ACTA ARITHMETICA
169.1 (2015)

On a conjecture of Sárközy and Szemerédi

by

YONG-GAO CHEN and JIN-HUI FANG (Nanjing)

Two infinite sequences A, B of non-negative integers are called *infinite additive complements* if their sum contains all sufficiently large integers. For a set T of non-negative integers, let $T(x)$ be the counting function of T. That is, $T(x) = |T \cap [0, x]|$.

It is easy to see that, for infinite additive complements A, B, we have

$$\liminf_{x\to\infty} \frac{A(x)B(x)}{x} \geq 1.$$

In 1994, Sárközy and Szemerédi [14] proved the following deep result which was conjectured by Danzer in 1964 ([2], see also [5, p. 10] and [9, p. 75]).

THEOREM (Sárközy and Szemerédi, 1994). *For infinite additive complements A, B, if*

$$\limsup_{x\to\infty} \frac{A(x)B(x)}{x} \leq 1, \tag{0.1}$$

then

$$A(x)B(x) - x \to \infty \quad \text{as } x \to \infty. \tag{0.2}$$

Sárközy and Szemerédi [14, p. 245] posed the following conjecture.

CONJECTURE 0.1. *There exist infinite additive complements A, B satisfying* (0.1) *such that*

$$A(x)B(x) - x = O(\min\{A(x), B(x)\}). \tag{0.3}$$

In this paper, we disprove this conjecture. In fact, the following stronger result is proved.

2010 *Mathematics Subject Classification*: Primary 11B13, 11B34; Secondary 05A17.
Key words and phrases: additive complements, sequences, counting functions.

DOI: 10.4064/aa169-1-3

Theorem 0.2. *For infinite additive complements A, B, if* (0.1) *holds, then, for any given $M > 1$, we have*

$$A(x)B(x) - x \geq (\min\{A(x), B(x)\})^M$$

for all sufficiently large integers x.

For related results, one may refer to [1], [6], [7], [8], [10], [12] and [13].

1. Preliminary lemmas

Lemma 1.1 (Narkiewicz [11]). *For infinite additive complements A, B, if* (0.1) *holds, then either*

$$\lim_{x\to\infty} \frac{A(2x)}{A(x)} = 1 \quad \text{or} \quad \lim_{x\to\infty} \frac{B(2x)}{B(x)} = 1.$$

Lemma 1.2. *Let $S = \{s_1, s_2, \dots\}$ and $T = \{t_1, t_2, \dots\}$ be finite sequences of integers, and let $r(S,T,n)$ denote the number of solutions $n = s_i + t_j$, $s_i \in S$, $t_j \in T$, and $\delta(S,T,n)$ denote the number of solutions $n = t_j - s_i$, $s_i \in S$, $t_j \in T$. Then*

$$\Big(\sum_{r(S,T,n)\geq 1} (r(S,T,n) - 1)\Big)^2 \geq \sum_{\delta(S,T,n)\geq 1} (\delta(S,T,n) - 1).$$

Proof. Let

$$\begin{aligned} M_1 = \{(i_1, j_1, i_2, j_2) : s_{i_1}, s_{i_2} \in S,\ t_{j_1}, t_{j_2} \in T,\ i_1 \neq i_2 \text{ or } j_1 \neq j_2, \\ s_{i_1} + t_{j_1} = s_{i_2} + t_{j_2}\}, \\ M_2 = \{(i_1, j_1, i_2, j_2) : s_{i_1}, s_{i_2} \in S,\ t_{j_1}, t_{j_2} \in T,\ i_1 \neq i_2 \text{ or } j_1 \neq j_2, \\ t_{j_2} - s_{i_1} = t_{j_1} - s_{i_2}\}. \end{aligned}$$

