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Combinatorial relations for Euler–Zagier sums

by

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1. Introduction. Let $k_1,\ldots,k_n$ be positive integers with $k_n\ge 2$. The *Euler–Zagier sum* $\zeta(k_1,\ldots,k_n)$ (of weight $k_1+\ldots+k_n$ and depth n) is the sum of the convergent series

$$\sum_{1\le m_1<\ldots<m_n}\frac{1}{m_1^{k_1}\ldots m_n^{k_n}}, \tag{1.1}$$

where the sum is over n-tuples of positive integers. These sums were introduced in [8] and [15] independently, but their origins go back to Euler. He investigated the properties of $\zeta(k_1,k_2)$ and discovered a number of relations, for example $\zeta(1,2)=\zeta(3)$. During the last decade, many important studies on Euler–Zagier sums (henceforth abbreviated as EZSs) have been made (see for example [1–6, 8–11, 14, 15]).

In [13] we introduced a new class of combinatorial relations for Tornheim's double series defined in [12] and discussed in [9], by means of an elementary method. In the present paper, we apply this method to prove some relations for EZSs, and give the proof of the following result which was conjectured in [6] (see also [2–5, 9, 12]).

THEOREM. *The Euler–Zagier sum $\zeta(k_1,\ldots,k_n)$ can be expressed as a rational linear combination of products of Euler–Zagier sums of depth lower than n, provided its depth n and its weight $\sum_{j=1}^{n}k_j$ are of different parity.*

The case $n=2$ of this theorem was proved in [12] and the explicit formula for $\zeta(k_1,k_2)$ was given in [2, 9]. The case $n=3$ was proved in [6].

We give some notation and lemmas in Section 2. In Section 3, we give another proof of the above result for EZSs of depth 2. In Section 4, we give the proof of the Theorem.

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2. Preliminaries. We use the same notation as in [13]. Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{Z}$ the ring of rational integers, $\mathbb{Q}$ the field of rational numbers, and $\mathbb{R}$ the field of real numbers. Throughout this paper we fix $\delta \in \mathbb{R}$ with $\delta > 0$. For $u \in \mathbb{R}$ with $1 \le u \le 1+\delta$ and $k \in \mathbb{N}$, we define

$$\phi(s;u) := \sum_{m\ge 1} \frac{(-u)^{-m}}{m^s} \qquad (s \in \mathbb{Z}). \tag{2.1}$$

If $u > 1$ then $\phi(s;u)$ is convergent for any $s \in \mathbb{Z}$. For $u = 1$, let $\phi(s) := \phi(s;1) = (2^{1-s}-1)\zeta(s)$. Corresponding to $\phi(s;u)$, we define a set $\{\varepsilon_m(u)\}$ of numbers by

$$F(x;u) := \frac{(1+u)e^x}{e^x+u} = \sum_{m=0}^{\infty} \varepsilon_m(u)\,\frac{x^m}{m!}. \tag{2.2}$$

In particular when $u = 1$, we have

$$F(x;1) = \frac{2e^x}{e^x+1} = \sum_{m=0}^{\infty} E_m(1)\,\frac{x^m}{m!},$$

where $E_m(X)$ is the mth Euler polynomial (see for example [7]). Hence

$$\varepsilon_{2j}(1) = E_{2j}(1) = 0 \qquad (j \in \mathbb{N}). \tag{2.3}$$

It follows from (2.2) that if $u \in [1, 1+\delta]$ then

$$\liminf_{m\to\infty} \left(\frac{|\varepsilon_m(u)|}{m!}\right)^{-1/m} \ge \pi, \tag{2.4}$$

and that the following lemma holds.

Lemma 1 ([13, Lemma 1]). *For $k \in \mathbb{N}\cup\{0\}$ and $u \in (1, 1+\delta]$,*

$$\phi(-k;u) = -\frac{1}{1+u}\,\varepsilon_k(u). \tag{2.5}$$

Hence we formally define

$$\varepsilon_k(u) := -(1+u)\phi(-k;u) \qquad (k \in \mathbb{Z},\ k<0;\ u \in [1,1+\delta]). \tag{2.6}$$

For $\theta \in \mathbb{R}$, let

$$G(\theta;u) := i^{-1}\sum_{j=0}^{\infty} \varepsilon_{2j+1}(u)\,\frac{(i\theta)^{2j+1}}{(2j+1)!};\quad H(\theta;u) := \sum_{j=1}^{\infty} \varepsilon_{2j}(u)\,\frac{(i\theta)^{2j}}{(2j)!}, \tag{2.7}$$

where $i=\sqrt{-1}$. If $u\in(1,1+\delta]$, then by (2.2) and (2.7), we obtain

$$\begin{aligned}G(\theta;u)&=-(1+u)\sum_{j=1}^{\infty}(-u)^{-j}\,\frac{e^{ij\theta}-e^{-ij\theta}}{2i}\\&=-(1+u)\sum_{j=1}^{\infty}(-u)^{-j}\sin(j\theta),\\H(\theta;u)&=-(1+u)\sum_{j=1}^{\infty}(-u)^{-j}\,\frac{e^{ij\theta}+e^{-ij\theta}}{2}-1\\&=-(1+u)\sum_{j=1}^{\infty}(-u)^{-j}\cos(j\theta)-1.\end{aligned}\tag{2.8}$$

By (2.3) and (2.7), we have

$$H(\theta;u)\to 0\qquad(u\to 1;\ \theta\in(-\pi,\pi)).\tag{2.9}$$

For $u\in[1,1+\varepsilon]$, $n\in\mathbb{N}$ and $(k_1,\dots,k_n)\in\mathbb{N}^n$, we define

$$\phi(k_1,\dots,k_n;u):=\sum_{1\le m_1<\dots<m_n}\frac{(-u)^{-m_n}}{m_1^{k_1}\dots m_n^{k_n}}.\tag{2.10}$$

When $u>1$, we define $\phi(k_1,\dots,k_n;u)$ for any $k_n\in\mathbb{Z}$ by (2.10). Furthermore, for $k_n\ge 2$ and $u\in[1,1+\delta]$, we define

$$\zeta(k_1,\dots,k_n;u):=\sum_{1\le m_1<\dots<m_n}\frac{u^{-m_n}}{m_1^{k_1}\dots m_n^{k_n}}.\tag{2.11}$$

