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On Waring's problem: two cubes and two minicubes

by

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1. Introduction. Davenport proved in [3] that almost every natural number can be expressed as a sum of four positive integral cubes. It is now known, courtesy of Brüdern [1] and Wooley [11], that when N is sufficiently large, the number of positive integers at most N that fail to be written in such a way is slightly smaller than $N^{37/42}$. Since any integer congruent to 4 (mod 9) is never a sum of three cubes, the number of summands here cannot in general be reduced. A heuristic argument shows, however, that one of the four cubes is almost redundant. This motivates the work of Brüdern and Wooley (see [2]) on the representation of almost all positive integers as a sum of four cubes, one of which is small (henceforth we call this a *minicube*). They have shown that, with n being the natural number to be represented, such a minicube can be as small as $n^{5/36}$ without obstructing the existence of representations. This raises the question as to whether we can restrict not only one, but two (or even three) of the cubes in such a representation to be minicubes, and still get an almost all result. The purpose of this paper is to investigate representations of natural numbers by sums of four cubes, two of which are small.

When n is a positive integer and $\theta > 0$, write $r_\theta(n)$ for the number of solutions to the equation

$$n = x_1^3 + x_2^3 + y_1^3 + y_2^3, \tag{1.1}$$

where x_1, x_2, y_1, y_2 are natural numbers satisfying $y_1, y_2 \le n^\theta$. Plainly any one of these variables satisfying this equation must be at most $n^{1/3}$, so a trivial upper bound for θ is $1/3$. A formal application of the circle method suggests that when $1/6 < \theta < 1/3$, we should have

$$r_\theta(n) \sim \frac{\Gamma(4/3)^2}{\Gamma(2/3)} \mathfrak{S}(n) n^{2\theta - 1/3},$$

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with $\mathfrak{S}(n)$ being the familiar singular series associated with the representation of positive integers as sums of four cubes. Recalling the estimate $\mathfrak{S}(n) \gg 1$ (see Exercise 3 of Section 4.6 of [8]), we therefore anticipate that $r_\theta(n) \geq 1$ as long as n is large enough and $\theta > 1/6$. We establish this for almost all n, in §§2–5, for values of θ rather smaller than $2/9$.

Theorem 1.1. *Whenever* $192/869 \leq \theta \leq 2/9$, *we have* $r_\theta(n) \geq 1$ *for almost all integers* n.

In some sense, the sum of two cubes and two minicubes at most n^θ employed in the representation (1.1) carries the same weight as $2+6\theta$ cubes. Thus Theorem 1.1 asserts that almost every natural number n is the sum of at most 3.326 cubes.

In §§6–8, we establish the following asymptotic formula for $r_\theta(n)$.

Theorem 1.2. *Whenever* $1/4 < \theta < 1/3$, *the asymptotic formula*

$$r_\theta(n) = \frac{\Gamma(4/3)^2}{\Gamma(2/3)}\mathfrak{S}(n)n^{2\theta-1/3} + O(n^{2\theta-1/3}(\log n)^{-1})$$

holds for almost all positive integers n.

This result can be compared with Brüdern and Wooley's result (see Theorem 1.2 of [2]) on representations as sums of three cubes and a minicube. Our range of permissible values of θ is identical to that obtained in the latter paper.

We establish Theorems 1.1 and 1.2 using the Hardy–Littlewood method. We begin in §2 by laying the foundations for the application of this method. This leads to a lower bound for the contribution from the major arcs in §3. Some auxiliary mean value estimates vital to the proof of Theorem 1.1 are then introduced in §4. Bessel's inequality is used to relate the exceptional set to a minor arc estimate in §5. Following three pruning processes, the proof of Theorem 1.1 is complete. The derivation of the asymptotic formula in Theorem 1.2 is covered in §§6–8 and essentially follows by conventional means.

Throughout this paper, we use ε to denote an arbitrarily small positive constant. The implicit constants in Vinogradov's well-known notations $\ll$ and $\gg$ depend at most on ε. Whenever ε appears in a statement, either implicitly or explicitly, we assert that the statement is true for each $\varepsilon > 0$. Note that the "value" of ε changes from statement to statement. The letter ϖ always denotes a prime, and any variable denoted by the letter p (with or without subscripts) is a prime that is congruent to 2 (mod 3). As usual, we write $e(z) = e^{2\pi i z}$.

2. Existence of representations. We begin our proof of Theorem 1.1 by introducing the basic ingredients for the application of the Hardy–Littlewood method.

Fix a large integer N. Let θ be a positive number with $\theta \le 1/3$. Define

$$P = (N/4)^{1/3}, \quad R = P^{3\theta}, \quad Y = P^{11/79}, \quad L = (\log P)^{10}. \tag{2.1}$$

We take η to be a sufficiently small (but fixed) positive number, and then define the set of smooth numbers

$$\mathcal{A}(R) = \{m \in [1,R] \cap \mathbb{Z} : \varpi \mid m \Rightarrow \varpi \le R^\eta\}.$$

Also, when $\alpha \in [0,1)$, define the generating functions

$$f(\alpha) = \sum_{P<x\le 2P} e(\alpha x^3) \quad \text{and} \quad h(\alpha) = \sum_{y\in\mathcal{A}(R)} e(\alpha y^3). \tag{2.2}$$

When X and Z are positive numbers, define

$$\mathcal{A}^*(X,Z) = \{n \in \mathbb{Z} \cap [1,X] : \varpi \mid n \Rightarrow \varpi \le Z^\eta\}, \tag{2.3}$$

$$\mathcal{B}(X,Z) = \mathcal{A}^*(2X,Z) \setminus \mathcal{A}^*(X,Z). \tag{2.4}$$

Note that $\mathcal{A}(X) = \mathcal{A}^*(X,X)$. Fix $\tau > 0$ with the property that $\tau^{-1} > 852 + 16\sqrt{2833} \approx 1703.6$. Define $J = \lfloor \frac{1}{2}\tau \log P \rfloor$, and when $\alpha \in \mathbb{R}$, write

$$K(\alpha) = \sum_{2^{-J}Y<p\le Y} \sum_{w\in\mathcal{B}(P/p,2P/Y)} e(\alpha p^3 w^3). \tag{2.5}$$

For all $\theta > 0$ and integers n with $N < n \le 2N$, let $\rho_\theta(n)$ denote the number of integral solutions to the equation

$$n = x^3 + (pw)^3 + y_1^3 + y_2^3,$$

with

$$P < x \le 2P, \quad 2^{-J}Y < p \le Y, \quad w \in \mathcal{B}(P/p, 2P/Y), \quad y_1, y_2 \in \mathcal{A}(R).$$

It is apparent that $r_\theta(n) \ge \rho_\theta(n)$, and our goal is to establish a lower bound for $\rho_\theta(n)$ that produces the desired lower bound for $r_\theta(n)$. To this end, for any measurable subset $\mathfrak{B}$ of $[0,1)$, define

$$\rho_\theta(n;\mathfrak{B}) = \int_{\mathfrak{B}} f(\alpha)K(\alpha)h(\alpha)^2 e(-n\alpha)\,d\alpha. \tag{2.6}$$

By orthogonality, we have $\rho_\theta(n) = \rho_\theta(n;[0,1))$ for all integers n with $N < n \le 2N$.

We analyse this integral using the Hardy–Littlewood method. When $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $0 \le a \le q \le L$ and $(a,q) = 1$, define

$$\mathfrak{P}(q,a) = \{\alpha \in [0,1) : |\alpha - a/q| \le LN^{-1}\}. \tag{2.7}$$

In addition, for any positive number X, when $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $0 \le a \le q \le X$ and $(a,q) = 1$, define

$$\mathfrak{M}(q,a;X) = \{\alpha \in [0,1) : |q\alpha - a| \le XP^{-3}\}. \tag{2.8}$$

With this in mind, we define the *major arcs* $\mathfrak{P}$ to be the union of the arcs $\mathfrak{P}(q,a)$ with $a \in \mathbb{Z}$, $q \in \mathbb{N}$ satisfying $0 \le a \le q \le L$ and $(a,q) = 1$. Similarly, when $1 \leq X \leq N^{1/2}$, define the *major arcs* $\mathfrak{M}(X)$ to be the union of the arcs $\mathfrak{M}(q, a; X)$ with $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfying $0 \leq a \leq q \leq X$ and $(a, q) = 1$. Their respective complements in $[0, 1)$ are the *minor arcs* $\mathfrak{p}$ and $\mathfrak{m}(X)$. The major arcs $\mathfrak{P}$ are of central interest in our argument, with $\mathfrak{M}(X)$ employed as a tool for pruning the minor arcs $\mathfrak{p}$ later.