Then $M_1 = M_2$ and

$$\begin{aligned} |M_1| &= \sum_n r(S,T,n)(r(S,T,n) - 1) \\ &= \sum_{r(S,T,n)\geq 1} (r(S,T,n) - 1)^2 + \sum_{r(S,T,n)\geq 1} (r(S,T,n) - 1), \\ |M_2| &= \sum_n \delta(S,T,n)(\delta(S,T,n) - 1) \\ &= \sum_{\delta(S,T,n)\geq 1} (\delta(S,T,n) - 1)^2 + \sum_{\delta(S,T,n)\geq 1} (\delta(S,T,n) - 1). \end{aligned}$$

It is clear that

$$\Big(\sum_{r(S,T,n)\ge 1}(r(S,T,n)-1)\Big)^2 \ge \sum_{r(S,T,n)\ge 1}(r(S,T,n)-1)^2$$
$$\ge \frac{1}{2}\Big(\sum_{r(S,T,n)\ge 1}(r(S,T,n)-1)^2+\sum_{r(S,T,n)\ge 1}(r(S,T,n)-1)\Big)=\frac{1}{2}|M_1|$$
$$=\frac{1}{2}|M_2|=\frac{1}{2}\Big(\sum_{\delta(S,T,n)\ge 1}(\delta(S,T,n)-1)^2+\sum_{\delta(S,T,n)\ge 1}(\delta(S,T,n)-1)\Big)$$
$$\ge \sum_{\delta(S,T,n)\ge 1}(\delta(S,T,n)-1). \ \blacksquare$$

Remark. Similarly,

$$\Big(\sum_{\delta(S,T,n)\ge 1}(\delta(S,T,n)-1)\Big)^2 \ge \sum_{r(S,T,n)\ge 1}(r(S,T,n)-1).$$

2. Proof of Theorem 0.2. We will prove the following general theorem.

Theorem 2.1. *Let A and B be infinite additive complements such that* (0.1) *holds. Suppose that h is a function on $(0,\infty)$ satisfying:*

(a) *$h(x)\to\infty$ as $x\to\infty$;*
(b) *$h(\min\{A(x),B(x)\})\le \frac{2}{3}\sqrt{x}$ for all sufficiently large integers x.*

Then

$$A(x)B(x)-x\ge h(\min\{A(x),B(x)\}) \tag{2.1}$$

for all sufficiently large integers x.

Firstly we derive Theorem 0.2 from Theorem 2.1. Suppose that Theorem 2.1 is true. Take $h(x)=x^M$. By Lemma 1.1, we may assume that

$$\lim_{x\to\infty}\frac{A(2x)}{A(x)}=1.$$

Then $A(x)\le x^{1/(2M+2)}$ for all sufficiently large x. Thus

$$h(\min\{A(x),B(x)\})\le h(A(x))=A(x)^M\le x^{M/(2M+2)}<\tfrac{2}{3}\sqrt{x}$$

for all sufficiently large x. Now Theorem 0.2 follows from Theorem 2.1.

Proof of Theorem 2.1. Let $f_x(n)$ be the number of solutions of $a+b=n$, $a\in A$, $a\le x$, $b\in B$ and $b\le x$. Since A, B are infinite additive complements, we have

$$f_x(n)\ge 1,\qquad n_0\le n\le x.$$

Hence

$$A(x)B(x) \geq x - n_0. \tag{2.2}$$

By (0.1) and (2.2), we have

$$\lim_{x\to\infty} \frac{A(x)B(x)}{x} = 1. \tag{2.3}$$

By Lemma 1.1, we may assume that

$$\lim_{x\to\infty} \frac{A(2x)}{A(x)} = 1. \tag{2.4}$$

By (2.3) and (2.4), we have

$$\lim_{x\to\infty} \frac{B(2x)}{B(x)} = \lim_{x\to\infty} \frac{B(2x)A(2x)}{2x} \frac{2x}{A(x)B(x)} \frac{A(x)}{A(2x)} = 2. \tag{2.5}$$

By (2.4) and (2.5),

$$A(x) < x^{1/4}, \qquad B(x) > x^{3/4} \tag{2.6}$$

for all sufficiently large x. Then

$$\min\{A(x), B(x)\} = A(x)$$

for all sufficiently large x.

