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The non-abelian normal CM-fields of degree 36 with class number one

by

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1. Introduction. A. M. Odlyzko proved that there are only finitely many normal CM-fields with class number one ([O]). J. Hoffstein proved that their degrees are less than or equal to 436 ([H]). Yamamura has determined all the imaginary abelian number fields with class number one ([Y]). According to [LLO] and [P] there are seven non-abelian normal CM-fields of degree 24 with class number one. We determine all normal CM-fields of degree $4p^2$ with class number one, where p is an odd prime number.

Theorem 1. (1) *If K is a non-abelian normal CM-field of degree $4p^2$ with relative class number one, then $p = 3$ and the Galois group of K is isomorphic to $D_{12} \times C_3$, the direct product of the dihedral group of order* 12 *and the cyclic group of order* 3.

(2) *There are three non-abelian normal CM-fields K of degree* 36 *with relative class number one, namely the composita $K = EMk$ given in Table* 1 *below, where $E = \mathbb{Q}(\alpha_E)$ with $P_E(\alpha_E) = 0$ is a non-normal totally real cubic field of discriminant d_E, M is an imaginary biquadratic bicyclic field $M = \mathbb{Q}(\sqrt{-m_0}, \sqrt{-m_1})$, and k is a cyclic cubic field of conductor f_k. Moreover, these three fields have class number one.*

Table 1

d_E	$P_E(X)$	(m_0, m_1)	f_k	h_{K^+}
756	$X^3 - 6X - 2$	$(3, 7)$	9	1
1620	$X^3 - 12X - 14$	$(3, 15)$	9	1
4312	$X^3 + X^2 - 16X - 8$	$(2, 11)$	7	1

The class numbers h_{K^+} of the maximal totally real subfields K^+ are obtained by using Kash ([K]). Throughout this paper the following notations will be used. For a number field K we let h_K, d_K, O_K, ω_K, and ζ_K denote

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the class number, the absolute value of the discriminant, the ring of algebraic integers, the number of roots of unity in K, and the Dedekind zeta function of K, respectively. If K is a CM-field, we let h_K^-, K^+, and $Q_K \in \{1,2\}$ denote the relative class number, the maximal totally real subfield, and the Hasse unit index of K, respectively (see [W]). We let $\mathfrak{F}_{K_1/K_2}$ denote the conductor of an abelian extension K_1/K_2. For a primitive Hecke character χ we let $\mathfrak{F}_\chi$ denote the finite part of its conductor.

2. Normal CM-fields of degree $4p^2$

PROPOSITION 1. *Let K be a non-abelian normal CM-field of degree $4p^2$. Then either K is a dihedral CM-field of degree $4p^2$, or K is a dicyclic CM-field of degree $4p^2$, or $K = N_1N_2$ is a compositum of two normal CM-fields N_1 and N_2 of degree $4p$ such that $M = N_1 \cap N_2$ is an imaginary abelian quartic field* (*notice that each N_i is either abelian*, *or dihedral*, *or dicyclic*).

Proof. Let S_p be a p-Sylow subgroup of G the Galois group of K over $\mathbb{Q}$, M the fixed subfield of S_p, and let $c \in G$ denote the complex conjugation. Since c is in the center of G (see [LOO, Lemma 2]), the fixed subfield M^+ of the subgroup $\langle c, S_p\rangle$ is a quadratic field contained in M. Hence, if M were not normal then its normal closure $\widehat{M}$ would be a dihedral octic subfield of the normal field K of degree $4p^2$. A contradiction. Therefore, M is a normal quartic field, hence an imaginary abelian quartic field, and S_p is a normal subgroup of G. Since S_p is of order p^2, it is abelian. If S_p is cyclic, then K/M^+ is cyclic of degree $2p^2$ and $K/\mathbb{Q}$ is either dihedral or dicyclic.