Note that $\zeta(k_1,\dots,k_n;1)$ coincides with $\zeta(k_1,\dots,k_n)$. As a generalization of (2.10) and (2.11), we define

$$\begin{aligned}R_p(\theta;k_1,\dots,k_{n-1};a,b;u):=i^{1-p}\sum_{\nu=0}^{b}\binom{a-1+b-\nu}{b-\nu}\frac{(-\theta)^{\nu}}{\nu!}\\\times\sum_{m_1<\dots<m_n}\frac{(-u)^{-m_n}\sin^{(\nu+p)}(m_n\theta)}{m_1^{k_1}\dots m_{n-1}^{k_{n-1}}m_n^{a+b-\nu}},\end{aligned}\tag{2.12}$$

for $\theta\in\mathbb{R}$, $a\in\mathbb{N}$ and $p,b\in\mathbb{N}\cup\{0\}$, where we denote the lth derivative of a function $f(\theta)$ by $f^{(l)}(\theta)$. Since $\sin^{(\nu+2)}\theta=-\sin^{(\nu)}\theta$, we have

$$R_p(\theta;k_1,\dots,k_{n-1};a,b;u)=R_{p+2}(\theta;k_1,\dots,k_{n-1};a,b;u).\tag{2.13}$$

Since $\sin^{(2j)}(m\pi)=0$ and $\sin^{(2j+1)}(m\pi)=(-1)^{j+m}$ for $j,m\in\mathbb{N}$, we immediately obtain the following. Note that (just as elsewhere in this paper) an empty sum is to be interpreted as zero.

LEMMA 2. *If* $a+(b-1)-2[(b-1)/2]\geq 2$, *then*

$$R_0(\pi;k_1,\ldots,k_{n-1};a,b;u) = -\sum_{j=0}^{[(b-1)/2]}\binom{a-2+b-2j}{b-2j-1} \times \zeta(k_1,\ldots,k_{n-1},a+b-2j-1;u)\frac{(i\pi)^{2j+1}}{(2j+1)!}. \tag{2.14}$$

If $a+b-2[b/2]\geq 2$, *then*

$$R_1(\pi;k_1,\ldots,k_{n-1};a,b;u) = \sum_{j=0}^{[b/2]}\binom{a-1+b-2j}{b-2j}\zeta(k_1,\ldots,k_{n-1},a+b-2j;u)\frac{(i\pi)^{2j}}{(2j)!}. \tag{2.15}$$

In particular, $R_0(\pi;k_1,\ldots,k_{n-1};a,0;1)=0$ *and* $R_1(\pi;k_1,\ldots,k_{n-1};a,0;1) = \zeta(k_1,\ldots,k_{n-1},a)$ *when* $a\geq 2$.

Let $f_p(x;a,\theta):=i\sin^{(p)}(\theta x)x^{-a}$ for $p,a\in\mathbb{N}\cup\{0\}$, and calculate its bth derivative $(d^b/dx^b)f_p(x;a,\theta)$. Then, in the same way as in [13, Lemma 6], we obtain

$$\sum_{\nu=0}^{b}\binom{a-1+b-\nu}{b-\nu}\frac{(-\theta)^\nu}{\nu!}\frac{\sin^{(\nu+p)}(\theta x)}{x^{a+b-\nu}} = i^{p-1}\sum_{N\geq 0}\binom{a-1+b-N}{b}\frac{(i\theta)^N}{N!}\lambda_{p-1+N}x^{-a-b+N}, \tag{2.16}$$

where we let

$$\lambda_j := \frac{1+(-1)^j}{2}. \tag{2.17}$$

LEMMA 3. *With the above notation, for* $u\in(1,1+\delta]$ *we have*

$$R_p(\theta;k_1,\ldots,k_{n-1};a,b;u) = \sum_{N=0}^{\infty}\binom{a-1+b-N}{b}\phi(k_1,\ldots,k_{n-1},a+b-N;u)\lambda_{p-1+N}\frac{(i\theta)^N}{N!}. \tag{2.18}$$

In particular, for $c\in\mathbb{N}$ *we have*

$$R_{p+c}(\theta;k_1,\ldots,k_{n-1};a+c,b;u) = \sum_{m=-c}^{\infty}(-1)^b\binom{m-a}{b}\phi(k_1,\ldots,k_{n-1},a+b-m;u)\lambda_{p-1+m}\frac{(i\theta)^{m+c}}{(m+c)!}. \tag{2.19}$$

Proof. By (2.10), (2.12) and (2.16), we immediately obtain (2.18). (2.19) can be proved by putting $m = N - c$ and by using the well-known relation

$$\binom{-X}{j} = (-1)^j \binom{X+j-1}{j}. \qquad \blacksquare$$

LEMMA 4. *With the above notation, for $u \in (1, 1+\delta]$ we have*

$$\begin{aligned}(2.20)\qquad & i^{-1} R_p(\theta; k_1, \ldots, k_{n-1}; a, b; u) G(\theta; u) \\ & \qquad - R_{p+1}(\theta; k_1, \ldots, k_{n-1}; a, b; u) H(\theta; u) \\ &= \sum_{m=0}^{\infty} \biggl\{ (1+u) \sum_{\nu=0}^{b} \binom{\nu - m - 1}{\nu} \binom{a-1+b-\nu}{b-\nu} \\ & \quad \times \phi(k_1, \ldots, k_{n-1}, a+b-\nu, \nu - m; u) \\ & \quad + \binom{a-1+b-m}{b} \phi(k_1, \ldots, k_{n-1}, a+b-m; u) \biggr\} \lambda_{p+m} \frac{(i\theta)^m}{m!}.\end{aligned}$$