If (2.1) does not hold, then

$$A(x)B(x) - x < h(A(x)) \tag{2.7}$$

for infinitely many positive integers x.

Now we cancel the multiplicities of B (B is a sequence, and some integers may appear in B many times). Let B' be the set of all integers of B. Then B' can be seen as a strictly increasing sequence. Thus $B'(\ell+1) \leq B'(\ell)+1$ for all integers ℓ. By (2.3), we have $B(x) < \infty$ for all $x > 0$. This implies that each integer appears in B at most finitely many times. So B' is an infinite set.

Since the sum of A and B contains all sufficiently large integers, it follows that so does the sum of A and B'. That is, A and B' are also infinite additive complements. It is clear that

$$\limsup_{x\to\infty} \frac{A(x)B'(x)}{x} \leq \limsup_{x\to\infty} \frac{A(x)B(x)}{x} \leq 1. \tag{2.8}$$

Similar to (2.3), we have

$$\lim_{x\to\infty} \frac{A(x)B'(x)}{x} = 1. \tag{2.9}$$

By (2.4) and (2.9), as in (2.5),

$$\lim_{x\to\infty} \frac{B'(2x)}{B'(x)} = 2. \tag{2.10}$$

By (2.4) and (2.10), we find that

$$A(x) < x^{1/4}, \qquad B'(x) > x^{3/4} \tag{2.11}$$

for all sufficiently large x. Then $\min\{A(x), B'(x)\} = A(x)$ for all sufficiently large x.

Since

$$A(x)B'(x) - x \le A(x)B(x) - x$$

for all integers x, it follows from (2.7) that

$$A(x)B'(x) - x < h(A(x)) \tag{2.12}$$

for infinitely many positive integers x.

Suppose that $x_1 < x_2 < \cdots$ are all positive integers with

$$A(x_k)B'(x_k) - x_k < h(A(x_k)). \tag{2.13}$$

By the assumption on h,

$$h(A(x_k)) \le \tfrac{2}{3}\sqrt{x_k} < x_k^{1/2}. \tag{2.14}$$

By (2.11) and (2.14),

$$B'(x_k) - 2h(A(x_k)) > x_k^{3/4} - 2x_k^{1/2} \to \infty \quad \text{as } k \to \infty. \tag{2.15}$$

Let u_k be the largest integer with

$$B'(u_k) \le B'(x_k) - 2h(A(x_k)).$$

It follows from (2.15) that u_k exists for sufficiently large k and $u_k \to \infty$ as $k \to \infty$. Since $h(A(x_k)) \to \infty$ as $k \to \infty$, we know that $u_k < x_k$ for all sufficiently large integers k. By the definition of u_k, we have

$$B'(u_k) + 1 \ge B'(u_k + 1) > B'(x_k) - 2h(A(x_k)).$$

Thus

$$2h(A(x_k)) \le B'(x_k) - B'(u_k) < 2h(A(x_k)) + 1. \tag{2.16}$$

By the assumption on h and (2.11),

$$0 \le \lim_{k\to\infty} \frac{2h(A(x_k))}{B'(x_k)} \le \lim_{k\to\infty} \frac{2x_k^{1/2}}{x_k^{3/4}} = 0.$$

It follows from (2.16) that

$$\lim_{k\to\infty} \frac{B'(u_k)}{B'(x_k)} = 1. \tag{2.17}$$

Thus, by (2.10) and (2.17),

$$\lim_{k\to\infty} \frac{B'(u_k)}{B'\left(\frac{1}{2}x_k\right)} = \lim_{k\to\infty} \frac{B'(u_k)}{B'(x_k)} \lim_{k\to\infty} \frac{B'(x_k)}{B'\left(\frac{1}{2}x_k\right)} = 2.$$

