aligned}$$

Following Berndt [4, pp. 119–120] we set

$$x = 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}, \tag{3.5}$$

$$z = \varphi^2(q). \tag{3.6}$$

From Berndt's catalogue of formulae for theta fuctions [4, p. 122] we have

$$\varphi(q) = \sqrt{z}, \tag{3.7}$$

$$\varphi(-q) = \sqrt{z}(1-x)^{1/4}. \tag{3.8}$$

Following Cheng and Williams [5, p. 564] we set

$$g = (1-x)^{1/4}. \tag{3.9}$$

LEMMA 3.2. *For $k \in \mathbb{N}$ and $q \in \mathbb{C}$ with $|q| < 1$, we have*

$$A_k(q) = g^kz^4.$$

Proof. By Lemma 3.1 and (3.7)–(3.9), we have

$$A_k(q) = (\sqrt{z})^{8-k}(\sqrt{z}(1-x)^{1/4})^k = (1-x)^{k/4}z^4 = g^kz^4. \ \blacksquare$$

LEMMA 3.3.

$$\sum_{n=1}^{\infty}c_{16}(n)q^n = \left(\frac{1}{32}\,g + \frac{3}{112}\,g^2 + \frac{1}{224}\,g^3 - \frac{1}{224}\,g^5 - \frac{3}{112}\,g^6 - \frac{1}{224}\,g^7\right)z^4.$$

Proof. This follows from (1.5) and Lemma 3.2. ■

LEMMA 3.4. *For $q \in \mathbb{C}$ with $|q| < 1$, we have*

$$\text{(i)} \qquad \sum_{\substack{n=1 \\ n\equiv 1 \pmod 4}}^{\infty} c_8(n)q^n = \frac{1}{64}(g + 2g^2 + g^3 - g^5 - 2g^6 - g^7)z^4.$$

$$\text{(ii)} \qquad \sum_{\substack{n=1 \\ n\equiv 3 \pmod 4}}^{\infty} c_8(n)q^n = \frac{1}{64}(-g + 2g^2 - g^3 + g^5 - 2g^6 + g^7)z^4.$$

$$\text{(iii)} \qquad \sum_{\substack{n=1 \\ n\equiv 0 \pmod 2}}^{\infty} c_8(n/2)q^n = \frac{1}{128}(g - g^3 - g^5 + g^7)z^4.$$

Proof. Part (i) is [3, Theorem 2.3(i)]. Part (ii) is [3, Theorem 2.3(ii)]. By [3, Theorem 2.4] and (1.7), we have

$$\sum_{\substack{n=1 \\ n\equiv 0 \pmod 2}}^{\infty} c_8(n/2)q^n = \sum_{\substack{n=1 \\ n\equiv 2 \pmod 4}}^{\infty} c_8(n/2)q^n = \frac{1}{128}(g - g^3 - g^5 + g^7)z^4. \blacksquare$$

THEOREM 3.1. *For $n \in \mathbb{N}$,*

$$c_{16}(n) = \begin{cases} \frac{12}{7} c_8(n/2) & \text{if } n \equiv 0 \pmod 2, \\ c_8(n) & \text{if } n \equiv 1 \pmod 4, \\ \frac{-1}{7} c_8(n) & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Proof. By Lemmas 3.3 and 3.4 we have