Now, assume that $S_p = C_p \times C_p$ is not cyclic. Let S_2 be a 2-Sylow subgroup of G. Since S_2 acts on the set of the $p+1$ subgroups C of order p of S_p by $s_2 \cdot C = s_2 C s_2^{-1}$ for $s_2 \in S_2$, the number n of such subgroups of order p which are invariant under the action of S_2 satisfies $p+1 \equiv n \pmod 2$ and $n \neq 0$, hence is ≥ 2. Therefore, there exist at least two distinct normal subgroups G_1 and G_2 of order p in G. By denoting N_i the fixed subfield of G_i, we obtain the desired result. ∎

PROPOSITION 2. (1) (Theorem 5 of [LOO]) *Let $k \subseteq K$ be two CM-fields. If $[K : k]$ is odd*, *then $Q_K = Q_k$ and h_k^- divides h_K^-.*

(2) (Theorem 7 of [LOO]) *There is no dicyclic CM-field of degree $4p$ with relative class number one.*

(3) (Proposition 5.2 of [Lef]) *If N is a CM-field of Galois group $D_{2^m p^2}$, the dihedral group of order $2^m p^2$, with $m \geq 2$, then $h_N^- > 1$.*

COROLLARY 1. *If K is a normal CM-field of degree $4p^2$ with relative class number one*, *then* $\mathrm{Gal}(K/\mathbb{Q})$ *is isomorphic to $C_p \times D_{4p}$.*

Proof. According to Proposition 2 if K is either a dihedral CM-field of degree $4p^2$ or a dicyclic CM-field of degree $4p^2$, then $h_K^- > 1$. If $K = N_1N_2$

is a compositum of two normal CM-fields N_1 and N_2 of degree $4p$ such that $N_1 \cap N_2$ is an abelian quartic field and if any one of N_i's is dicyclic, then $h_K^- > 1$. If an imaginary abelian number field has relative class number one, then its degree over $\mathbb{Q}$ is not greater than 24; therefore K cannot be an abelian number field ([CK]). Using [LOO] and [Lef] we verify that there is no pair of dihedral CM-fields N_1 and N_2 of degree $4p$ such that $N_1 \cap N_2$ is a quartic number field and $h_{N_1}^- = h_{N_2}^- = 1$. In conclusion, K is a compositum of a dihedral CM-field N_1 of degree $4p$ and an abelian number field N_2 of degree $4p$ such that $N_1 \cap N_2$ is an imaginary bicyclic biquadratic number field. ∎

3. Normal CM-fields with Galois group $C_p \times D_{4p}$. Let K be a normal CM-field of degree $4p^2$ and Galois group $C_p \times D_{4p}$. Then $K = N_1 N_2$ is a compositum of a dihedral CM-field N_1 of degree $4p$ and of an imaginary abelian field N_2 of degree $4p$ which are both cyclic extensions of an imaginary biquadratic field $M = \mathbb{Q}(\sqrt{-m_0}, \sqrt{-m_1})$. If $h_K^- = 1$ then $h_{N_1}^- = h_{N_2}^- = 1$, by Proposition 2(1). Hence, $p = 3$ by [Lef, Th. 4.1] and [CK, Table I]. Thus Theorem 1(1) is proved.

Let us now proceed with the proof of Theorem 1(2). According to [Lef, Th. 4.1] or [LOO, Table 1] (in which the 15th entry should be $[(8505, X^3 - 27X - 51)\ (7, 15)\ 2]$) and according to [CK, Table I], K must be one of the 13 composita $K = N_1 N_2$ of degree 36 given in Table 2 below, where $N_1 = EM$ is a dihedral CM-field of degree 12, $N_2 = kM$ an imaginary abelian field of degree 12, $E = \mathbb{Q}(\alpha_E)$ with $P_E(\alpha_E) = 0$ is a totally real cubic field, k a real cyclic cubic field, and $M = \mathbb{Q}(\sqrt{-m_0}, \sqrt{-m_1})$ an imaginary biquadratic field. In Table 2 we denote by L the real quadratic subfield of M so that $\mathrm{Gal}(K/L) = C_2 \times C_3 \times C_3$ and use the following Dirichlet characters to describe the cyclic cubic fields k:

$$\chi_7(3) = e^{2i\pi/6}, \qquad \psi_9(2) = e^{2i\pi/3}, \qquad \chi_{13}(2) = e^{2i\pi/12}.$$

Now, we use the following proposition to prove that 3^2 divides h_K^- for the seven CM-fields numbered 1, 3, 5, 6, 7, 9 and 10 in Table 2:

PROPOSITION 3 (see Proposition 8 of [LOO]). *Let N/M be a cyclic extension of degree p of CM-fields. Assume that N^+/M^+ is a cyclic extension of degree p. Let t be the number of prime ideals of M^+ which split in M/M^+ and are ramified in N^+/M^+. Then $p^{t-1} h_M^-$ divides h_N^-.*

For example let us consider the field N numbered 1. Then $L = \mathbb{Q}(\sqrt{3})$, 7 is inert in L, the ideal $7O_L$ splits completely in N_1, and $7O_L$ is ramified in N_2^+. Namely, there are at least 3 prime ideals which split in N_1/N_1^+ and are ramified in K^+/N_1^+. Hence $3^2 \mid h_N^-$. In the last column of Table 2 we list the appropriate prime ideals. The prime ideal $\mathfrak{p}_7$ is the prime ideal lying above 7 in L.

Table 2

No.	d_E	$P_E(X)$	(m_0, m_1)	k	Prime ideals
1	2700	$X^3-15X-20$	$(1,3)$	χ_7^2	$7O_L$
2				ψ_9	
3	756	X^3-6X-2	$(3,7)$	χ_7^2	$\mathfrak{p}_7$
4				ψ_9	
5				χ_{13}^4	$13O_L$
6				$\chi_7^4\psi_9$	$\mathfrak{p}_7$
7	1620	$X^3-12X-14$	$(3,15)$	χ_7^2	$7O_L$
8				ψ_9	
9	1944	X^3-9X-6	$(2,3)$	χ_7^2	$7O_L$
10				χ_{13}^4	$13O_L$
11	4312	$X^3+X^2-16X-8$	$(2,11)$	χ_7^2	
12	8505	$X^3-27X-51$	$(7,15)$	χ_7^2	
13				ψ_9	

In the next section we will compute the relative class numbers of the six remaining fields numbered 2, 4, 8, 11, 12 and 13 in Table 2, thus proving Theorem 1(2) (see the results of our computations in Table 3 below).

4. Computation of relative class numbers. Let N_3/M and N_4/M be two other cubic subextensions of the abelian extension K/M. Then N_3 and N_4 are non-normal isomorphic CM-fields. Therefore, $h_{N_4}^- = h_{N_3}^-$.

Now, using abelian L-series associated with the abelian extension K/L, one can easily prove that

$$(*) \qquad \zeta_K/\zeta_{K^+} = \Big(\prod_{i=1}^{4} \zeta_{N_i}/\zeta_{N_i^+}\Big)/(\zeta_M/\zeta_L)^3.$$

Since $[K:M]$ and $[N_i:M]$ for $1\le i\le 4$ are odd, we have $Q_{N_i}=Q_K=Q_M$, h_M^- divides h_K^- and each $h_{N_i}^-$ for $1\le i\le 4$. Since $\omega_K=\omega_{N_2}$ and $\omega_{N_i}=\omega_M$ for $1\le i\le 3$, $(*)$ yields (see [Lou1])

$$(**) \qquad h_K^-/h_M^- = \prod_{i=1}^{4} h_{N_i}^-/h_M^-.$$

Finally, let χ denote any one of the two sextic characters associated with the sextic cyclic extension N_3/L. Then

$$(***) \qquad h_{N_4}^-/h_M^- = h_{N_3}^-/h_M^- = \frac{d_L N_{L/\mathbb{Q}}(\mathfrak{F}_\chi)}{(2\pi)^4}|L(1,\chi)|^2 = \left|\frac{1}{4}L(0,\chi)\right|^2.$$

Notice that if $h_K^- = 1$, then $h_M^- = h_{N_1}^- = h_{N_2}^- = 1$. Hence, according to $(**)$ and $(***)$ we deduce that $h_K^- = (h_{N_3}^-)^2$ is a perfect square. Now we can take $\chi = \psi_1\psi_2\chi_M$ where ψ_1, ψ_2 and χ_M are the primitive abelian Hecke characters associated with the abelian extensions N_1^+/L, N_2^+/L, and M/L, respectively. It remains to evaluate $L(1,\chi) = L(1,\psi_1\psi_2\chi_M)$.