So $\frac{1}{2}x_k < u_k < x_k$ for all sufficiently large integers k. Thus

$$A\big(\tfrac{1}{2}x_k\big) \le A(u_k) \le A(x_k) \tag{2.18}$$

for all sufficiently large integers k. By (2.4) and (2.18) we have

$$\lim_{k\to\infty} \frac{A(u_k)}{A(x_k)} = 1.$$

Thus, by (2.9) and (2.17),

$$\lim_{k\to\infty} \frac{u_k}{x_k} = \lim_{k\to\infty} \frac{u_k}{A(u_k)B'(u_k)} \frac{A(u_k)B'(u_k)}{A(x_k)B'(x_k)} \frac{A(x_k)B'(x_k)}{x_k} = 1. \tag{2.19}$$

Let $w_k = x_k - u_k$. Then, by (2.19), we have $w_k = o(x_k)$. By (2.16),

$$\begin{aligned} 2h(A(x_k)) &\le B'(x_k) - B'(u_k) = B'(u_k + w_k) - B'(u_k) \\ &\le B'(u_k) + w_k - B'(u_k) = w_k. \end{aligned}$$

It follows from $h(A(x_k)) \to \infty$ as $k \to \infty$ that $w_k \to \infty$ as $k \to \infty$. It is clear that (2.16) is equivalent to

$$2h(A(x_k)) \le B'(x_k) - B'(x_k - w_k) < 2h(A(x_k)) + 1. \tag{2.20}$$

Now we prove that $A(x_k) = A(w_k)$ for all sufficiently large integers k. Let $f'_x(n)$ be the number of solutions of $a + b = n$, $a \in A$, $a \le x$, $b \in B'$ and $b \le x$. Since A, B' are infinite additive complements, we have

$$f'_x(n) \ge 1, \quad n'_0 \le n \le x. \tag{2.21}$$

Hence

$$A(x)B'(x) \ge x - n'_0. \tag{2.22}$$

By (2.13), (2.20) and (2.21), we have

$$\begin{aligned} h(A(x_k)) &> A(x_k)B'(x_k) - x_k = \sum_{n=0}^{2x_k} f'_{x_k}(n) - x_k \\ &\ge \sum_{n=n'_0+1}^{x_k} f'_{x_k}(n) + \sum_{\substack{w_k<a\le x_k \\ a\in A}} \sum_{\substack{x_k-w_k<b\le x_k \\ b\in B'}} 1 - x_k \\ &\ge \sum_{n=n'_0+1}^{x_k} 1 + \sum_{\substack{w_k<a\le x_k \\ a\in A}} \sum_{\substack{x_k-w_k<b\le x_k \\ b\in B'}} 1 - x_k \\ &= (A(x_k) - A(w_k))\big(B'(x_k) - B'(x_k - w_k)\big) - n'_0 \\ &\ge 2(A(x_k) - A(w_k))h(A(x_k)) - n'_0. \end{aligned}$$

Thus

$$0 \le A(x_k) - A(w_k) \le \frac{1}{2} + \frac{n'_0}{2h(A(x_k))} < 1$$

for all sufficiently large integers k. So $A(x_k) = A(w_k)$ for all sufficiently large integers k. Since $w_k = o(x_k)$, we have $2w_k < x_k$ for all sufficiently large integers k. As $w_k < 2w_k < x_k$ and $A(x_k) = A(w_k)$ for all sufficiently large integers k, we get $A(x_k) = A(2w_k)$ for all sufficiently large integers k.

Define

$$\begin{aligned} D &= \{(b,a) : b \in B',\, a \in A,\, b \le x_k - w_k,\, b - a > w_k\},\\ D_1 &= \{(b,a) : b \in B',\, a \in A,\, 2w_k < b \le x_k - w_k,\, b - a > w_k\},\\ D_2 &= \left\{(b,a) : b \in B',\, a \in A,\, \tfrac{3}{2}w_k < b \le 2w_k,\, b - a > w_k\right\}. \end{aligned}$$

Then $D_1 \cap D_2 = \emptyset$, $D_1 \cup D_2 \subset D$. Hence $|D| \ge |D_1| + |D_2|$.