3. Major arc estimate. The familiar approach to estimating the major arc contribution $\rho_\theta(n; \mathfrak{P})$, which we largely follow, is to approximate the generating functions in the integrand of (2.6) by some suitably well-behaved functions. First, when $a \in \mathbb{Z}$ and $q \in \mathbb{N}$, let

$$S(q, a) = \sum_{r=1}^{q} e(ar^3/q). \tag{3.1}$$

Also, when β is a real number and Z is a positive number, write

$$v(\beta; Z) = \int_{Z}^{2Z} e(\beta\gamma^3)\, d\gamma. \tag{3.2}$$

In particular, write $v(\beta)$ for $v(\beta; P)$. Recall from Theorem 4.1 of [8] that when $\alpha \in \mathbb{R}$, $a \in \mathbb{Z}$ and $q \in \mathbb{N}$, we have

$$f(\alpha) = q^{-1}S(q, a)v(\alpha - a/q) + O(q^{1/2+\varepsilon}(1 + P^3|\alpha - a/q|)^{1/2}). \tag{3.3}$$

In particular, for all $\alpha \in \mathfrak{P}(q, a) \subseteq \mathfrak{P}$, one obtains from (2.7) the relation

$$f(\alpha) = q^{-1}S(q, a)v(\alpha - a/q) + O(L^{1+\varepsilon}). \tag{3.4}$$

Meanwhile, it follows from Lemma 8.5 of [10] that for all $\alpha \in \mathfrak{P}(q, a) \subseteq \mathfrak{P}$,

$$h(\alpha) = q^{-1}S(q, a)h(0) + O(RL^{-5}). \tag{3.5}$$

The analysis which leads to the asymptotic formula for $\tilde{K}(\alpha)$ preceding equation (4.4) in [2] shows that there exists a positive constant C with the property that for all $\alpha \in \mathfrak{P}(q, a) \subseteq \mathfrak{P}$, we have

$$K(\alpha) = Cq^{-1}S(q, a)v(\alpha - a/q) + O(PL^{-5}). \tag{3.6}$$

When $\beta \in \mathbb{R}$, define

$$u(\beta) = Ch(0)^2 v(\beta)^2. \tag{3.7}$$

Successive applications of (3.4), (3.5) and (3.6) then yield the relation

$$f(\alpha)K(\alpha)h(\alpha)^2 = (q^{-1}S(q, a))^4 u(\alpha - a/q) + O(P^2R^2L^{-5}) \tag{3.8}$$

for all $\alpha \in \mathfrak{P}(q, a)$. For all positive integers q and n, write

$$A(q, n) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} (q^{-1}S(q, a))^4 e(-an/q). \tag{3.9}$$

In addition, when n is a natural number, define

$$\mathfrak{S}(n;L)=\sum_{1\le q\le L}A(q,n), \tag{3.10}$$

$$J(n;L)=\int_{-L/N}^{L/N}u(\beta)e(-n\beta)\,d\beta. \tag{3.11}$$

From (2.7), the measure of $\mathfrak{P}$ is $O(L^3/N)$, so integrating both sides of (3.8) against $e(-n\alpha)$ over $\alpha\in\mathfrak{P}$, and appealing to (3.9)–(3.11), yields

$$\rho_\theta(n;\mathfrak{P})=\mathfrak{S}(n;L)J(n;L)+O(P^{-1}R^2L^{-2}). \tag{3.12}$$

Next recall the estimate

$$v(\beta)\ll P(1+P^3|\beta|)^{-1}, \tag{3.13}$$

obtained via integration by parts. This ensures that the singular integral

$$J(n)=\int_{-\infty}^{\infty}u(\beta)e(-n\beta)\,d\beta \tag{3.14}$$

converges absolutely and uniformly in n. Also, since $h(0)\ll R$, we deduce from (3.7), (3.11) and (3.13) that

$$J(n)-J(n;L)\ll R^2P^2\int_{L/N}^{\infty}(1+P^3\beta)^{-2}\,d\beta\ll R^2P^{-1}L^{-1}. \tag{3.15}$$

Finally, the singular integral can be evaluated as follows. On applying the methods outlined on pp. 21–22 of [4], it is readily seen that

$$\int_{-\infty}^{\infty}v(\beta)^2e(-\beta n)\,d\beta=\frac{1}{9}\int_{V(n)}\xi^{-2/3}(n-\xi)^{-2/3}\,d\xi,$$

where $V(n)$ is the interval of all real numbers ξ satisfying the inequalities $P^3\le\xi\le 8P^3$ and $P^3\le n-\xi\le 8P^3$. Consequently, by (3.7), (3.14) and the fact that $h(0)\gg R$, we infer that whenever n satisfies $N<n\le 2N$, we have

$$J(n)\gg R^2P^{-1}. \tag{3.16}$$

Meanwhile, Theorem 4.3 of [8] ensures that the singular series

$$\mathfrak{S}(n)=\sum_{q=1}^{\infty}A(q,n) \tag{3.17}$$

converges absolutely. Also, equation (1.3) of [5] shows that $1\ll\mathfrak{S}(n)\ll(\log\log n)^4$. In addition, the analysis of the singular series $\mathfrak{S}(n)$ provided in [2] demonstrates that

$$\mathfrak{S}(n)-\mathfrak{S}(n;L)\ll L^{-1/16} \tag{3.18}$$

for all but $O(NL^{-1/16})$ integers n with $N<n\le 2N$.

Equations (3.12), (3.15), (3.18), (3.16) and (2.1) together thus lead to the asymptotic lower bound

$$\rho_\theta(n;\mathfrak{P}) \gg \mathfrak{S}(n)n^{2\theta-1/3} + O(n^{2\theta-1/3}(\log n)^{-1/16}),$$

valid for all integers n with $N < n \le 2N$, with at most $O(N(\log N)^{-1/16})$ exceptions. We summarise this conclusion in the following proposition.

PROPOSITION 3.1. *For all but $O(N(\log N)^{-1/16})$ integers n with $N < n \le 2N$, we have $\rho_\theta(n;\mathfrak{P}) \gg n^{2\theta-1/3}$.*

4. Auxiliary estimates. We now establish several mean value estimates of generating functions that are required in the following section to evaluate the minor arc contribution $\rho_\theta(n;\mathfrak{p})$.

When $\alpha \in \mathbb{R}$ and $Q \ge 1$, write

$$g(\alpha) = \sum_{Q<y\le 2Q} e(\alpha y^3).$$

We record for future reference the following lemma.

LEMMA 4.1. *Whenever $R \le Q^{2/3}$, we have*

$$\int_0^1 |g(\alpha)^2 h(\alpha)^6|\, d\alpha \ll QR^{13/4-\tau}.$$

Proof. This is Lemma 2.1 of [2]. ∎

The above gives rise to the following corollaries.

COROLLARY 4.2. *Let*

$$T_1 = \int_0^1 |f(\alpha)^2 h(\alpha)^6|\, d\alpha \quad \text{and} \quad T_2 = \int_0^1 |K(\alpha)^2 h(\alpha)^6|\, d\alpha.$$

Then whenever $R \le P^{2/3}$, we have

$$T_2 \le T_1 \ll PR^{13/4-\tau}.$$

Proof. The second inequality is immediate from Lemma 4.1 on taking $Q = P$. It remains to establish the first inequality. We first show that every natural number m admits at most one representation in the form pw, where $2^{-J}Y < p \le Y$ and $w \in \mathcal{B}(P/p, 2P/Y)$. Indeed, suppose $p_1w_1 = p_2w_2$ with $2^{-J}Y < p_1, p_2 < Y$ and $w_i \in \mathcal{B}(P/p_i, 2P/Y)$ for $i = 1, 2$. If $p_1 \ne p_2$, then $p_1 \mid w_2$. Since $w_2 \in \mathcal{B}(P/p_2, 2P/Y)$, we see from (2.4) and (2.3) that $2^{-J}Y < p_1 \le (2P/Y)^\eta$. According to (2.1), this is absurd as long as η is small enough. Thus $p_1 = p_2$ and $w_1 = w_2$.