Proof. By (2.8), (2.9) and the well-known relation

$$\sin^{(k)} \alpha \cdot \sin \beta - \sin^{(k+1)} \alpha \cdot \cos \beta = -\sin^{(k+1)}(\alpha + \beta),$$

we can verify that the left-hand side of (2.20) is equal to

$$\frac{1+u}{i^p} \sum_{\nu=0}^{b} \binom{a-1+b-\nu}{b-\nu} \frac{(-\theta)^\nu}{\nu!} \sum_{m_1 < \ldots < m_n} \frac{(-u)^{-m_{n+1}} \sin^{(\nu+p+1)}(m_{n+1}\theta)}{m_1^{k_1} \ldots m_{n-1}^{k_{n-1}} m_n^{a+b-\nu}} + R_{p+1}(\theta; k_1, \ldots, k_{n-1}; a, b; u).$$

It follows from the relation $\sin^{(\nu)} \theta = i^{\nu-1}(e^{i\theta} + (-1)^{\nu-1} e^{-i\theta})/2$, the Maclaurin expansion of e^x, and the binomial theorem that the first term is equal to

$$(1+u) \sum_{m=0}^{\infty} \sum_{\nu=0}^{b} \binom{m}{\nu} (-1)^\nu \binom{a-1+b-\nu}{b-\nu} \times \phi(k_1, \ldots, k_{n-1}, a+b-\nu, \nu-m; u) \lambda_{p+m} \frac{(i\theta)^m}{m!}.$$

The other term can be calculated by Lemma 3. ∎

For $u \in (1, 1+\delta]$, $m \in \mathbb{Z}$, $n, a \in \mathbb{N}$, $b \in \mathbb{N} \cup \{0\}$ and $(k_1, \ldots, k_n) \in \mathbb{N}^n$, we define

$$\begin{aligned} A_m(k_1, \ldots, k_{n-1}; a, b; u) &:= (1+u) \sum_{\nu=0}^{b} \binom{\nu-m-1}{\nu} \binom{a-1+b-\nu}{b-\nu} \\ & \qquad \times \phi(k_1, \ldots, k_{n-1}, a+b-\nu, \nu-m; u) \\ & \qquad + \binom{a-1+b-m}{b} \phi(k_1, \ldots, k_{n-1}, a+b-m; u). \end{aligned}$$

In particular when $m \le -1$, we can define

$$A_m(k_1,\dots,k_{n-1};a,b;1) := \lim_{u\to 1} A_m(k_1,\dots,k_{n-1};a,b;u). \tag{2.21}$$

Lemma 4 states that

$$\begin{aligned} i^{-1}R_p(\theta;k_1,\dots,k_{n-1};a,b;u)G(\theta;u) \\ - R_{p+1}(\theta;k_1,\dots,k_{n-1};a,b;u)H(\theta;u) \\ = \sum_{m=0}^{\infty} A_m(k_1,\dots,k_{n-1};a,b;u)\lambda_{p+m}\frac{(i\theta)^m}{m!}. \end{aligned} \tag{2.22}$$

LEMMA 5. *With the above notation, for $c \in \mathbb{N}$ we have*

$$\begin{aligned} \sum_{\nu=0}^{b} \binom{a-1+b-\nu}{b-\nu} R_{p+c+1}(\theta;k_1,\dots,k_{n-1},a+b-\nu;c,\nu;u) \\ + \frac{1}{1+u} R_{p+c+1}(\theta;k_1,\dots,k_{n-1};a+c,b;u) \\ = \frac{1}{1+u}\sum_{m=-c}^{\infty} A_m(k_1,\dots,k_{n-1};a,b;u)\lambda_{p+m}\frac{(i\theta)^{m+c}}{(m+c)!}. \end{aligned} \tag{2.23}$$

Proof. By applying Lemma 3 to the left-hand side of (2.23) and using (2.21), we obtain the asserted formula. ∎

3. The case of depth 2. By Lemmas 1 and 3, we have

$$\begin{aligned} R_{k+1+\mu}(\theta;\,;k,0;u) &= -\frac{1}{1+u}\sum_{m=-k}^{\infty} \varepsilon_m(u)\lambda_{m+\mu}\frac{(i\theta)^{m+k}}{(m+k)!} \\ &= -\frac{1}{1+u}\sum_{N=0}^{\infty} \varepsilon_{N-k}(u)\lambda_{N-k+\mu}\frac{(i\theta)^N}{N!} \end{aligned} \tag{3.1}$$

for $k \in \mathbb{N}$, $\mu \in \mathbb{N}\cup\{0\}$ and $u \in (1, 1+\delta]$. Let

$$\begin{aligned} I_1(\theta;k;u) &:= R_{k+1}(\theta;\,;k,0;u) + \frac{1}{1+u}\sum_{j=0}^{k} \varepsilon_{j-k}(u)\lambda_{k+j}\frac{(i\theta)^j}{j!} \\ &= -\frac{1}{1+u}\sum_{r=1}^{\infty} \varepsilon_{2r}(u)\frac{(i\theta)^{2r+k}}{(2r+k)!}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} J_1(\theta;k;u) &:= R_k(\theta;\,;k,0;u) \\ &= -\frac{1}{1+u}\sum_{r=[-k/2]}^{\infty} \varepsilon_{2r+1}(u)\frac{(i\theta)^{2r+1+k}}{(2r+1+k)!}. \end{aligned} \tag{3.3}$$

By (2.3) and (2.4), $I_1(\theta;k;u)$ and $J_1(\theta;k;u)$ are uniformly convergent with respect to $u \in (1, 1+\delta]$ when $\theta \in (-\pi, \pi)$, and

$$I_1(\theta;k;u) \to 0 \qquad (u \to 1). \tag{3.4}$$

LEMMA 6. *With the above notation,*

$$\begin{aligned} &i^{-1} I_1(\theta;k;u)G(\theta;u) - J_1(\theta;k;u)H(\theta;u) \\ &\quad = \sum_{m=0}^{\infty} \left\{ A_m(;k,0;u) + \sum_{j=0}^{k} \binom{m}{j} \varepsilon_{j-k}(u)\phi(j-m;u)\lambda_{k+j} \right\} \lambda_{k+1+m} \frac{(i\theta)^m}{m!}. \end{aligned}$$

Proof. It follows from (2.22), (3.2) and (3.3) that

$$\begin{aligned} i^{-1} I_1(\theta;k;u)G(\theta;u) &- J_1(\theta;k;u)H(\theta;u) \\ &= \sum_{m=0}^{\infty} A_m(;k,0;u)\lambda_{k+1+m} \frac{(i\theta)^m}{m!} \\ &\quad + \frac{1}{i(1+u)} \left(\sum_{j=0}^{k} \varepsilon_{j-k}(u)\lambda_{k+j} \frac{(i\theta)^j}{j!} \right) G(\theta;u). \end{aligned}$$