THEOREM 2 ([Lou1, Theorem 4]). *Let $\gamma = 0.577215\ldots$ denote Euler's constant and let B be positive. Set*

$$K_1(B) = 1 + 4\sum_{n\geq 0}\left(\gamma + \log B - \frac{1}{2n+2} - \sum_{k=1}^{n}\frac{1}{k}\right)\frac{B^{2n+2}}{(2n+2)(n!)^2}$$

and

$$K_2(B) = \pi B + 4\sum_{n\geq 0}\left(\gamma + \log B - \frac{1}{2n+1} - \sum_{k=1}^{n}\frac{1}{k}\right)\frac{B^{2n+2}}{(2n+1)(n!)^2}.$$

Then $B > 0$ implies $0 \leq K_2(B) \leq K_1(B) \leq 2e^{-B}$.

Let $L(s,\chi) = \sum_{n\geq 1} a_n(\chi)/n^s$ be a Hecke L-series associated with a primitive Hecke character on a ray class group of a real quadratic number field L. Assume that χ is ramified at the infinite places ∞_1 and ∞_2 of L. Let $\infty_1\infty_2\mathfrak{F}_\chi$ be the conductor of χ, set $f_\chi = N_{L/\mathbb{Q}}(\mathfrak{F}_\chi)$ and $A_\chi = \sqrt{d_L f_\chi/\pi^2}$. Let W_χ be the Artin root number associated with this L-series, i.e., let W_χ be the complex number of absolute value 1 such that

$$F(1-s,\chi) = W_\chi F(s,\overline{\chi})$$

where $F(s,\chi) = A_\chi^s \Gamma^2((s+1)/2)L(s,\chi)$.

Then we have the following absolutely convergent series expansion:

$$L(1,\chi) = \sum_{n\geq 1}\frac{a_n(\chi)}{n}K_1(n/A_\chi) + W_\chi\sum_{n\geq 1}\frac{\overline{a_n(\chi)}}{n}K_2(n/A_\chi).$$

If we let $S_{M,\chi}$ denote the value obtained by disregarding the indices $n > M$ in the above series, then

$$|L(1,\chi) - S_{M,\chi}| \leq 4(\log(Me)+2)^2 e^{-M/A_\chi}.$$

It remains to compute $a_n(\chi)$ and W_χ.

4.1. *Computation of $\chi(I) = \psi_1\psi_2\chi_M(I)$ for non-zero integral ideals I.* For any Hecke character χ there exists a character $\widehat{\chi}$ on $(O_L/\mathfrak{F}_\chi)^*$ such that for any $\alpha \neq 0$ in O_L, $\chi((\alpha)) = \nu(\alpha)\widehat{\chi}(\alpha)$, where $\nu(\alpha)$ is the sign of $N_{L/\mathbb{Q}}(\alpha)$. This character $\widehat{\chi}$ is called the modular character associated with χ. We describe explicitly the modular character $\widehat{\psi_2}$ for the six fields of degree 36 given in Table 2. (For ψ_1 and χ_M see [Lou1, Section 4.2] and [Lef, Section 2].)

(1) $K = N_1N_2$: $N_1 = E(\sqrt{-1}, \sqrt{-3})$ with $d_E = 2^2 \cdot 3^3 \cdot 5$, $N_2 = k(\sqrt{-1}, \sqrt{-3})$ with $f_k = 3^2$, and $L = \mathbb{Q}(\sqrt{3})$. We have $\mathfrak{F}_{N_1^+/L} = (15)$, $\mathfrak{F}_{M/L} = \infty_1\infty_2$, and $\mathfrak{F}_{N_2^+/L} = (3\sqrt{3})$. Then $(O_L/(3\sqrt{3}))^* = \{\varepsilon_L^i 4^j \mid 0 \leq i \leq 5,\ 0 \leq j \leq 2\} \simeq \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, $\widehat{\psi}_2(\varepsilon_L) = 1$, and $\widehat{\psi}_2(4) = \zeta_3$, where $\varepsilon_L = 2 + \sqrt{3}$.