The first inequality in the corollary then follows by considering the diophantine equations associated with T_1 and T_2, noting from (2.4) and (2.3) that $pw > P$ for any pair (p, w) of variables in the sum (2.5) defining $K(\alpha)$, and finally using the observation in the last paragraph. ∎

We also quote the following useful lemma.

LEMMA 4.3. *Whenever* $1 \le Q \le P$, *we have*

$$\int_0^1 |f(\alpha)^2 g(\alpha)^4|\, d\alpha \ll P^{\varepsilon}(PQ^2 + P^{-1}Q^{9/2}).$$

Proof. This is the first estimate of Lemma 2.3 of [2]. ∎

The following result is a direct consequence of this lemma.

COROLLARY 4.4. *Whenever* $R \le P^{2/3}$, *we have the estimate*

$$\int_0^1 |K(\alpha)^2 h(\alpha)^4|\, d\alpha \ll P^{1+\varepsilon} R^2.$$

Proof. Using the argument of the proof of Corollary 4.2, we have

$$\int_0^1 |K(\alpha)^2 h(\alpha)^4|\, d\alpha \le \int_0^1 |f(\alpha)^2 h(\alpha)^4|\, d\alpha.$$

On considering the underlying diophantine equations, the desired conclusion follows from Lemma 4.3 and the assumption that $R \le P^{2/3}$. ∎

Next we define the mean value

$$U_1 = \int_0^1 |K(\alpha)|^8\, d\alpha. \tag{4.1}$$

By considering the underlying diophantine equation, it follows from Theorem 2 of [6] that

$$U_1 \ll P^5. \tag{4.2}$$

Introduce the function $f^* : [0,1) \to \mathbb{C}$ given by

(4.3)
$$f^*(\alpha) = \begin{cases} q^{-1}S(q,a)v(\alpha - a/q) & \text{when } \alpha \in \mathfrak{M}(q,a;P^{6/5}) \subseteq \mathfrak{M}(P^{6/5}), \\ 0 & \text{otherwise.} \end{cases}$$

Also, when $R \le X \le P^{6/5}$ and $t \ge 4$, let

$$U(X;t) = \int_{\mathfrak{M}(2X)\setminus\mathfrak{M}(X)} |f^*(\alpha)|^t\, d\alpha. \tag{4.4}$$

Finally, for all such t, define

$$U_2(t) = \int_{\mathfrak{M}(R)\setminus\mathfrak{P}} |f^*(\alpha)|^t\, d\alpha. \tag{4.5}$$

Then we have the following estimates.

Lemma 4.5. *Whenever $R \le X \le P^{6/5}$ and $t \ge 4$, we have*

$$U(X;t) \ll P^{t-3} X^{1-t/3}.$$

For all such t, we also have

$$U_2(t) \ll P^{t-3} L^{1-t/3}.$$

Proof. These two inequalities are established by the argument of Lemma 5.1 of [7]. ∎

When $\beta \in \mathbb{R}$ and Z is a positive number, write

$$w(\beta; Z) = \int_0^Z e(\beta\gamma^3)\, d\gamma. \tag{4.6}$$

For all such Z and $\alpha \in \mathbb{R}$, let

$$F^*(\alpha; Z) = \begin{cases} q^{-1} S(q,a) w(\alpha - a/q; 2P) & \text{when } \alpha \in \mathfrak{M}(q,a;Z), \\ 0 & \text{otherwise.} \end{cases} \tag{4.7}$$

For $\alpha \in \mathbb{R}$ and $\mathcal{B} \subseteq [1, R]$, let

$$j(\alpha; \mathcal{B}) = \sum_{x \in \mathcal{B}} e(\alpha x^3). \tag{4.8}$$

Finally, when $t, Z > 0$ and $\mathcal{B} \subseteq [1, R]$, put

$$U_3(t; Z; \mathcal{B}) = \int_{\mathfrak{M}(Z)} |f^*(\alpha)^t j(\alpha; \mathcal{B})^6|\, d\alpha, \tag{4.9}$$

$$U_4(t; Z; \mathcal{B}) = \int_{\mathfrak{M}(Z)} |F^*(\alpha; Z)^t j(\alpha; \mathcal{B})^6|\, d\alpha. \tag{4.10}$$

We give upper bounds for these two integrals in the following lemma.

Lemma 4.6. *Let $1 < Z \le R$ and $\mathcal{B} \subseteq [1, R]$. Then when $t > 2$, we have*

$$U_3(t; Z; \mathcal{B}) \ll_t P^{t-3} R^6.$$

On the other hand, when $t > 3$, we have the same upper bound for $U_4(t; Z; \mathcal{B})$.

Proof. This argument is largely akin to that given in the proof of Lemma 5.4 of [9]. Define the arithmetic function w multiplicatively by

$$w(\varpi^{3u+v}) = \begin{cases} 3\varpi^{-u-1/2} & \text{if } u \ge 0 \text{ and } v = 1, \\ \varpi^{-u-1} & \text{if } u \ge 0 \text{ and } v \in \{2, 3\}. \end{cases} \tag{4.11}$$

Then from [8, Lemmata 4.3, 4.4 and Theorem 4.2], we deduce that when $(a, q) = 1$, we have

$$q^{-1} S(q, a) \ll w(q). \tag{4.12}$$

When $\alpha \in \mathbb{R}$, $1 < Z \le R$, $\theta_1 > 2$ and $\theta_2 > 1$, let

(4.13)
$$\Upsilon(\alpha;\theta_1,\theta_2;Z) = \begin{cases} w(q)^{\theta_1}(1+P^3|\alpha - a/q|)^{-\theta_2} & \text{when } \alpha \in \mathfrak{M}(q,a;Z), \\ 0 & \text{otherwise.} \end{cases}$$

With θ_1, θ_2 and Z as above and $\mathcal{B} \subseteq [1,R]$, write

$$T(\theta_1,\theta_2;Z;\mathcal{B}) = \int_{\mathfrak{M}(Z)} |\Upsilon(\alpha;\theta_1,\theta_2;Z)j(\alpha;\mathcal{B})^6|\,d\alpha. \tag{4.14}$$

Equations (4.3), (4.12) and (3.13) imply that when $t > 2$, we have

$$\int_{\mathfrak{M}(Z)} |f^*(\alpha)^t j(\alpha;\mathcal{B})^6|\,d\alpha \ll P^t T(t,t;Z;\mathcal{B}). \tag{4.15}$$

On the other hand, Theorem 7.3 of [8] yields

$$w(\beta;U) \ll U(1+U^3|\beta|)^{-1/3} \tag{4.16}$$

for all positive numbers U. This, together with (4.7), (4.12), (4.13) and (4.14), gives rise to

$$\int_{\mathfrak{M}(Z)} |F^*(\alpha;Z)^t j(\alpha;\mathcal{B})^6|\,d\alpha \ll P^t T(t,t/3;Z;\mathcal{B}) \tag{4.17}$$

for all $t > 3$. Here we prove that when $\theta_1 > 2$, $\theta_2 > 1$, $1 < Z \le R$ and $\mathcal{B} \subseteq [1,R]$, we have

$$T(\theta_1,\theta_2;Z;\mathcal{B}) \ll_{\theta_1} P^{-3}R^6. \tag{4.18}$$

Substituting (4.13) and (4.8) into (4.14), and recalling the definition of $\mathfrak{M}(Z)$ given after (2.8), yields

$$T(\theta_1,\theta_2;Z;\mathcal{B}) \le \sum_{1\le q\le Z} w(q)^{\theta_1} \sum_{a=1}^{q} \int_{\mathfrak{M}(q,a;Z)} (1+P^3|\alpha - a/q|)^{-\theta_2} \Big|\sum_{y\in\mathcal{B}} e(y^3\alpha)\Big|^6\,d\alpha,$$

where we have removed the coprimality condition $(a,q) = 1$ in the a-summation. Making the change of variables $\alpha = a/q + \beta$ in the integral, we obtain

$$T(\theta_1,\theta_2;Z;\mathcal{B}) \le \sum_{1\le q\le Z} w(q)^{\theta_1} \int_{-\infty}^{\infty} (1+P^3|\beta|)^{-\theta_2} \sum_{a=1}^{q} \Big|\sum_{y\in\mathcal{B}} e(y^3(\beta + a/q))\Big|^6\,d\beta. \tag{4.19}$$

Here we have extended the range of integration from $|\beta| \le q^{-1}ZP^{-3}$ to the whole real line. This is valid as the completed integral converges absolutely.