For $(b,a) \in D_1$, we have $a < b - w_k \le x_k - 2w_k \le x_k$ and $b > 2w_k$. Since $A(x_k) = A(w_k)$ for all sufficiently large integers k, we have $a \le w_k$ for all sufficiently large integers k. Thus

$$D_1 = \{(b,a) : b \in B',\, a \in A,\, 2w_k < b \le x_k - w_k,\, a \le w_k\}$$

for all sufficiently large integers k. By (2.9) and (2.22), noting that $A(w_k) = A(x_k) = A(2w_k)$ for all sufficiently large integers k, we have

$$\begin{aligned} |D_1| &= \big(B'(x_k - w_k) - B'(2w_k)\big)A(w_k)\\ &= B'(x_k)A(w_k) - B'(2w_k)A(w_k) + \big(B'(x_k - w_k) - B'(x_k)\big)A(w_k)\\ &= B'(x_k)A(x_k) - B'(2w_k)A(2w_k) + \big(B'(x_k - w_k) - B'(x_k)\big)A(w_k)\\ &\ge x_k - n_0 - 2w_k + o(w_k) - \big(B'(x_k) - B'(x_k - w_k)\big)A(w_k). \end{aligned}$$

From $A(x_k) = A(w_k)$, (2.6), (2.20) and the assumption on h, we deduce

$$\begin{aligned} 0 &\le \big(B'(x_k) - B'(x_k - w_k)\big)A(w_k)\\ &< (2h(A(x_k)) + 1)A(w_k) = (2h(A(w_k)) + 1)A(w_k)\\ &\le (2w_k^{1/2} + 1)w_k^{1/4} = o(w_k). \end{aligned}$$

Hence $|D_1| \ge x_k - 2w_k + o(w_k)$.

Now we are going to estimate $|D_2|$. It is clear that

$$D_2 \supseteq \left\{(b,a) : b \in B',\, a \in A,\, \tfrac{3}{2}w_k < b \le 2w_k,\, a \le \tfrac{1}{2}w_k\right\}.$$

Thus

$$|D_2| \ge A\big(\tfrac{1}{2}w_k\big)\big(B'(2w_k) - B'\big(\tfrac{3}{2}w_k\big)\big).$$

It follows from $A(x_k) = A(w_k)$ and $w_k < \frac{3}{2}w_k < 2w_k < x_k$ that $A(w_k) = A\big(\frac{3}{2}w_k\big) = A(2w_k)$ for all sufficiently large integers k. By (2.4) and (2.9), we have

$$\begin{aligned}|D_2| &\ge A\bigl(\tfrac{1}{2}w_k\bigr)\bigl(B'(2w_k)-B'\bigl(\tfrac{3}{2}w_k\bigr)\bigr)\\ &= A(w_k)(1+o(1))\bigl(B'(2w_k)-B'\bigl(\tfrac{3}{2}w_k\bigr)\bigr)\\ &= (1+o(1))\bigl(A(w_k)B'(2w_k)-A(w_k)B'\bigl(\tfrac{3}{2}w_k\bigr)\bigr)\\ &= (1+o(1))\bigl(A(2w_k)B'(2w_k)-A\bigl(\tfrac{3}{2}w_k\bigr)B'\bigl(\tfrac{3}{2}w_k\bigr)\bigr)=\tfrac{1}{2}w_k+o(w_k).\end{aligned}$$

Thus

$$|D|\ge|D_1|+|D_2|\ge x_k-2w_k+\tfrac{1}{2}w_k+o(w_k). \tag{2.23}$$

Now we derive a contradiction. Let

$$S=\{a\in A: a\le x_k\},\qquad T=\{b\in B': b\le x_k\},\qquad g(n)=\sum_{\substack{(b,a)\in D\\ b-a=n}}1.$$