(2) $K = N_1N_2$: $N_1 = E(\sqrt{-3}, \sqrt{-7})$ with $d_E = 2^2 \cdot 3^3 \cdot 7$, $N_2 = k(\sqrt{-3}, \sqrt{-7})$ with $f_k = 3^2$, and $L = \mathbb{Q}(\sqrt{21})$. We have $\mathfrak{F}_{N_1^+/L} = (6)$, $\mathfrak{F}_{M/L} = \infty_1\infty_2$, and $\mathfrak{F}_{N_2^+/L} = \mathfrak{p}^3$ with $(3) = \mathfrak{p}^2$ and $\mathfrak{p} = \left(\frac{3+\sqrt{21}}{2}\right)$. Then $(O_L/\mathfrak{p}^3)^* = \{(-\varepsilon_L)^i 4^j \mid 0 \leq i \leq 5, 0 \leq j \leq 2\} \simeq \mathbb{Z}/6\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$, $\widehat{\psi}_2(-\varepsilon_L) = 1$, and $\widehat{\psi}_2(4) = \zeta_3$, where $\varepsilon_L = (5 + \sqrt{21})/2$.

(3) $K = N_1N_2$: $N_1 = E(\sqrt{-3}, \sqrt{-15})$ with $d_E = 2^2 \cdot 3^4 \cdot 5$, $N_2 = k(\sqrt{-3}, \sqrt{-15})$ with $f_k = 3^2$, and $L = \mathbb{Q}(\sqrt{5})$. We have $\mathfrak{F}_{N_1^+/L} = (18)$, $\mathfrak{F}_{M/L} = \infty_1\infty_2(3)$, and $\mathfrak{F}_{N_2^+/L} = (3)^2$. Then $(O_L/(3)^2)^* = \{\varepsilon_L^i 4^j \mid 0 \leq i \leq 23, 0 \leq j \leq 2\} \simeq \mathbb{Z}/24\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, $\widehat{\psi}_2(\varepsilon_L) = 1$, $\widehat{\psi}_2(4) = \zeta_3$, where $\varepsilon_L = (1 + \sqrt{5})/2$.

(4) $K = N_1N_2$: $N_1 = E(\sqrt{-2}, \sqrt{-11})$ with $d_E = 2^3 \cdot 7^2 \cdot 11$, $N_2 = k(\sqrt{-2}, \sqrt{-11})$ with $f_k = 7$, and $L = \mathbb{Q}(\sqrt{22})$. We have $\mathfrak{F}_{N_1^+/L} = (7)$, $\mathfrak{F}_{M/L} = \infty_1\infty_2$, $\mathfrak{F}_{N_2^+/L} = (7)$, and $(7) = \mathfrak{p}_7\mathfrak{p}_7'$ with $\mathfrak{p}_7 = 7\mathbb{Z} + (1 + \sqrt{22})\mathbb{Z}$ and $\mathfrak{p}_7' = 7\mathbb{Z} + (1 - \sqrt{22})\mathbb{Z}$. If $\alpha = x_\alpha + y_\alpha\sqrt{22}$, then $\alpha \equiv x_\alpha - y_\alpha \pmod{\mathfrak{p}_7}$ and $\alpha \equiv x_\alpha - y_\alpha \pmod{\mathfrak{p}_7'}$. Let ϕ be a character of order 3 on $(O_L/\mathfrak{p}_7)^*$. Then $\widehat{\psi}_2(\alpha) = \phi(\alpha\alpha')$, where $\alpha' = x_\alpha - y_\alpha\sqrt{22}$.

(5) $K = N_1N_2$: $N_1 = E(\sqrt{-7}, \sqrt{-15})$ with $d_E = 3^5 \cdot 5 \cdot 7$, $N_2 = k(\sqrt{-7}, \sqrt{-15})$ with $f_k = 7$, and $L = \mathbb{Q}(\sqrt{105})$. We have $\mathfrak{F}_{N_1^+/L} = (9)$, $\mathfrak{F}_{M/L} = \infty_1\infty_2$, $\mathfrak{F}_{N_2^+/L} = \mathfrak{p}_7$, and $(7) = \mathfrak{p}_7^2$, where $\mathfrak{p}_7 = 7\mathbb{Z} + \frac{7+\sqrt{105}}{2}\mathbb{Z}$. For any $\alpha = (x_\alpha + y_\alpha\sqrt{105})/2$ coprime to $\mathfrak{p}_7$ we have $\alpha \equiv (x_\alpha - 7y_\alpha)/2 \pmod{\mathfrak{p}_7}$. Let n_α be an integer such that $n_\alpha \equiv (x_\alpha - 7y_\alpha)/2 \pmod 7$. Then $\widehat{\psi}_2(\alpha) = \zeta_3^k$, where $k = \min\{j \geq 0 : n_\alpha \equiv 3^j \pmod 7\}$.