When $\boldsymbol{y} = (y_1,\dots,y_6)$ with integral coordinates $y_1,\dots,y_6$, write

$$\Psi(\boldsymbol{y}) = y_1^3 - y_2^3 + y_3^3 - y_4^3 + y_5^3 - y_6^3. \tag{4.20}$$

Expanding the innermost sum in (4.19) and then swapping the a- and $\boldsymbol{y}$-sums, we see that

$$\sum_{a=1}^{q}\Big|\sum_{y\in\mathcal{B}} e(y^3(\beta+a/q))\Big|^6 = \sum_{y_1,\dots,y_6\in\mathcal{B}} e(\beta\Psi(\boldsymbol{y}))\sum_{a=1}^{q} e(a\Psi(\boldsymbol{y})/q).$$

The a-sum here is zero unless $q\,|\,\Psi(\boldsymbol{y})$, in which case it equals q. Thus

$$\sum_{a=1}^{q}\Big|\sum_{y\in\mathcal{B}} e(y^3(\beta+a/q))\Big|^6 = q\sum_{\substack{y_1,\dots,y_6\in\mathcal{B}\\ q|\Psi(\boldsymbol{y})}} e(\beta\Psi(\boldsymbol{y})). \tag{4.21}$$

Let $\rho(q)$ be the number of solutions to the congruence $\Psi(\boldsymbol{y}) \equiv 0 \pmod q$, under the constraint that each coordinate of $\boldsymbol{y}$ is a positive integer not exceeding q. A trivial estimate then gives

$$\sum_{\substack{y_1,\dots,y_6\in\mathcal{B}\\ q|\Psi(\boldsymbol{y})}} e(\beta\Psi(\boldsymbol{y})) \le \sum_{\substack{1\le y_1,\dots,y_6\le R\\ q|\Psi(\boldsymbol{y})}} 1 \ll (R/q+1)^6\rho(q). \tag{4.22}$$

By orthogonality, it follows from (3.1) and (4.20) that

$$q\rho(q) = \sum_{a=1}^{q}|S(q,a)|^6 = \sum_{a=1}^{q}(q,a)^6\Big|S\Big(\frac{q}{(q,a)},\frac{a}{(q,a)}\Big)\Big|^6.$$

Hence Theorem 4.2 of [8] yields

$$q\rho(q) \ll \sum_{a=1}^{q}(q,a)^6((q/(q,a))^{2/3})^6 = q^4\sum_{a=1}^{q}(q,a)^2 \ll q^6. \tag{4.23}$$

From (4.21)–(4.23) and the assumption that $Z \le R$, we see that the double sum within the integral in (4.19) has the asymptotic upper bound

$$q(R/q+1)^6\rho(q) \ll (R/q+1)^6q^6 = (R+q)^6 \ll R^6.$$

Inserting this estimate into (4.19), and evaluating the remaining integral in the same equation, we obtain

$$T(\theta_1,\theta_2;Z;\mathcal{B}) \ll P^{-3}R^6\sum_{1\le q\le Z} w(q)^{\theta_1}. \tag{4.24}$$

It thus remains to show that the sum here is uniformly bounded over Z, up to a multiplicative constant possibly depending on θ_1.

As w is defined to be a multiplicative function, in the analysis of the q-sum in (4.24) it suffices to evaluate the values of $w(\varpi^{3u+v})^{\theta_1}$ for all primes ϖ. From the definition (4.11) of w, we readily confirm that for any prime ϖ, we have the bounds

$$w(\varpi^{3u+v})^{\theta_1} \ll_{\theta_1} \begin{cases} \varpi^{-\theta_1 u-\theta_1/2} & \text{if } u\ge 0 \text{ and } v=1,\\ \varpi^{-\theta_1 u-\theta_1} & \text{if } u\ge 0 \text{ and } v\in\{2,3\}.\end{cases}$$

This reveals that for each ϖ we have

$$\sum_{h=1}^{\infty} w(\varpi^h)^{\theta_1} \ll_{\theta_1} \varpi^{-\theta_1/2}.$$

This ensures the existence of a positive number $A = A(\theta_1)$ for which the q-sum in (4.24) is bounded above by the Euler product

$$\prod_{\varpi\le Z}\Big(1+\sum_{h=1}^{\infty} w(\varpi^h)^{\theta_1}\Big) \ll \prod_{\varpi\le Z}(1+A\varpi^{-\theta_1/2}) \ll \prod_{\varpi}(1+\varpi^{-\theta_1/2})^A \ll_{\theta_1} 1.$$

The given condition $\theta_1 > 2$ validates the last inequality above. The desired inequality (4.18) follows immediately.

The lemma thus follows by putting (4.18) into (4.15) and (4.17) in turn, and recalling the definitions (4.9) and (4.10) of U_3 and U_4 respectively. ■

5. Minor arc estimate. On recalling (2.1), it follows from Proposition 3.1 that for almost all integers n with $N < n \le 2N$, we have

$$\rho_\theta(n;\mathfrak{P}) \gg P^{-1}R^2.$$

We therefore seek to show that the minor arc contribution $\rho_\theta(n;\mathfrak{p})$ is $o(P^{-1}R^2)$ for almost all such n. For any measurable subset $\mathfrak{B}$ of $[0,1)$, write

$$S(\mathfrak{B}) = \int_{\mathfrak{B}} |f(\alpha)^2K(\alpha)^2h(\alpha)^4|\,d\alpha. \tag{5.1}$$

An application of Bessel's inequality then gives

$$\sum_{N<n\le 2N} |\rho_\theta(n;\mathfrak{p})|^2 \le S(\mathfrak{p}). \tag{5.2}$$

If we can show that $S(\mathfrak{p}) = o(PR^4)$, the desired bound for $\rho_\theta(n;\mathfrak{p})$ then follows from a simple averaging argument. We first decompose $\mathfrak{p}$ into the components

$$\begin{aligned} \mathfrak{m} &= \mathfrak{m}(PY^3), & \mathfrak{D} &= \mathfrak{M}(PY^3)\setminus\mathfrak{M}(P^{6/5}),\\ \mathfrak{U} &= \mathfrak{M}(P^{6/5})\setminus\mathfrak{M}(R), & \mathfrak{A} &= \mathfrak{M}(R)\setminus\mathfrak{P}. \end{aligned} \tag{5.3}$$

Then evidently

$$S(\mathfrak{p}) = S(\mathfrak{m}) + S(\mathfrak{D}) + S(\mathfrak{U}) + S(\mathfrak{A}). \tag{5.4}$$

For any positive number Y, define

$$I(Y) = \int_{\mathfrak{m}} |f(\alpha)^2K(\alpha)^6|\,d\alpha.$$

It is then a consequence of Corollary 3.2 of [2] that when Y is chosen as in (2.1), we have the bound

$$I(Y) \ll P^{19/4-\tau/2}Y^{-3/4}. \tag{5.5}$$

Proposition 5.1. *As long as $\theta \le 2/9$, we have*

$$S(\mathfrak{m}) \ll PR^4L^{-1/50}.$$

Proof. An application of Hölder's inequality to (5.1) reveals that

$$S(\mathfrak{m}) \le I(Y)^{1/3}\Big(\int_0^1 |f(\alpha)^2h(\alpha)^6|\,d\alpha\Big)^{2/3}.$$