The last term on the right-hand side is equal to

$$\sum_{m=0}^{\infty} \left(\sum_{j=0}^{k} \varepsilon_{j-k}(u)\phi(j-m;u)\lambda_{k+j}\lambda_{m-j+1} \right) \frac{(i\theta)^m}{m!},$$

since it follows from (2.8) that

$$G(\theta;u) = i(1+u) \sum_{l=0}^{\infty} \phi(-l;u)\lambda_{l+1} \frac{(i\theta)^l}{l!}.$$

By (2.17), we can verify that $\lambda_{p+r}\lambda_{q+r} = \lambda_{p+r}\lambda_{p+q}$ for any $p, q, r \in \mathbb{Z}$. Hence we obtain the asserted formula. ∎

For $m \in \mathbb{Z}$, $k \in \mathbb{N}$ and $u \in (1, 1+\delta]$, we define

$$\varepsilon_m(k;u) := -A_m(;k,0;u) - \sum_{j=0}^{k} \binom{m}{j} \varepsilon_{j-k}(u)\phi(j-k;u)\lambda_{k+j}. \tag{3.5}$$

Lemma 6 states that

$$\begin{aligned} &i^{-1} I_1(\theta;k;u)G(\theta;u) - J_1(\theta;k;u)H(\theta;u) \\ &\qquad = -\sum_{m=0}^{\infty} \varepsilon_m(k;u)\lambda_{k+1+m} \frac{(i\theta)^m}{m!}. \end{aligned} \tag{3.6}$$

It follows from (2.4), (3.2) and (3.3) that (3.6) is uniformly convergent with respect to $u \in (1, 1+\delta]$ when $\theta \in (-\pi, \pi)$, and

$$\liminf_{m\to\infty} \left(\frac{|\varepsilon_m(k;u)|}{m!}\right)^{-1/m} \geq \pi. \tag{3.7}$$

Furthermore, by (2.9) and (3.6), we have

$$\varepsilon_m(k;u)\lambda_{k+1+m} \to 0 \qquad (u \to 1;\ m \in \mathbb{N}\cup\{0\}). \tag{3.8}$$

On the other hand, by (2.21), we can define

$$\varepsilon_m(k;1) := \lim_{u\to 1} \varepsilon_m(k;u) \tag{3.9}$$

for $m \in \mathbb{Z}$ with $m \leq -1$.

Proposition 1. *For $k, l \in \mathbb{N}$, $\mu \in \{0,1\}$, $u \in (1, 1+\delta]$ and $\theta \in (-\pi, \pi)$,*

$$\begin{aligned} &R_{k+l+\mu}(\theta;k;l,0;u) + \frac{1}{1+u} R_{k+l+\mu}(\theta;;k+l,0;u) \\ &\qquad + \frac{1}{1+u}\sum_{j=0}^{k} \varepsilon_{j-k}(u)(-1)^j \lambda_{k+j} R_{k+l+\mu}(\theta;;l,j;u) \\ &\qquad = -\frac{1}{1+u}\sum_{m=-l}^{\infty} \varepsilon_m(k;u)\lambda_{k+1+m+\mu}\frac{(i\theta)^{m+l}}{(m+l)!}. \end{aligned} \tag{3.10}$$

In particular when $l \geq 2$, (3.10) holds for $\theta \in [-\pi, \pi]$.

Proof. By (3.5), Lemma 5 with $(n,p,a,b,c) = (1, k+1+\mu, k, 0, l)$ and Lemma 3, (2.19), with $(n,p,a,b,c) = (1, k+\mu, 0, j, l)$, we obtain (3.10). By (3.7), we obtain the last assertion. ∎

Suppose $l \geq 2$. Then it follows from (3.7), (3.8) and Proposition 1 that the right-hand side of (3.10) is uniformly convergent with respect to $u \in (1, 1+\delta]$ when $\theta = \pi$. Hence by letting $u \to 1$ on both sides of (3.10), we have

$$\begin{aligned} &R_{k+l}(\pi;k;l,0;1) + \frac{1}{2}R_{k+l}(\pi;;k+l,0;1) \\ &\qquad + \frac{1}{2}\sum_{j=0}^{k} \varepsilon_{j-k}(1)(-1)^j \lambda_{k+j} R_{k+l}(\pi;;l,j;1) \\ &\qquad = -\frac{1}{2}\sum_{m=-l}^{-1} \varepsilon_m(k;1)\lambda_{k+1+m}\frac{(i\pi)^{m+l}}{(m+l)!} \\ &\qquad = -\frac{1}{2}\sum_{r=0}^{l-1} \varepsilon_{r-l}(k;1)\lambda_{k+1+l+r}\frac{(i\pi)^{r}}{r!}. \end{aligned} \tag{3.11}$$

Now we assume that $k+l\equiv 1 \pmod 2$. By substituting (2.6) and (2.15) into (3.11), we have

$$\begin{aligned}(3.12)\qquad &\zeta(k,l)+\frac{1}{2}\zeta(k+l)\\ &\quad -(-1)^k\sum_{\mu=0}^{[k/2]}\phi(2\mu)\sum_{\nu=0}^{[(k-2\mu)/2]}\binom{l-1+k-2\mu-2\nu}{k-2\mu-2\nu}\\ &\qquad\qquad\times\zeta(l+k-2\mu-2\nu)\frac{(i\pi)^{2\nu}}{(2\nu)!}\\ &=-\frac{1}{2}\sum_{r=0}^{l-1}\varepsilon_{r-l}(k;1)\lambda_r\frac{(i\pi)^r}{r!}=-\frac{1}{2}\sum_{j=0}^{[(l-1)/2]}\varepsilon_{2j-l}(k;1)\frac{(i\pi)^{2j}}{(2j)!}.\end{aligned}$$

We replace l with $h+1$ in (3.11) when $h\in\mathbb{N}$ and $k+h+1\equiv 0 \pmod 2$. Then by Lemma 2, we have

$$\begin{aligned}(3.13)\qquad &-(-1)^k\sum_{\mu=0}^{[k/2]}\phi(2\mu)\sum_{\nu=0}^{[(k-2\mu-1)/2]}\binom{h-1+k-2\mu-2\nu}{k-2\mu-2\nu-1}\\ &\qquad\qquad\times\zeta(h+k-2\mu-2\nu)\frac{(i\pi)^{2\nu+1}}{(2\nu+1)!}\\ &=-\frac{1}{2}\sum_{r=0}^{h}\varepsilon_{r-h-1}(k;1)\lambda_{r+1}\frac{(i\pi)^r}{r!}=-\frac{1}{2}\sum_{j=0}^{[(h-1)/2]}\varepsilon_{2j-h}(k;1)\frac{(i\pi)^{2j+1}}{(2j+1)!}.\end{aligned}$$

On the other hand, we recall the following lemma.