$$\begin{aligned} \sum_{n=1}^{\infty} c_{16}(n)q^n &= \left(\frac{1}{32}g + \frac{3}{112}g^2 + \frac{1}{224}g^3 - \frac{1}{224}g^5 - \frac{3}{112}g^6 - \frac{1}{224}g^7\right)z^4 \\ &= \left(\frac{1}{64}g + \frac{1}{32}g^2 + \frac{1}{64}g^3 - \frac{1}{64}g^5 - \frac{1}{32}g^6 - \frac{1}{64}g^7\right)z^4 \\ &\quad - \frac{1}{7}\left(-\frac{1}{64}g + \frac{1}{32}g^2 - \frac{1}{64}g^3 + \frac{1}{64}g^5 - \frac{1}{32}g^6 + \frac{1}{64}g^7\right)z^4 \\ &\quad + \frac{12}{7}\left(\frac{1}{128}g - \frac{1}{128}g^3 - \frac{1}{128}g^5 + \frac{1}{128}g^7\right)z^4 \\ &= \sum_{\substack{n=1 \\ n\equiv 1 \pmod 4}}^{\infty} c_8(n)q^n - \frac{1}{7}\sum_{\substack{n=1 \\ n\equiv 3 \pmod 4}}^{\infty} c_8(n)q^n + \frac{12}{7}\sum_{\substack{n=1 \\ n\equiv 0 \pmod 2}}^{\infty} c_8(n/2)q^n. \end{aligned}$$

Equating coefficients of q^n, we obtain the assertion. ∎

We note that the first equality of Theorem 3.1 is Corollary 2.1 of [2]. We also observe that Theorem 3.1 can be expressed as

$$c_{16}(n) = \frac{1}{7}\left(3 + 4\left(\frac{-4}{n}\right)\right)c_8(n) + \frac{12}{7}c_8(n/2). \tag{3.10}$$

4. Proof of Theorem 1.1. We just prove part (i) as the remaining parts can be proved similarly. Appealing to (1.3), Proposition 1.1(i)–(ii), Definition 2.1 and Propositions 2.1–2.3, we obtain

$$\begin{aligned}
N(1^7, 4; n) &= \sum_{m=0}^{n} N(1^3, 4; m)N(1^4; n-m) \\
&= N(1^4; n) + N(1^3, 4; n) + \sum_{m=1}^{n-1} N(1^3, 4; m)N(1^4; n-m) \\
&= 8\sigma(n) - 32\sigma(n/4) \\
&\quad + \left(4 + 2\left(\frac{-4}{n}\right)\right)\sigma(n) - 20\sigma(n/4) + 24\sigma(n/8) - 32\sigma(n/16) \\
&\quad + \sum_{m=1}^{n-1}(4\sigma(m) - 20\sigma(m/4) + 24\sigma(m/8) - 32\sigma(m/16)) \\
&\quad \times (8\sigma(n-m) - 32\sigma((n-m)/4)) \\
&\quad + \sum_{m=1}^{n-1} 2\left(\frac{-4}{m}\right)\sigma(m)(8\sigma(n-m) - 32\sigma((n-m)/4)) \\
&= \left(12 + 2\left(\frac{-4}{m}\right)\right)\sigma(n) - 52\sigma(n/4) + 24\sigma(n/8) - 32\sigma(n/16) \\
&\quad + 32X(1,1;n) - 160X(4,1;n) + 192X(8,1;n) - 256X(16,1;n) \\
&\quad - 128X(1,4;n) + 640X(4,4;n) - 768X(8,4;n) \\
&\quad + 1024X(16,4;n) + 16\sum_{m=1}^{n-1}\left(\frac{-4}{m}\right)\sigma(m)\sigma(n-m) \\
&\quad - 64\sum_{m=1}^{n-1}\left(\frac{-4}{m}\right)\sigma(m)\sigma((n-m)/4) \\
&= \left(12 + 2\left(\frac{-4}{m}\right)\right)\sigma(n) - 52\sigma(n/4) + 24\sigma(n/8) - 32\sigma(n/16) \\
&\quad + 32W_1(n) - 288W_4(n) + 192W_8(n) - 256W_{16}(n)
\end{aligned}$$

$$+ 640W_1(n/4) - 768W_2(n/4) + 1024W_4(n/4)$$
$$+ 16\left(\frac{-4}{n}\right)\left(\frac{1}{48}\,\sigma_3(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma(n) + \frac{3}{16}\,c_8(n)\right)$$
$$+ 16c_8(n/2) - 64\left(\frac{-4}{n}\right)W_4(n).$$

The asserted result now follows from Proposition 1.2 and Theorem 3.1. ∎

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