The restriction $\theta \le 2/9$ enables the application of Corollary 4.2 here. Together with (5.5) and (2.1), this gives

$$S(\mathfrak{m}) \ll (P^{19/4-\tau/2}Y^{-3/4})^{1/3}(PR^{13/4-\tau})^{2/3} = P^{175/79-\tau/6}R^{13/6-2\tau/3}.$$

Note that when $\theta \ge 192/869$, this bound is indeed sufficient for this proposition. ∎

Next we evaluate $S(\mathfrak{D})$. For any non-negative integer l, if the dyadic interval $(2^lP^{6/5}, 2^{l+1}P^{6/5}]$ lies within the interval $(P^{6/5}, PY^3]$, then $l < c\log P$, where $c = 86/(395\log 2)$. By introducing another dissection in the shape

$$\mathfrak{k}(X) = \mathfrak{M}(2X)\setminus\mathfrak{M}(X), \tag{5.6}$$

we have

$$\mathfrak{D} \subseteq \bigcup_{0\le l<c\log P} \mathfrak{k}(2^lP^{6/5}), \tag{5.7}$$

whence it suffices to consider $S(\mathfrak{k}(X))$ when $X \in [P^{6/5}, PY^3]$. With $\lambda = 3/34 - \tau/4$ and X as above, we record for future reference the bound

$$\int_{\mathfrak{k}(X)} |f(\alpha)^2K(\alpha)^5|\,d\alpha \ll P^{4+\lambda}Y^{-1-\lambda}(PY^3X^{-1})^{1/2}, \tag{5.8}$$

which is provided by equation (5.6) of [2]. This inequality is the ignition spark for the following lemma.

Lemma 5.2. *Whenever $P^{6/5} \le X \le PY^3$ and $\theta \le 2/9$, we have*

$$S(\mathfrak{k}(X)) \ll P^{13/6+\lambda/3+\varepsilon}Y^{1/6-\lambda/3}R^{13/6-2\tau/3}.$$

Proof. Equation (5.1) reveals that

$$S(\mathfrak{k}(X)) \le \Big(\sup_{\alpha\in\mathfrak{k}(X)}|f(\alpha)|\Big)^{1/3}\int_{\mathfrak{k}(X)} |f(\alpha)^{5/3}K(\alpha)^2h(\alpha)^4|\,d\alpha. \tag{5.9}$$

On recalling that $X \ge P^{6/5}$, successive applications of (3.3), (3.13), Theorem 4.2 of [8], (5.6) as well as the definition of $\mathfrak{m}(X)$ after (2.8), give rise to the bound

$$\sup_{\alpha\in\mathfrak{k}(X)}|f(\alpha)| \ll PX^{-1/3} + X^{1/2+\varepsilon} \ll X^{1/2+\varepsilon}.$$

This together with another application of Hölder's inequality to (5.9) leads to

$$S(\mathfrak{k}(X)) \ll \Big(X^{1/2+\varepsilon} \int\limits_{\mathfrak{k}(X)} |f(\alpha)^2 K(\alpha)^5|\, d\alpha\Big)^{1/3} T_2^{1/6} T_1^{1/2},$$

where T_1 and T_2 are defined in Corollary 4.2. Applications of that corollary as well as (5.8) thus yield

$$S(\mathfrak{k}(X)) \ll P^{\varepsilon}(X^{1/2}P^{4+\lambda}Y^{-1-\lambda}(PY^3X^{-1})^{1/2})^{1/3}(PR^{13/4-\tau})^{2/3}.$$

A modicum of computation confirms that this is indeed the bound in the statement of the lemma. ∎

The relation (5.7), together with Lemma 5.2, (2.1) and the value of λ given before (5.8), reveals that

$$\begin{aligned} S(\mathfrak{D}) &\le \sum_{0\le l<c\log P} S(\mathfrak{k}(2^l P^{6/5})) \ll P^{13/6+\lambda/3}Y^{1/6-\lambda/3}R^{13/6} \\ &= P^{175/79-17\tau/237}R^{13/6}. \end{aligned}$$

A modest calculation leads us to the following proposition.

Proposition 5.3. *Provided that* $192/869 \le \theta \le 2/9$, *we have*

$$S(\mathfrak{D}) \ll PR^4L^{-1/50}.$$

The treatment of $S(\mathfrak{U})$ is similar. For any non-negative integer l, if

$$(2^lR, 2^{l+1}R] \subseteq (R, P^{6/5}],$$

then l satisfies the constraint $2^{l+1}R \le P^{6/5}$. This together with (2.1) implies that $0 \le l < c_\theta \log P$, where $c_\theta = (6/5 - 3\theta)/\log 2$. It follows that

$$\mathfrak{U} \subseteq \bigcup_{0\le l<c_\theta \log P} \mathfrak{k}(2^lR). \tag{5.10}$$

This necessitates the estimation of $S(\mathfrak{k}(X))$ in the case where $R \le X \le P^{6/5}$. This is provided by the following lemma.

Lemma 5.4. *Whenever* $R \le X \le P^{6/5}$ *and* $\theta \le 2/9$, *we have*

$$S(\mathfrak{k}(X)) \ll P^{\varepsilon}(P^{7/3}R^{13/6-2\tau/3}X^{-5/12} + XPR^2).$$

Proof. By (3.3), (5.6) and (2.8), when $\alpha \in \mathfrak{k}(X)$, we have

$$f(\alpha) = f^*(\alpha) + O(X^{1/2+\varepsilon}),$$

where f^* is defined in (4.3). This together with (5.1) implies that

$$S(\mathfrak{k}(X)) \ll I_1(X) + X^{1+\varepsilon}\int_0^1 |K(\alpha)^2h(\alpha)^4|\, d\alpha, \tag{5.11}$$

where

$$I_1(X) = \int_{\mathfrak{k}(X)} |f^*(\alpha)^2 K(\alpha)^2 h(\alpha)^4| \, d\alpha.$$

By Corollary 4.4, when $\theta \le 2/9$, the second term in (5.11) is $O(XP^{1+\varepsilon}R^2)$. Meanwhile, an application of Hölder's inequality yields

$$I_1(X) \le T_2^{2/3} U_1^{1/12} U(X;8)^{1/4},$$

where T_2 is given in Corollary 4.2, and U_1 and $U(X;t)$ are defined respectively in (4.1) and (4.4). Successive applications of Lemma 4.5, Corollary 4.2 and (4.2) give rise to

$$I_1(X) \ll (PR^{13/4-\tau})^{2/3}(P^5)^{1/12}(P^5X^{-5/3})^{1/4} = P^{7/3}R^{13/6-2\tau/3}X^{-5/12}.$$

The lemma then follows by inserting this bound into (5.11). ∎

The relation (5.10) implies that

$$S(\mathfrak{U}) \le \sum_{0 \le l < c_\theta \log P} S(\mathfrak{k}(2^l R)).$$

Applying Lemma 5.4 to each term in the sum, and recalling (2.1), gives

$$\begin{aligned} S(\mathfrak{U}) &\ll P^\varepsilon \sum_{0 \le l < c_\theta \log P} (P^{7/3}R^{13/6-2\tau/3}(2^l R)^{-5/12} + (2^l R)PR^2) \\ &\ll P^\varepsilon (P^{7/3}R^{7/4-2\tau/3} + P^{11/5}R^2). \end{aligned}$$

A modest assumption on θ, together with (2.1), implies the following result.