Lemma 7 ([13, Lemma 8]). *Suppose $\{P_m\}$ and $\{Q_m\}$ are the sequences which satisfy*

$$\sum_{j=0}^{[m/2]}P_{m-2j}\frac{(i\pi)^{2j}}{(2j+1)!}=Q_m\qquad\text{for any } m\in\mathbb{N}\cup\{0\}.$$

Then

$$P_m=-2\sum_{\nu=0}^{m}\phi(m-\nu)\lambda_{m-\nu}Q_\nu\qquad\text{for any } m\in\mathbb{N}\cup\{0\}.$$

We denote by $\mathcal{Z}_1$ the $\mathbb{Q}$-algebra generated by $\{\zeta(k)\mid k\in\mathbb{N},\,k\geq 2\}$. If we apply Lemma 7 with $m=h-1$, $P_m=-\frac{1}{2}\varepsilon_{-m-1}(k;1)\lambda_{m+k}$ and

$$\begin{aligned}Q_m=&-(-1)^k\lambda_{m+k}\sum_{\mu=0}^{[k/2]}\phi(2\mu)\sum_{\nu=0}^{[(k-2\mu-1)/2]}\binom{m+k-2\mu-2\nu}{k-2\mu-2\nu-1}\\ &\qquad\times\zeta(m+1+k-2\mu-2\nu)\frac{(i\pi)^{2\nu}}{(2\nu+1)!},\end{aligned}$$

then by (3.13) and by the relation $\phi(s)=(2^{1-s}-1)\zeta(s)$, we have

$$\varepsilon_j(k;1)\lambda_{k+1+j}\in\mathcal{Z}_1 \qquad (j\in\mathbb{Z} \text{ with } j\le -1). \tag{3.14}$$

Hence, by (3.12), we have $\zeta(k,l)\in\mathcal{Z}_1$ when $k+l$ is odd. This gives another proof of the Theorem for the EZSs of depth 2 in Section 1. For example, by putting $(k,h)=(1,1)$ in (3.13), we have $\varepsilon_{-1}(1;1)=-\zeta(3)$. By putting $(k,l)=(1,2)$ in (3.12), we obtain Euler's formula $\zeta(1,2)=\zeta(3)$.

4. The case of depth n. In this section, we assume that $u\in[1,1+\delta]\cap\mathbb{Q}$. For $n\in\mathbb{N}$, we denote by $\mathcal{Z}_n$ the $\mathbb{Q}$-algebra generated by

$$\{\zeta(k_1,\dots,k_r)\mid (k_1,\dots,k_r)\in\mathbb{N}^r \text{ with } k_r\ge 2,\ 1\le r\le n\}.$$

As a generalization of $\mathcal{Z}_n$, we denote by $V_n(p)$ the $\mathcal{Z}_n$-module generated by

$$\Bigg\{R_p(\theta;k_1,\dots,k_{r-1};a,b;u)\,\Bigg|\,(k_1,\dots,k_{r-1})\in\mathbb{N}^{r-1},\ 1\le r\le n,\ a\in\mathbb{N},$$
$$b\in\mathbb{N}\cup\{0\} \text{ with } a+(b-1)-2\left[\frac{b-1}{2}\right]\ge 2,\ u\in[1,1+\delta]\cap\mathbb{Q}\Bigg\}$$

for $p\in\mathbb{N}\cup\{0\}$ and $n\in\mathbb{N}$. By (2.13), we have $V_n(p)=V_n(p+2)$. Lemma 2 shows that if $g(\theta;u)\in V_n(\mu)$ then $g(\pi;1)/(i\pi)^{1-\mu}\in\mathcal{Z}_n$ for $\mu\in\{0,1\}$.

Now we define the $\mathcal{Z}_n$-linear operator $\Delta^{(l)}:V_n(p)\to V_{n+1}(p+l+1)$ for $l,n\in\mathbb{N}$ and $p\in\mathbb{N}\cup\{0\}$ by

$$\begin{aligned}\Delta^{(l)}&(R_p(\theta;k_1,\dots,k_{r-1};a,b;u))\\ &:=\sum_{\nu=0}^{b}\binom{a-1+b-\nu}{b-\nu}R_{p+l+1}(\theta;k_1,\dots,k_{r-1},a+b-\nu;l,\nu;u)\\ &\quad+\frac{1}{1+u}R_{p+l+1}(\theta;k_1,\dots,k_{r-1};a+l,b;u),\end{aligned} \tag{4.1}$$

where $r\in\mathbb{N}$ with $r\le n$. For $k,l\in\mathbb{N}$, $\mu\in\{0,1\}$, $\theta\in[-\pi,\pi]$ and $u\in[1,1+\delta]\cap\mathbb{Q}$, we define

$$\begin{aligned}\Gamma_{2,\mu}(\theta;k,l;u)&:=\Delta^{(l)}(R_{k+1+\mu}(\theta;;k,0;u))\\ &\quad+\frac{1}{1+u}\sum_{j=0}^{k}\varepsilon_{j-k}(u)(-1)^j\lambda_{k+j}R_{k+l+\mu}(\theta;;l,j;u).\end{aligned} \tag{4.2}$$