(6) $K = N_1N_2$: $N_1 = E(\sqrt{-7}, \sqrt{-15})$ with $d_E = 3^5 \cdot 5 \cdot 7$, $N_2 = k(\sqrt{-3}, \sqrt{-15})$ with $f_k = 3^2$, and $L = \mathbb{Q}(\sqrt{105})$. Then $\mathfrak{F}_{N_1^+/L}$ and $\mathfrak{F}_{M/L}$ are the same as (5). We have $\mathfrak{F}_{N_2^+/L} = \mathfrak{p}_3^3$ with $(3) = \mathfrak{p}_3^2$, $(O_L/\mathfrak{p}_3^3)^* = \{\varepsilon_L^i 4^j \mid 0 \leq i \leq 5,\ 0 \leq j \leq 2\} \simeq \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, $\widehat{\psi}_2(\varepsilon_L) = 1$, and $\widehat{\psi}_2(4) = \zeta_3$, where $\varepsilon_L = 41 + 4\sqrt{105}$.

4.2. *Computation of* $a_n(\chi)$. Set $\chi = \psi_1\psi_2\chi_M$. Since $n \mapsto a_n(\chi) = \sum_{N_{L/\mathbb{Q}}(I)=n} \chi(I)$ is multiplicative we only have to explain how to compute $a_{l^k}(\chi)$ on powers of primes ([Lou1]). Let L_0 and L_1 be two imaginary quadratic subfields of M and let $\chi_{L_0}, \chi_{L_1}, \chi_L$ be quadratic Dirichlet characters associated with L_0, L_1, and L, respectively.

Set $f = \operatorname{lcm}(N_{L/\mathbb{Q}}(\mathfrak{F}_{N_1^+/L}),\, N_{L/\mathbb{Q}}(\mathfrak{f}_{M/L}),\, N_{L/\mathbb{Q}}(\mathfrak{F}_{N_2^+/L}))$ and

$$\eta_l = \begin{cases} 0 & \text{if } \chi_{L_0}(l) = \chi_{L_1}(l) = 0, \\ -1 & \text{if } \chi_L(l) \neq -1 \text{ and if either } \chi_{L_0}(l) = -1 \text{ or } \chi_{L_1}(l) = -1, \\ 1 & \text{otherwise,} \end{cases}$$

where $\mathfrak{f}_{M/L}$ denotes the finite part of the conductor $\mathfrak{F}_{M/L}$. If $l \mid f$, then $a_{l^k}(\chi) = 0$. Assume that $(l, f) = 1$.

(a) If $\chi_L(l) = -1$, then $\psi_1(l) = \chi_M(l) = 1$ and

$$a_{l^k}(\chi) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \psi_2(l)^{k/2} & \text{if } k \text{ is even.} \end{cases}$$

(b) If $\chi_L(l) = 0$ and $(l) = \mathfrak{L}^2$, then $\psi_1(\mathfrak{L}) = 1$ and $a_{l^k}(\chi) = \eta_l^k \psi_2(\mathfrak{L})^k$.

(c) If $\chi_L(l) = 1$ and $(l) = \mathfrak{L}\mathfrak{L}'$, then $1 = \psi_1((l)) = \psi_1(\mathfrak{L})\psi_1(\mathfrak{L}')$ and $\psi_2((l)) = \psi_2(\mathfrak{L})\psi_2(\mathfrak{L}')$. We have

$$\begin{aligned} a_{l^k}(\chi) &= \sum_{i=0}^{k} \chi(\mathfrak{L})^i \chi(\mathfrak{L}')^{k-i} \\ &= \eta_l^k \sum_{i=0}^{k} \psi_1(\mathfrak{L})^i \psi_2(\mathfrak{L})^i \psi_1(\mathfrak{L}')^{k-i} \psi_2(\mathfrak{L}')^{k-i} \\ &= \eta_l^k \sum_{i=0}^{k} \psi_1(\mathfrak{L})^{2i-k} \psi_2((l))^{k-i} \psi_2(\mathfrak{L})^{2i-k}. \end{aligned}$$

Then $\mathfrak{L}^{h_L} = (\alpha)$ for some integer. We have $\psi_1(\mathfrak{L}) = \widehat{\psi}_1(\alpha)^{h'}$ and $\psi_2(\mathfrak{L}) = \widehat{\psi}_2(\alpha)^{h'}$ if $(h_L, 3) = 1$ and $h_L h' \equiv 1 \pmod 3$.