Proposition 5.5. *Whenever* $1/5 < \theta \le 2/9$, *we have*

$$S(\mathfrak{U}) \ll PR^4 L^{-1/50}.$$

It remains to investigate $S(\mathfrak{A})$. Recall the definition (4.3) of f^*. By (3.3) and (2.8), when $\alpha \in \mathfrak{M}(R)$, we have

$$f(\alpha) = f^*(\alpha) + O(R^{1/2+\varepsilon}).$$

Using this and (5.3) in (5.1) gives

$$S(\mathfrak{A}) \ll R^{1+\varepsilon} \int_0^1 |K(\alpha)^2 h(\alpha)^4| \, d\alpha + \int_{\mathfrak{A}} |f^*(\alpha)^2 K(\alpha)^2 h(\alpha)^4| \, d\alpha.$$

Corollary 4.4 implies that when $\theta \le 2/9$, we have

$$S(\mathfrak{A}) \ll P^{1+\varepsilon}R^3 + \int_{\mathfrak{A}} |f^*(\alpha)^2 K(\alpha)^2 h(\alpha)^4| \, d\alpha. \tag{5.12}$$

An application of Hölder's inequality yields

$$\int_{\mathfrak{A}} |f^*(\alpha)^2 K(\alpha)^2 h(\alpha)^4|\,d\alpha \le U_2(16/3)^{1/12}\Big(\int_{\mathfrak{M}(R)} |f^*(\alpha)^{7/3} h(\alpha)^6|\,d\alpha\Big)^{2/3} U_1^{1/4}, \tag{5.13}$$

where U_1 and $U_2(t)$ are defined as in (4.1) and (4.5) respectively. According to (4.9), (4.8) and (2.2), the integral here is just $U_3(7/3;R;\mathcal{B})$ with $\mathcal{B} = \mathcal{A}(R)$. Applying Lemmata 4.5, 4.6 and (4.2) to (5.13) gives

$$\int_{\mathfrak{A}} |f^*(\alpha)^2 K(\alpha)^2 h(\alpha)^4|\,d\alpha \ll (P^{7/3}L^{-7/9})^{1/12}(P^{-2/3}R^6)^{2/3}(P^5)^{1/4} \ll PR^4L^{-7/108}.$$

Putting this back into (5.12), we obtain the following proposition.

Proposition 5.6. *For any positive number θ with $\theta \le 2/9$, we have*

$$S(\mathfrak{A}) \ll PR^4L^{-7/108}.$$

Propositions 5.1, 5.3, 5.5 and 5.6, together with (5.2) and (5.4), thus imply that whenever $192/869 \le \theta \le 2/9$, we have

$$\sum_{N<n\le 2N} |\rho_\theta(n;\mathfrak{p})|^2 \ll PR^4L^{-1/50}.$$

A simple averaging argument then reveals that for all except $O(NL^{-1/100})$ integers n with $N < n \le 2N$, we have

$$\rho_\theta(n;\mathfrak{p}) \ll P^{-1}R^2L^{-1/200}.$$

This together with (2.1) yields the following proposition.

Proposition 5.7. *Whenever* $192/869 \le \theta \le 2/9$, *we have*

$$\rho_\theta(n;\mathfrak{p}) \ll n^{2\theta-1/3}(\log n)^{-1/20}$$

for all integers n with $N < n \le 2N$, with at most $O(N(\log N)^{-1/10})$ exceptions.

Theorem 1.1 follows from Propositions 3.1 and 5.7, upon recalling that $r_\theta(n) \ge \rho_\theta(n)$, and finally by summing over dyadic intervals.

6. The asymptotic formula. Our goal in this section is to establish Theorem 1.2. Let N be a large integer, let $P = (N/4)^{1/3}$, and take R to be a parameter in the interval $[N^\theta, (2N)^\theta]$. Observe that if n is the integer in (1.1) with $N < n \le 2N$, then at least one of x_1, x_2, y_1 and y_2 is greater than P. Since $y_1, y_2 \le n^\theta$ with $\theta < 1/3$, neither y_1 nor y_2 exceeds P. So one of x_1 and x_2 is greater than P. For all integers n with $N < n \le 2N$, we thus define $\sigma_\theta(n)$ to be the number of solutions to (1.1) with

$$1 \le x_1, x_2 \le 2P, \quad \max\{x_1, x_2\} > P, \quad 1 \le y_1, y_2 \le R.$$

In our analysis of $\sigma_\theta(n)$, we adapt the argument which leads to equation (2.2) of [12]. When $\alpha \in [0,1)$, write

$$F(\alpha) = \sum_{1\le x\le 2P} e(\alpha x^3), \quad F_0(\alpha) = \sum_{1\le x\le P} e(\alpha x^3)$$

and

$$G(\alpha) = \sum_{1\le y\le R} e(\alpha y^3). \tag{6.1}$$

For any measurable set $\mathfrak{B} \subseteq [0,1)$, write

$$\sigma_\theta(n;\mathfrak{B}) = \int_{\mathfrak{B}} (F(\alpha)^2 - F_0(\alpha)^2)G(\alpha)^2 e(-n\alpha)\, d\alpha. \tag{6.2}$$

Then by orthogonality, we have $\sigma_\theta(n) = \sigma_\theta(n;[0,1))$ for all integers n with $N < n \le 2N$.

Take $L = (\log P)^{100}$. Recall the respective definitions (2.7) and (2.8) of $\mathfrak{P}$ and $\mathfrak{M}(X)$. For all integers a and q with $0 \le a \le q \le P^{3/4}$ and $(a,q) = 1$, introduce the major arc

$$\mathfrak{N}(q,a) = \mathfrak{M}(q,a;P^{3/4}). \tag{6.3}$$

Write $\mathfrak{N}$ for the union of all these major arcs $\mathfrak{N}(q,a)$, and let $\mathfrak{n} = [0,1)\setminus\mathfrak{N}$ be the corresponding minor arc.

7. Major arc estimate. As in §2, we first seek approximations for the generating functions in the integral defining $\sigma_\theta(n;\mathfrak{P})$. Recall the definition (4.6) of $w(\beta;Z)$. By Theorem 4.1 of [8], when $\alpha \in [0,1)$, $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a,q) = 1$, we have the relation

$$F(\alpha) = q^{-1}S(q,a)w(\alpha - a/q;2P) + O(q^{1/2+\varepsilon}(1 + P^3|\alpha - a/q|)^{1/2}). \tag{7.1}$$

In particular, whenever $\alpha \in \mathfrak{P}(q,a) \subseteq \mathfrak{P}$, we get

$$F(\alpha) = q^{-1}S(q,a)w(\alpha - a/q;2P) + O(L^{1+\varepsilon}). \tag{7.2}$$

By the same token, when $\alpha \in \mathfrak{P}(q,a)$, we have

$$F_0(\alpha) = q^{-1}S(q,a)w(\alpha - a/q;P) + O(L^{1+\varepsilon}), \tag{7.3}$$

$$G(\alpha) = q^{-1}S(q,a)w(\alpha - a/q;R) + O(L^{1+\varepsilon}). \tag{7.4}$$

When β is a real number, write

$$W(\beta) = (w(\beta;2P)^2 - w(\beta;P)^2)w(\beta;R)^2. \tag{7.5}$$

We deduce from (7.2)–(7.5) that

$$(F(\alpha)^2 - F_0(\alpha)^2)G(\alpha)^2 = (q^{-1}S(q,a))^4 W(\alpha - a/q) + O(P^2RL^{1+\varepsilon}). \tag{7.6}$$

Define $A(q,n)$ and $\mathfrak{S}(n;L)$ as in (3.9) and (3.10) respectively, and write

$$\mathcal{J}(n;L) = \int_{-L/N}^{L/N} W(\beta)e(-n\beta)\,d\beta. \tag{7.7}$$

Recall from (2.7) that the measure of $\mathfrak{P}$ is $O(L^3/N)$. Integrating both sides of (7.6) against $e(-n\alpha)$ over all $\alpha \in \mathfrak{P}$, and using (6.2), (3.10) and (7.7), thus gives

$$\sigma_\theta(n;\mathfrak{P}) = \mathfrak{S}(n;L)\mathcal{J}(n;L) + O(P^{-1}RL^{4+\varepsilon}). \tag{7.8}$$

Recall the definition (3.2) of $v(\beta;Z)$. From (7.5), we get

$$\begin{aligned} W(\beta) &= ((v(\beta;P) + w(\beta;P))^2 - w(\beta;P)^2)w(\beta;R)^2 \\ &= (v(\beta;P)^2 + 2v(\beta;P)w(\beta;P))w(\beta;R)^2. \end{aligned} \tag{7.9}$$