Then, for all integers n,

$$f'_{x_k}(n)=r(S,T,n),\qquad g(n)\le\delta(S,T,n),$$

where $r(S,T,n)$ and $\delta(S,T,n)$ are defined as in Lemma 1.2. By that lemma,

$$\begin{aligned}\Bigl(\sum_{f'_{x_k}(n)\ge1}(f'_{x_k}(n)-1)\Bigr)^2 &= \Bigl(\sum_{r(S,T,n)\ge1}(r(S,T,n)-1)\Bigr)^2\\ &\ge\sum_{\delta(S,T,n)\ge1}(\delta(S,T,n)-1)\ge\sum_{g(n)\ge1}(g(n)-1).\end{aligned}$$

Noting that $w_k<b-a\le x_k-w_k$ for all $(b,a)\in D$, we get

$$\sum_{g(n)\ge1}1\le\sum_{w_k<n\le x_k-w_k}1=x_k-2w_k. \tag{2.24}$$

It follows from (2.23) and (2.24) that

$$\begin{aligned}\sum_{g(n)\ge1}(g(n)-1) &= \sum_{g(n)\ge1}g(n)-\sum_{g(n)\ge1}1=|D|-\sum_{g(n)\ge1}1\\ &\ge x_k-2w_k+\tfrac{1}{2}w_k+o(w_k)-(x_k-2w_k)=\tfrac{1}{2}w_k+o(w_k).\end{aligned}$$

Thus

$$\sum_{f'_{x_k}(n)\ge1}(f'_{x_k}(n)-1)\ge\frac{\sqrt{2}}{2}\sqrt{w_k}\,(1+o(1)). \tag{2.25}$$

Since

$$\begin{aligned}&\sum_{n=0}^{n'_0}f'_{x_k}(n)+\sum_{n=n'_0+1}^{x_k}(f'_{x_k}(n)-1)+\sum_{n=x_k+1}^{2x_k}f'_{x_k}(n)\\ &\qquad=\sum_{n=0}^{2x_k}f'_{x_k}(n)-x_k+n'_0=A(x_k)B'(x_k)-x_k+n'_0,\end{aligned}$$

it follows that

$$\sum_{f'_{x_k}(n)\ge 1} (f'_{x_k}(n)-1) \le A(x_k)B'(x_k) - x_k + n'_0.$$

Thus, by (2.13), $A(x_k)=A(w_k)$ and the assumption on h, for all sufficiently large integers k, we have

$$\begin{aligned}\sum_{f'_{x_k}(n)\ge 1} (f'_{x_k}(n)-1) &\le A(x_k)B'(x_k) - x_k + n'_0\\ &< h(A(x_k)) + n'_0 = h(A(w_k)) + n'_0 \le \frac{2}{3}\sqrt{w_k} + n'_0.\end{aligned}$$

It follows from (2.25) that

$$\frac{\sqrt{2}}{2}\sqrt{w_k}(1+o(1)) < \frac{2}{3}\sqrt{w_k} + n_0$$

for all sufficiently large integers k, a contradiction.

This completes the proof of Theorem 2.1. ∎

3. Additive complements with more than two sequences. Infinite sequences $A_1,\dots,A_r$ of non-negative integers are called *infinite additive complements* if their sum contains all sufficiently large integers.

It is easy to see that, for infinite additive complements $A_1,\dots,A_r$, we have

$$\liminf_{x\to\infty} \frac{A_1(x)\cdots A_r(x)}{x} \ge 1.$$

THEOREM 3.1. *For infinite additive complements $A_1,\dots,A_r$, if*

$$\limsup_{x\to\infty} \frac{A_1(x)\cdots A_r(x)}{x} \le 1,$$

then, for any given $M>1$, we have

$$A_1(x)\cdots A_r(x) - x \ge \left(\min\left\{\frac{A_1(x)\cdots A_r(x)}{A_1(x)},\dots,\frac{A_1(x)\cdots A_r(x)}{A_r(x)}\right\}\right)^M$$

for all sufficiently large integers x.