4.3. *Numerical computation of approximations of the Artin root numbers.* We recall Louboutin's ideas for computing approximations of the Artin root numbers (see [Lou1, Section 5]). Set

$$G(x, \chi) = \frac{1}{2i\pi} \int_{\alpha - i\infty}^{\alpha + i\infty} F(s, \chi) x^{-s}\, ds \qquad \text{with } x > 0 \text{ and } \alpha > 1$$

and

$$H(B) = -4 \sum_{n \geq 0} \frac{B^{2n+1}}{(n!)^2} \left(\gamma + \log B - \sum_{k=1}^{n} \frac{1}{k} \right) \qquad \text{for } B > 0$$

(where $\gamma = 0.577\ldots$ denotes Euler's constant). Then

$$G(x, \chi) = \sum_{n \geq 1} a_n(\chi) H(nx/A_\chi)$$

(where A_χ is as in Theorem 2), and we can therefore easily compute good approximations of $G(1, \chi)$. If we can deduce from these approximations that

$G(1,\chi) \neq 0$, then since

$$G(x,\chi) = \frac{W_\chi}{x} G(1/x, \overline{\chi}),$$

we have $W_\chi = G(1,\chi)/\overline{G(1,\chi)}$ and obtain good approximations of W_χ which can be used, together with Theorem 2, to compute good approximations of the integer h_K^-, hence to compute its exact value. According to our numerical computations for the six fields given in Table 2 we have $G(1,\chi) \neq 0$ and we obtain Table 3.

Table 3

d_E	(m_0, m_1)	f_k	W_χ	W_χ^6	$L(1,\chi)$	$L(0,\chi)/4$	h_K^-
2700	$(1,3)$	9	$-0.642787\ldots$ $-0.766044\ldots i$	$\frac{1-\sqrt{-3}}{2}$	$1.013996\ldots$ $-0.564536\ldots i$	$\frac{-1-3\sqrt{-3}}{2}$	7^2
756	$(3,7)$	9	$-0.642787\ldots$ $-0.766044\ldots i$	$\frac{1-\sqrt{-3}}{2}$	$0.816374\ldots$ $-0.143948\ldots i$	$\frac{-1-\sqrt{-3}}{2}$	1
1620	$(3,15)$	9	$-0.766044\ldots$ $+0.642787\ldots i$	$\frac{-1+\sqrt{-3}}{2}$	$0.921696\ldots$ $+0.335470\ldots i$	$\frac{-1+\sqrt{-3}}{2}$	1
4312	$(2,11)$	7	$0.386513\ldots$ $+0.922283\ldots i$	$\frac{71+39\sqrt{-3}}{2\cdot 7^2}$	$0.232372\ldots$ $+0.554478\ldots i$	1	1
8505	$(7,15)$	7	$-0.895953\ldots$ $-0.444148\ldots i$	$\frac{-13+3\sqrt{-3}}{14}$	$1.325166\ldots$ $+0.969708\ldots i$	$-10+\sqrt{-3}$	103^2
		9	$0.766044\ldots$ $+0.642787\ldots i$	$\frac{-1-\sqrt{-3}}{2}$	$1.788302\ldots$ $+0.532666\ldots i$	$4+\sqrt{-3}$	19^2

REMARK. Since $h_M^- = h_{N_1}^- = h_{N_2}^- = 1$ for those six fields, we have $h_K^- = (h_{N_3}^-/h_M^-)^2 = \left|\frac{1}{4}L(0,\chi)\right|^4$. According to Siegel–Klingen's Theorem ([Hid, Cor. 1 in §2.5]) $L(0,\chi) \in \mathbb{Q}(\sqrt{-3})$. Using [Lou2] we can determine explicitly $L(0,\chi)$. Moreover, we can prove that $W_\chi^6 \in \mathbb{Q}(\sqrt{-3})$.

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