Combining this with (4.16), (3.13) and a trivial estimate for $w(\beta;R)$, we get

$$W(\beta) \ll P^2R^2(1+P^3|\beta|)^{-4/3} \tag{7.10}$$

for all real β. This confirms the absolute and uniform convergence over n of the singular integral

$$\mathcal{J}(n) = \int_{-\infty}^{\infty} W(\beta)e(-n\beta)\,d\beta. \tag{7.11}$$

Also, for all positive integers n, we deduce from (7.7) and (7.10) that

$$|\mathcal{J}(n) - \mathcal{J}(n;L)| \ll P^2R^2 \int_{L/N}^{\infty} (1+P^3\beta)^{-4/3}\,d\beta \ll P^{-1}R^2L^{-1/3}. \tag{7.12}$$

The value of the singular integral can be computed as follows. Putting (7.9) into (7.11) gives

$$\mathcal{J}(n) = \int_{-\infty}^{\infty} (v(\beta;P)^2 + 2v(\beta;P)w(\beta;P))w(\beta;R)^2 e(-n\beta)\,d\beta. \tag{7.13}$$

By a change of variables in (3.2) and (4.6), when Z is any positive number and β is real, we have

$$v(\beta;Z) = Zv(\beta Z^3;1) \quad \text{and} \quad w(\beta;Z) = Zw(\beta Z^3;1). \tag{7.14}$$

Putting these two equalities into (7.13), and replacing β by βP^{-3} in the same equation, we get

$$\begin{aligned} \mathcal{J}(n) &= P^{-1}R^2 \\ &\quad \times \int_{-\infty}^{\infty} (v(\beta;1)^2 + 2v(\beta;1)w(\beta;1))w(\beta R^3P^{-3};1)^2 e(-n\beta P^{-3})\,d\beta. \end{aligned} \tag{7.15}$$

Using a first order Taylor approximation, we have

$$w(\xi;1) = 1 + O(\min\{1, |\xi|\})$$

for all real ξ. Meanwhile, the inequalities (3.13) and (4.16) imply that

$$v(\beta;1)^2 + 2v(\beta;1)w(\beta;1) \ll |\beta|^{-4/3}$$

for all real β. Coupled with (7.15), these two relations give rise to the equality

$$\mathcal{J}(n) = P^{-1}R^2(\mathcal{J}^*(n) + E), \tag{7.16}$$

where

$$\mathcal{J}^*(n) = \int_{-\infty}^{\infty} (v(\beta;1)^2 + 2v(\beta;1)w(\beta;1))e(-n\beta P^{-3})\, d\beta \tag{7.17}$$

and

$$E \ll \int_{0}^{\infty} \beta^{-4/3} \min\{1, \beta R^3 P^{-3}\}\, d\beta \ll P^{-1}R. \tag{7.18}$$

Replacing β by βP^3 in (7.17), and using (7.14), we have

$$\mathcal{J}^*(n) = P(\mathcal{J}_1(n) + 2\mathcal{J}_2(n)), \tag{7.19}$$

where

$$\mathcal{J}_1(n) = \int_{-\infty}^{\infty} v(\beta;P)^2 e(-n\beta)\, d\beta,$$

$$\mathcal{J}_2(n) = \int_{-\infty}^{\infty} v(\beta;P)w(\beta;P)e(-n\beta)\, d\beta.$$

Using the methods outlined on pp. 21–22 of [4], we have

$$\mathcal{J}_1(n) = \left(\frac{1}{3}\right)^2 n^{-1/3} \int_a^b (\sigma(1-\sigma))^{-2/3}\, d\sigma,$$

where

$$a = \max\left\{P^3/n, 1 - 8P^3/n\right\} \quad \text{and} \quad b = \min\{1 - P^3/n, 8P^3/n\}.$$

On recalling that $P = (N/4)^{1/3}$ and $N < n \le 2N$, we check that $n < 9P^3$. This gives $a = P^3/n$ and $b = 1 - P^3/n$, hence

$$\mathcal{J}_1(n) = \left(\frac{1}{3}\right)^2 n^{-1/3} \int_{P^3/n}^{1-P^3/n} (\sigma(1-\sigma))^{-2/3}\, d\sigma. \tag{7.20}$$

In similar fashion, we obtain

$$\mathcal{J}_2(n) = \left(\frac{1}{3}\right)^2 n^{-1/3} \int_{1-P^3/n}^{1} (\sigma(1-\sigma))^{-2/3}\, d\sigma. \tag{7.21}$$

Combining (7.19), (7.20) and (7.21) gives

$$\mathcal{J}^*(n) = \left(\frac{1}{3}\right)^2 (Pn^{-1/3})\Big(\int_{P^3/n}^{1} (\sigma(1-\sigma))^{-2/3}\, d\sigma + \int_{1-P^3/n}^{1} (\sigma(1-\sigma))^{-2/3}\, d\sigma\Big).$$

Replacing σ by $1-\sigma$ in the latter integral then yields

$$\mathcal{J}^*(n) = \left(\frac{1}{3}\right)^2 (Pn^{-1/3}) \int_0^1 (\sigma(1-\sigma))^{-2/3}\, d\sigma = Pn^{-1/3}\Gamma(4/3)^2/\Gamma(2/3).$$

This, together with (7.16), (7.18) and our choice of parameters at the beginning of §6, gives rise to the asymptotic formula

$$\mathcal{J}(n) = \frac{\Gamma(4/3)^2}{\Gamma(2/3)} R^2 n^{-1/3}(1 + O(P^{-1}R)).$$

Meanwhile, the singular series $\mathfrak{S}(n)$ as defined in (3.17) stays absolutely and uniformly convergent. The estimate (3.18) remains true for all but $O(NL^{-1/16})$ integers n with $N < n \le 2N$. This together with (7.12) and (7.8) leads to the following proposition.

Proposition 7.1. *Let θ be a positive number with $\theta < 1/3$. Then for all but $O(N(\log N)^{-6})$ integers n with $N < n \le 2N$, we have*

$$\sigma_\theta(n; \mathfrak{P}) = \frac{\Gamma(4/3)^2}{\Gamma(2/3)} \mathfrak{S}(n) R^2 n^{-1/3} + O(P^{-1}R^2(\log N)^{-6}).$$

8. Minor arc estimate. As in §5, we wish to obtain a bound for the mean square value

$$\sum_{N<n\le 2N} |\sigma_\theta(n;\mathfrak{p})|^2,$$

which is $o(PR^4)$. When $\mathfrak{B}$ is a measurable subset of $[0,1)$, write

$$\Xi(\mathfrak{B}) = \int_{\mathfrak{B}} |(F(\alpha)^2 - F_0(\alpha)^2)^2 G(\alpha)^4|\, d\alpha. \tag{8.1}$$

An application of Bessel's inequality gives

$$\sum_{N<n\le 2N} |\sigma_\theta(n;\mathfrak{p})|^2 \le \Xi(\mathfrak{p}). \tag{8.2}$$

The decomposition $\mathfrak{p} = \mathfrak{n} \cup (\mathfrak{N} \setminus \mathfrak{P})$ implies that

$$\Xi(\mathfrak{p}) = \Xi(\mathfrak{n}) + \Xi(\mathfrak{N} \setminus \mathfrak{P}). \tag{8.3}$$

Define f as in (2.2). From the factorisation

$$F(\alpha)^2 - F_0(\alpha)^2 = f(\alpha)(F(\alpha) + F_0(\alpha)),$$

we can expand $\Xi(\mathfrak{n})$ in (8.1) as

$$\Xi(\mathfrak{n}) \ll \int_{\mathfrak{n}} |f(\alpha)^2 F(\alpha)^2 G(\alpha)^4|\, d\alpha + \int_{\mathfrak{n}} |f(\alpha)^2 F_0(\alpha)^2 G(\alpha)^4|\, d\alpha. \tag{8.4}$$