Proof. Given i with $1\le i\le r$, let $A=A_i$ and

$$\begin{aligned}B &= A_1+\dots+A_{i-1}+A_{i+1}+\dots+A_r\\ &= \Big\{\sum_{j=1,\, j\ne i}^{r} a_j : a_j\in A_j\ (1\le j\le r,\, j\ne i)\Big\}.\end{aligned}$$

Since $A_1,\dots,A_r$ are infinite additive complements, so are A and B. It is clear that

$$B(x) \le \frac{A_1(x)\cdots A_r(x)}{A_i(x)}.$$

Hence

$$\limsup_{x\to\infty} \frac{A(x)B(x)}{x} \le \limsup_{x\to\infty} \frac{A_1(x)\cdots A_r(x)}{x} \le 1.$$

This implies that (0.1) holds. Since A, B are infinite additive complements, we have

$$\liminf_{x\to\infty} \frac{A(x)B(x)}{x} \ge 1.$$

Thus

$$\lim_{x\to\infty} \frac{A(x)B(x)}{x} = 1. \tag{3.1}$$

By Lemma 1.1, either

$$\lim_{x\to\infty} \frac{A(2x)}{A(x)} = 1 \quad \text{or} \quad \lim_{x\to\infty} \frac{B(2x)}{B(x)} = 1.$$

By (3.1),

$$\lim_{x\to\infty} \frac{A(2x)B(2x)}{A(x)B(x)} = \lim_{x\to\infty} \frac{A(2x)B(2x)}{2x} \lim_{x\to\infty} \frac{2x}{A(x)B(x)} = 2.$$

Thus, either

$$\lim_{x\to\infty} \frac{A(2x)}{A(x)} = 1 \quad \text{or} \quad \lim_{x\to\infty} \frac{A(2x)}{A(x)} = 2.$$

Hence, for every i,

$$\lim_{x\to\infty} \frac{A_i(2x)}{A_i(x)} \in \{1, 2\}.$$

Let

$$\alpha_i = \lim_{x\to\infty} \frac{A_i(2x)}{A_i(x)}, \qquad i = 1, \dots, r.$$

Since $A_1, \dots, A_r$ are infinite additive complements and

$$\limsup_{x\to\infty} \frac{A_1(x)\cdots A_r(x)}{x} \le 1,$$

it follows that

$$\lim_{x\to\infty} \frac{A_1(x)\cdots A_r(x)}{x} = 1. \tag{3.2}$$

Hence $\alpha_1 \cdots \alpha_r = 2$. Since $\alpha_i \in \{1, 2\}$, exactly one of the α_i is 2. Without loss of generality, we may assume that

$$\alpha_1 = \cdots = \alpha_{r-1} = 1, \qquad \alpha_r = 2.$$

Now, we take $A = A_r$ and $B = A_1 + \cdots + A_{r-1}$. Then

$$\lim_{x\to\infty} \frac{A(2x)}{A(x)} = 2 \quad \text{and} \quad \lim_{x\to\infty} \frac{B(2x)}{B(x)} = 1.$$

So $A(x) > B(x)$ for all $x \geq x_0$. By Theorem 0.2,

$$A(x)B(x) - x \geq B(x)^{2M}$$

for all sufficiently large x. It follows from (3.1) and (3.2) that

$$\lim_{x\to\infty} \frac{A_1(x)\cdots A_{r-1}(x)}{B(x)} = 1.$$

Thus there exists $u_0 \geq x_0$ such that

$$B(x)^2 \geq A_1(x)\cdots A_{r-1}(x), \qquad x \geq u_0.$$

Noting that $B(x) \leq A_1(x)\cdots A_{r-1}(x)$, we arrive at

$$\begin{aligned} A_1(x)\cdots A_r(x) - x \geq A(x)B(x) - x &\geq B(x)^{2M} \\ &\geq (A_1(x)\cdots A_{r-1}(x))^M, \qquad x \geq u_0. \end{aligned}$$

This completes the proof of Theorem 3.1. ∎

4. Final remarks. We pose several problems for further research.

PROBLEM 4.1. *Is there a non-decreasing function $l(x)$ with $l(x) \to \infty$ as $x \to \infty$ such that, for infinite additive complements A, B, if* (0.1) *holds, then*

$$A(x)B(x) - x \geq l(x)$$

for all sufficiently large integers x?