Then Proposition 1 states that

$$\Gamma_{2,\mu}(\theta;k,l;u)=-\frac{1}{1+u}\sum_{m=-l}^{\infty}\varepsilon_m(k;u)\lambda_{k+1+m+\mu}\frac{(i\theta)^{m+l}}{(m+l)!}.$$

By (3.7), (3.8) and (3.14), we have

$$\liminf_{m\to\infty}\left(\frac{|\varepsilon_m(k;u)|}{m!}\right)^{-1/m} \geq \pi,$$
$$\varepsilon_m(k;u)\lambda_{k+1+m} \to 0 \qquad (u\to 1;\ m\in\mathbb{N}\cup\{0\}),$$
$$\varepsilon_m(k;1)\in\mathcal{Z}_1 \qquad (m\in\mathbb{Z} \text{ with } m\leq -1,\ m\equiv k+1 \pmod 2).$$

PROPOSITION 2. *For $n\in\mathbb{N}$ with $n\geq 2$, $(k_1,\dots,k_n)\in\mathbb{N}^n$ with $k_n\geq 2$, $\mu\in\{0,1\}$, $u\in[1,1+\delta]\cap\mathbb{Q}$ and $\theta\in[-\pi,\pi]$, there exist $\Gamma_{n,\mu}(\theta;k_1,\dots,k_n;u)$ $\in V_n(\sum_{j=1}^n(k_j+1)+\mu)$ and $\{\varepsilon_m(k_1,\dots,k_{n-1};u)\}_{m\in\mathbb{Z}}$ such that the following five conditions hold*:

$$\Gamma_{n,\mu}(\theta;k_1,\dots,k_n;u) - R_{\sum_{j=1}^n(k_j+1)+\mu}(\theta;k_1,\dots,k_{n-1};k_n,0;u) \in V_{n-1}\Big(\sum_{j=1}^n(k_j+1)+\mu\Big), \tag{4.3}$$

$$\Gamma_{n,\mu}(\theta;k_1,\dots,k_n;u) = -\frac{1}{1+u}\sum_{m=-k_n}^{\infty}\varepsilon_m(k_1,\dots,k_{n-1};u) \times \lambda_{\sum_{j=1}^{n-1}(k_j+1)+m+\mu}\frac{(i\theta)^{m+k_n}}{(m+k_n)!}, \tag{4.4}$$

$$\liminf_{m\to\infty}\left(\frac{|\varepsilon_m(k_1,\dots,k_{n-1};u)|}{m!}\right)^{-1/m} \geq \pi, \tag{4.5}$$

$$\varepsilon_m(k_1,\dots,k_{n-1};u)\lambda_{\sum_{j=1}^{n-1}(k_j+1)+m} \to 0 \qquad (u\to 1;\ m\in\mathbb{N}\cup\{0\}), \tag{4.6}$$

$$\varepsilon_m(k_1,\dots,k_{n-1};1)\lambda_{\sum_{j=1}^{n-1}(k_j+1)+m} \in \mathcal{Z}_{n-1} \qquad (m\in\mathbb{Z} \text{ with } m\leq -1). \tag{4.7}$$

Proof. We argue by induction on n. The case of $n=2$ is just what is mentioned above. Assume that $\Gamma_{n,\mu}(\theta;k_1,\dots,k_n;u)\in V_n(\sum_{j=1}^n(k_j+1)+\mu)$ and $\{\varepsilon_m(k_1,\dots,k_{n-1};u)\}_{m\in\mathbb{Z}}$ satisfy (4.3)–(4.7). Suppose $u>1$ and let $p=\sum_{j=1}^n(k_j+1)$. By the assumption, we can write $\Gamma_{n,0}(\theta;k_1,\dots,k_n;u)\in V_n(p)$ as the following finite sum:

$$\Gamma_{n,0}(\theta;k_1,\dots,k_n;u) = \sum_{\tau} C_\tau R_p(\theta;l_{\tau,1},\dots,l_{\tau,N_\tau};a_\tau,b_\tau;u), \tag{4.8}$$

where $C_\tau\in\mathcal{Z}_n$ for any τ. By Lemma 3 and (4.4), we see that

$$\Gamma_{n,1}(\theta;k_1,\dots,k_n;u) = \sum_{\tau} C_\tau R_{p+1}(\theta;l_{\tau,1},\dots,l_{\tau,N_\tau};a_\tau,b_\tau;u).$$

Let

$$(4.9)\qquad I_n(\theta;k_1,\ldots,k_n;u) := \Gamma_{n,0}(\theta;k_1,\ldots,k_n;u) + \frac{1}{1+u}\sum_{m=-k_n}^{-1}\varepsilon_m(k_1,\ldots,k_{n-1};u)\lambda_{p-(k_n+1)+m}\frac{(i\theta)^{m+k_n}}{(m+k_n)!}$$

$$= \Gamma_{n,0}(\theta;k_1,\ldots,k_n;u) + \frac{1}{1+u}\sum_{j=0}^{k_n-1}\varepsilon_{j-k_n}(k_1,\ldots,k_{n-1};u)\lambda_{p+j+1}\frac{(i\theta)^j}{j!},$$

$$(4.10)\qquad J_n(\theta;k_1,\ldots,k_n;u) := \Gamma_{n,1}(\theta;k_1,\ldots,k_n;u).$$

By (4.4)–(4.6), we have $I_n(\theta;k_1,\ldots,k_n;u)\to 0$ $(u\to 1)$ for $\theta\in(-\pi,\pi)$. As in the proof of Lemma 6, it follows from (2.22) and (4.8) that

$$(4.11)\qquad i^{-1}I_n(\theta;k_1,\ldots,k_n;u)G(\theta;u) - J_n(\theta;k_1,\ldots,k_n;u)H(\theta;u)$$
$$= \sum_{m=0}^{\infty}\Big\{\sum_{\tau}C_\tau A_m(l_{\tau,1},\ldots,l_{\tau,N_\tau};a_\tau,b_\tau;u) + \sum_{j=0}^{k_n-1}\varepsilon_{j-k_n}(k_1,\ldots,k_{n-1};u)\phi(j-m;u)\lambda_{p+j+1}\Big\}\lambda_{p+m}\frac{(i\theta)^m}{m!},$$