An application of Hölder's inequality yields

$$\int_{\mathfrak{n}} |f(\alpha)^2F(\alpha)^2G(\alpha)^4|\,d\alpha \le \Big(\sup_{\alpha\in\mathfrak{n}}|F(\alpha)|\Big)^2 \int_0^1 |f(\alpha)^2G(\alpha)^4|\,d\alpha. \tag{8.5}$$

A modified version of Weyl's inequality (see, for instance, Lemma 1 of [6]) confirms that

$$\sup_{\alpha\in\mathfrak{n}}|F(\alpha)| \ll P^{3/4+\varepsilon}. \tag{8.6}$$

Hence (8.5), (8.6) and Lemma 4.3 imply that

$$\begin{aligned}\int_{\mathfrak{n}} |f(\alpha)^2F(\alpha)^2G(\alpha)^4|\,d\alpha &\ll (P^{3/4})^2P^{\varepsilon}(PR^2+P^{-1}R^{9/2})\\ &= P^{\varepsilon}(P^{5/2}R^2+P^{1/2}R^{9/2}).\end{aligned}$$

An almost identical argument yields the same upper bound for the other integral in (8.4). Hence whenever $1/4<\theta<1/3$, we have

$$\Xi(\mathfrak{n}) \ll P^{\varepsilon}(P^{5/2}R^2+P^{1/2}R^{9/2}) \ll PR^4L^{-1/10}. \tag{8.7}$$

As for the contribution from the set of arcs $\mathfrak{N}\setminus\mathfrak{P}$, first note that (8.1) gives

$$\Xi(\mathfrak{N}\setminus\mathfrak{P}) \ll \int_{\mathfrak{N}\setminus\mathfrak{P}} |F(\alpha)^4G(\alpha)^4|\,d\alpha + \int_{\mathfrak{N}\setminus\mathfrak{P}} |F_0(\alpha)^4G(\alpha)^4|\,d\alpha. \tag{8.8}$$

Recalling the respective definitions (4.7) and (6.3) of F^* and $\mathfrak{N}(q,a)$, we deduce from (7.1) that

$$F(\alpha) = F^*(\alpha;P^{3/4}) + O(P^{3/8+\varepsilon})$$

for all $\alpha\in\mathfrak{N}$. This together with an application of Hua's lemma thus yields

$$\begin{aligned}&\int_{\mathfrak{N}\setminus\mathfrak{P}} |F(\alpha)^4G(\alpha)^4|\,d\alpha\\ &\quad\ll \int_{\mathfrak{N}\setminus\mathfrak{P}} |F^*(\alpha;P^{3/4})^4G(\alpha)^4|\,d\alpha + O\Big(P^{3/2+\varepsilon}\int_0^1 |G(\alpha)|^4\,d\alpha\Big)\\ &\quad= \int_{\mathfrak{N}\setminus\mathfrak{P}} |F^*(\alpha;P^{3/4})^4G(\alpha)^4|\,d\alpha + O(P^{3/2+\varepsilon}R^2).\end{aligned} \tag{8.9}$$

An application of Hölder's inequality yields

$$\begin{aligned}&\int_{\mathfrak{N}\setminus\mathfrak{P}} |F^*(\alpha;P^{3/4})^4G(\alpha)^4|\,d\alpha\\ &\quad\le \Big(\int_{\mathfrak{N}} |F^*(\alpha;P^{3/4})^{7/2}G(\alpha)^6|\,d\alpha\Big)^{2/3}\Big(\int_{\mathfrak{N}\setminus\mathfrak{P}} |F^*(\alpha;P^{3/4})|^5\,d\alpha\Big)^{1/3}.\end{aligned} \tag{8.10}$$

From (4.8), (4.10) and (6.1), the first integral here is $U_4(7/2, P^{3/4}; \mathcal{B})$ with $\mathcal{B} = [1, R] \cap \mathbb{Z}$. Now that $\theta > 1/4$, the assumptions $P = (N/4)^{1/3}$ and $N^\theta < R \le (2N)^\theta$, made in §6, imply that $P^{3/4} < R$. This allows us to apply Lemma 4.6, producing the bound

$$\int_{\mathfrak{N}} |F^*(\alpha; P^{3/4})^{7/2} G(\alpha)^6| \, d\alpha \ll P^{1/2} R^6. \tag{8.11}$$

Now the argument of Lemma 5.1 of [7] provides the bound

$$\int_{\mathfrak{N}\setminus\mathfrak{P}} |F^*(\alpha; P^{3/4})|^5 \, d\alpha \ll P^2 L^{-2/3}. \tag{8.12}$$

Inserting the estimates (8.11) and (8.12) into (8.10), and using (8.9), we obtain

$$\begin{aligned}\int_{\mathfrak{N}\setminus\mathfrak{P}} & |F(\alpha)^4 G(\alpha)^4| \, d\alpha \\ & \ll (P^{1/2} R^6)^{2/3} (P^2 L^{-2/3})^{1/3} + P^{3/2+\varepsilon} R^2 \ll P R^4 L^{-2/9}.\end{aligned}$$

A parallel argument yields the same upper bound for the remaining integral in (8.8). We therefore obtain

$$\Xi(\mathfrak{N} \setminus \mathfrak{P}) \ll P R^4 L^{-2/9}.$$

This, together with (8.2), (8.3) and (8.7), implies the following.

PROPOSITION 8.1. *As long as* $1/4 < \theta < 1/3$, *we have*

$$\sum_{N<n\le 2N} |\sigma_\theta(n; \mathfrak{p})|^2 \ll P R^4 L^{-1/10}.$$

We conclude this section with a proof of Theorem 1.2. A simple averaging argument from the above proposition, along with our choice of parameters in §6, implies that for such θ, the inequality

$$|\sigma_\theta(n; \mathfrak{p})| \ll P^{-1} R^2 L^{-1/40} \ll P^{-1} R^2 (\log N)^{-2} \tag{8.13}$$

holds for all integers n with $N < n \le N + N(\log N)^{-2}$, with $O(N(\log N)^{-5})$ exceptions. Coupled with the conclusion of Proposition 7.1, the upper bound (8.13) implies that

$$\sigma_\theta(n) = \sigma_\theta(n; [0,1)) = \frac{\Gamma(4/3)^2}{\Gamma(2/3)} \mathfrak{S}(n) R^2 n^{-1/3} + O(P^{-1} R^2 (\log N)^{-2}) \tag{8.14}$$

for all but $O(N(\log N)^{-5})$ integers n with $N < n \le N + N(\log N)^{-2}$. For all such n and θ, we have

$$N^\theta < n^\theta \le N^\theta + O(N^\theta (\log N)^{-2}),$$

whence there exists a positive constant A such that

$$N^\theta < n^\theta \le N^\theta + AN^\theta(\log N)^{-2}.$$

Recall the definition (1.1) of $r_\theta(n)$ as well as equation (1.3) of [5], which gives $1 \ll \mathfrak{S}(n) \ll (\log\log n)^4$. Putting first $R = N^\theta$ and then $R = N^\theta + AN^\theta(\log N)^{-2}$ into (8.14), we arrive at the relation

$$r_\theta(n) = \frac{\Gamma(4/3)^2}{\Gamma(2/3)}\mathfrak{S}(n)N^{2\theta-1/3} + O(N^{2\theta}n^{-1/3}(\log N)^{\varepsilon-2}).$$

Hence when $1/4 < \theta < 1/3$, the asymptotic formula

$$r_\theta(n) = \frac{\Gamma(4/3)^2}{\Gamma(2/3)}\mathfrak{S}(n)n^{2\theta-1/3} + O(n^{2\theta-1/3}(\log n)^{-1})$$

holds for all but $O(N(\log N)^{-5})$ integers n with $N < n \le N + N(\log N)^{-2}$. There are altogether $O((\log N)^3)$ intervals of such type that cover $[1, 2N]$. Summing over all such intervals leads to the same asymptotic formula with the total number of exceptions encountered being $O(N(\log N)^{-2})$. The conclusion of Theorem 1.2 thus holds for all real numbers θ in the range $(1/4, 1/3)$.

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