The following Problem 4.2 is a special case of Problem 4.1.

PROBLEM 4.2. *Is there a positive real number θ such that, for infinite additive complements A, B, if* (0.1) *holds, then*

$$A(x)B(x) - x \geq x^\theta$$

for all sufficiently large integers x?

PROBLEM 4.3. *For each integer $r \geq 3$, find infinite additive complements $A_1, \ldots, A_r$ such that*

$$\lim_{x\to\infty} \frac{A_1(x)\cdots A_r(x)}{x} = 1.$$

For $r = 2$, Danzer [2] solved Problem 4.3, which gives a negative answer to a conjecture of Erdős (see [3], [4]).

Chen and Fang [6], [8] proved that, for infinite additive complements A, B, if

$$\limsup_{x\to\infty} \frac{A(x)B(x)}{x} < 3 - \sqrt{3} \quad \text{or} \quad \limsup_{x\to\infty} \frac{A(x)B(x)}{x} > 2,$$

then $A(x)B(x) - x \to \infty$ as $x \to \infty$. On the other hand, Chen and Fang [1] proved that, for any $\varepsilon > 0$, there exist infinite additive complements A, B

such that

$$2-\varepsilon<\limsup_{x\to\infty}\frac{A(x)B(x)}{x}<2$$

and $A(x)B(x)-x=1$ for infinitely many positive integers x.

Acknowledgments. This work was supported by the National Natural Science Foundation of China, Grant Nos. 11371195 and 11201237, and a Project Funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions.

References

[1] Y.-G. Chen and J.-H. Fang, *On additive complements. II*, Proc. Amer. Math. Soc. 139 (2011), 881–883.

[2] L. Danzer, *Über eine Frage von G. Hanani aus der additiven Zahlentheorie*, J. Reine Angew. Math. 214/215 (1964), 392–394.

[3] P. Erdős, *Some unsolved problems*, Michigan Math. J. 4 (1957), 291–300.

[4] P. Erdős, *Some unsolved problems*, Magyar Tud. Akad. Mat. Kutató Int. Közl. 6 (1961), 221–254.

[5] P. Erdős and R. L. Graham, *Old and New Problems and Results in Combinatorial Number Theory*, Monogr. Enseign. Math. 28, Univ. de Genève, Genève, 1980.

[6] J.-H. Fang and Y.-G. Chen, *On additive complements*, Proc. Amer. Math. Soc. 138 (2010), 1923–1927.

[7] J.-H. Fang and Y.-G. Chen, *On finite additive complements*, Discrete Math. 313 (2013), 595–598.

[8] J.-H. Fang and Y.-G. Chen, *On additive complements. III*, J. Number Theory 141 (2014), 83–91.

[9] H. Halberstam and K. F. Roth, *Sequences*, 2nd ed., Springer, New York, 1983.

[10] S. Z. Kiss, E. Rozgonyi and Cs. Sándor, *On additive complement of a finite set*, J. Number Theory 136 (2014), 195–203.

[11] W. Narkiewicz, *Remarks on a conjecture of Hanani in additive number theory*, Colloq. Math. 7 (1959/60), 161–165.

[12] M. B. Nathanson, *Problems in additive number theory, IV: Nets in groups and shortest length g-adic representations*, Int. J. Number Theory 7 (2011), 1999–2017.

[13] I. Z. Ruzsa, *Additive completion of lacunary sequences*, Combinatorica 21 (2001), 279–291.

[14] A. Sárközy and E. Szemerédi, *On a problem in additive number theory*, Acta Math. Hungar. 64 (1994), 237–245.

Yong-Gao Chen
School of Mathematical Sciences
and Institute of Mathematics
Nanjing Normal University
Nanjing 210023, P.R. China
E-mail: ygchen@njnu.edu.cn

Jin-Hui Fang
Department of Mathematics
Nanjing University of
Information Science & Technology
Nanjing 210044, P.R. China
E-mail: fangjinhui1114@163.com

Received on 6.10.2014 (7950)