since $\lambda_{p+j+1}\lambda_{j-m+1} = \lambda_{p+j+1}\lambda_{p+m}$. So we define

$$(4.12)\qquad \varepsilon_m(k_1,\ldots,k_{n-1},k_n;u) := -\sum_{\tau}C_\tau A_m(l_{\tau,1},\ldots,l_{\tau,N_\tau};a_\tau,b_\tau;u) - \sum_{j=0}^{k_n-1}\varepsilon_{j-k_n}(k_1,\ldots,k_{n-1};u)\phi(j-m;u)\lambda_{p+j+1}$$

for $m\in\mathbb{Z}$. Then it follows from (2.7), (2.9), (4.4)–(4.7), (4.9), (4.10) and (4.12) that (4.11) is uniformly convergent with respect to $u\in(1,1+\delta]$ when $\theta\in(-\pi,\pi)$, and that

$$(4.13)\qquad \liminf_{m\to\infty}\left(\frac{|\varepsilon_m(k_1,\ldots,k_n;u)|}{m!}\right)^{-1/m} \geq \pi,$$

$$(4.14)\qquad \varepsilon_m(k_1,\ldots,k_n;u)\lambda_{\sum_{j=1}^n(k_j+1)+m}\to 0 \qquad (u\to 1;\ m\in\mathbb{N}\cup\{0\}).$$

By (2.21) and (4.7), we can define

$$\varepsilon_m(k_1,\ldots,k_n;1) := \lim_{u\to 1}\varepsilon_m(k_1,\ldots,k_n;u)$$

for $m\in\mathbb{Z}$ with $m\leq -1$. For $k_{n+1}\in\mathbb{N}$ and $\mu\in\{0,1\}$, we define

$$\text{(4.15)}\quad \Gamma_{n+1,\mu}(\theta;k_1,\ldots,k_{n+1};u) := \Delta^{(k_{n+1})}(\Gamma_{n,\mu}(\theta;k_1,\ldots,k_n;u)) + \frac{1}{1+u}\sum_{j=0}^{k_n-1}\varepsilon_{j-k_n}(k_1,\ldots,k_{n-1};u)(-1)^j \times \lambda_{p+j+1}R_{p+k_{n+1}+1+\mu}(\theta;;k_{n+1},j;u),$$

where $p=\sum_{j=1}^{n}(k_j+1)$. In the same way as in the proof of Proposition 1, it follows from (4.1), (4.11), (4.12), Lemmas 3 and 5 that

$$\text{(4.16)}\quad \Gamma_{n+1,\mu}(\theta;k_1,\ldots,k_{n+1};u) = -\frac{1}{1+u}\sum_{m=-k_{n+1}}^{\infty}\varepsilon_m(k_1,\ldots,k_n;u)\lambda_{p+m+\mu}\frac{(i\theta)^{m+k_{n+1}}}{(m+k_{n+1})!}$$

for $\mu\in\{0,1\}$. By (4.1), (4.3), (4.7) and (4.15), we have

$$\text{(4.17)}\quad \Gamma_{n+1,\mu}(\theta;k_1,\ldots,k_{n+1};u) - R_{q+\mu}(\theta;k_1,\ldots,k_n;k_{n+1},0;u)\in V_n(q+\mu),$$

where $q=\sum_{j=1}^{n+1}(k_j+1)$. For $(k_1,\ldots,k_{n+1})\in\mathbb{N}^{n+1}$ with $k_{n+1}\geq 2$ and $q\equiv 0 \pmod 2$, let

$$\text{(4.18)}\quad h(\theta;k_1,\ldots,k_{n+1};u) := \Gamma_{n+1,0}(\theta;k_1,\ldots,k_{n+1};u) - R_0(\theta;k_1,\ldots,k_n;k_{n+1},0;u).$$

Then by (4.17), we have $h(\theta;k_1,\ldots,k_{n+1};u)\in V_n(0)$. By combining (4.16) and (4.18), we have

$$\text{(4.19)}\quad R_0(\theta;k_1,\ldots,k_n;k_{n+1},0;u)+h(\theta;k_1,\ldots,k_{n+1};u) = -\frac{1}{1+u}\sum_{m=-k_{n+1}}^{\infty}\varepsilon_m(k_1,\ldots,k_n;u)\lambda_{\sum_{j=1}^{n}(k_j+1)+m}\frac{(i\theta)^{m+k_{n+1}}}{(m+k_{n+1})!}.$$

By the condition $k_{n+1}\geq 2$ and by (4.13), we can let $\theta=\pi$ and $u\to 1$ on both sides of (4.19). Then by Lemma 2 and (4.14), we have

$$\begin{aligned} h(\pi;k_1,\ldots,k_{n+1};1) &= -\frac{1}{2}\sum_{m=-k_{n+1}}^{-1}\varepsilon_m(k_1,\ldots,k_n;1)\lambda_{\sum_{j=1}^{n}(k_j+1)+m}\frac{(i\pi)^{m+k_{n+1}}}{(m+k_{n+1})!}\\ &= -\frac{1}{2}\sum_{r=0}^{k_{n+1}-1}\varepsilon_{r-k_{n+1}}(k_1,\ldots,k_n;1)\lambda_{r+1}\frac{(i\pi)^r}{r!}\\ &= -\frac{1}{2}\sum_{j=0}^{[(k_{n+1}-2)/2]}\varepsilon_{2j+1-k_{n+1}}(k_1,\ldots,k_n;1)\frac{(i\pi)^{2j+1}}{(2j+1)!}. \end{aligned}$$

If we put $m = k_{n+1} - 2 \in \mathbb{N} \cup \{0\}$, then

$$(4.20)\qquad -\frac{1}{2}\sum_{j=0}^{[m/2]} \varepsilon_{2j-m-1}(k_1,\ldots,k_n;1)\,\frac{(i\pi)^{2j+1}}{(2j+1)!} = h(\pi;k_1,\ldots,k_n,m+2;1).$$

By the condition $h(\theta;k_1,\ldots,k_{n+1};u) \in V_n(0)$ and (2.14), we can see that $h(\pi;k_1,\ldots,k_n,m+2;1)/(i\pi) \in \mathcal{Z}_n$ for any $m \in \mathbb{N} \cup \{0\}$. By Lemma 7 with

$$P_m = -\frac{1}{2}\,\varepsilon_{-m-1}(k_1,\ldots,k_n;1)\lambda_{\sum_{j=1}^n(k_j+1)+1+m},$$
$$Q_m = \frac{1}{i\pi}\,h(\pi;k_1,\ldots,k_n,m+2;1)\lambda_{\sum_{j=1}^n(k_j+1)+1+m}$$

for $m \in \mathbb{N} \cup \{0\}$, we have

$$(4.21)\qquad \varepsilon_m(k_1,\ldots,k_n;1)\lambda_{\sum_{j=1}^n(k_j+1)+m} \in \mathcal{Z}_n \qquad (m \in \mathbb{Z} \text{ with } m \le -1).$$

By (4.13)–(4.16) and (4.21) the proof is complete. ■

By using this result, we give the proof of the Theorem in Section 1 as follows.

Proof of the Theorem. Since $\sum_{j=1}^n (k_j+1) = \sum_{j=1}^n k_j + n \equiv 1 \pmod 2$, the condition (4.3) with $\mu = 0$ gives

$$\Gamma_{n,0}(\theta;k_1,\ldots,k_n;u) - R_1(\theta;k_1,\ldots,k_{n-1};k_n,0;u) \in V_{n-1}(1).$$

Hence by Lemma 2, we have

$$(4.22)\qquad \Gamma_{n,0}(\pi;k_1,\ldots,k_n;1) - \zeta(k_1,\ldots,k_n) \in \mathcal{Z}_{n-1}.$$

On the other hand, by (4.5), (4.6) and the condition $k_n \ge 2$, we can let $\theta = \pi$ and $u \to 1$ on both sides of (4.4) when $\mu = 0$. Then by (4.7), we have

$$\begin{aligned}(4.23)\qquad \Gamma_{n,0}&(\pi;k_1,\ldots,k_n;1)\\ &= -\frac{1}{2}\sum_{m=-k_n}^{-1} \varepsilon_m(k_1,\ldots,k_{n-1};1)\lambda_{\sum_{j=1}^{n-1}(k_j+1)+m}\,\frac{(i\pi)^{m+k_n}}{(m+k_n)!}\\ &= -\frac{1}{2}\sum_{N=0}^{k_n-1} \varepsilon_{N-k_n}(k_1,\ldots,k_{n-1};1)\lambda_N\,\frac{(i\pi)^N}{N!}\\ &= -\frac{1}{2}\sum_{r=0}^{[(k_n-1)/2]} \varepsilon_{2r-k_n}(k_1,\ldots,k_{n-1};1)\,\frac{(i\pi)^{2r}}{(2r)!} \in \mathcal{Z}_{n-1}.\end{aligned}$$

By combining (4.22) and (4.23), we have $\zeta(k_1,\ldots,k_n) \in \mathcal{Z}_{n-1}$. Hence we obtain the proof of the Theorem. ■

Example. By the above method, we can give the evaluation formula for $\zeta(k_1,\dots,k_n)$ when n and $\sum_{j=1}^n k_j$ are of different parity. For example,

$$\begin{aligned}(4.24)\quad &\zeta(1,2,2k+1)\\ &= \frac{1}{2}\,\zeta(3,2k+1)+\frac{2k+1}{2}\,\zeta(2,2k+2)-\frac{1}{2}\,\zeta(1,2k+3)\\ &\quad+\frac{1}{2}\,\zeta(3)\zeta(2k+1)+\frac{k+1}{2}\,\zeta(2k+4)\\ &\quad-\sum_{j=0}^{k}\sum_{\mu=0}^{k-j}(2^{1-2k+2j+2\mu}-1)\zeta(2k-2j-2\mu)\\ &\quad\times\Big\{\zeta(2,2\mu+2)+\frac{1}{2}\,\zeta(2\mu+4)\Big\}\frac{(-1)^j\pi^{2j}}{(2j)!}\end{aligned}$$

for $k\in\mathbb{N}$. When $k=1$, we have

$$\begin{aligned}(4.25)\qquad \zeta(1,2,3)&=\frac{1}{2}\,\zeta(3,3)+\frac{3}{2}\,\zeta(2,4)-\frac{1}{2}\,\zeta(1,5)+\frac{1}{2}\,\zeta(3)^2+\zeta(6)\\ &\quad-\frac{\pi^2}{6}\Big\{\zeta(2,2)+\frac{1}{2}\,\zeta(4)\Big\}+\frac{1}{2}\Big\{\zeta(2,4)+\frac{1}{2}\,\zeta(6)\Big\}.\end{aligned}$$

It follows from the known results, for example, Ohno's formula in [11], that (4.25) is equivalent to

$$\zeta(1,2,3)=3\zeta(3)^2-\frac{29}{6480}\,\pi^6.$$

Further we obtain

$$\begin{aligned}\zeta(1,2,5)&=\frac{7}{2}\,\zeta(5)\zeta(3)-\frac{\pi^2}{6}\,\zeta(3)^2-\frac{289}{1360800}\,\pi^8+\frac{7}{4}\,\zeta(2,6),\\ \zeta(1,2,7)&=\frac{9}{2}\,\zeta(7)\zeta(3)+\frac{9}{4}\,\zeta(5)^2-\frac{\pi^4}{90}\,\zeta(3)^2-\frac{377}{5613300}\,\pi^{10}\\ &\quad+\frac{9}{4}\,\zeta(2,8)-\frac{\pi^2}{6}\,\zeta(2,6).\end{aligned}$$

Remark. We cannot evaluate $\zeta(k_1,\dots,k_n)$ in a closed form by our method when n and $\sum_{j=1}^n k_j$ are of the same parity. One of the keys to proving the Theorem is the fact that

$$\lim_{u\to 1} I_n(\theta;k_1,\dots,k_n;u)=0$$

(see (4.4) and (4.9)). In this case, it does not hold.

Added in proof. I learned from Masanobu Kaneko that the Theorem was proved (but apparently not published) several years ago by D. Zagier, in a totally different way. His proof is reproduced in the paper: *Derivation and double shuffle relations for multiple zeta values*, by K. Ihara, M. Kaneko and D. Zagier (in preparation).

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