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Asymptotic formulas for the coefficients of certain automorphic functions

by

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1. Introduction. Let $\mathcal{H}$ denote the complex upper half-plane. Let k be an even integer and M_k be the space of modular forms of weight k on $\mathrm{SL}_2(\mathbb{Z})$. The Eisenstein series of weight $k \geq 4$ for $\mathrm{SL}_2(\mathbb{Z})$ is defined by

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n \geq 1} \sigma_{k-1}(n) q^n,$$

where $q = e^{2\pi i \tau}$ with $\tau \in \mathcal{H}$, $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ and $\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}$. It is well known that $E_k \in M_k$ for all $k \geq 4$. The Ramanujan delta function is given by

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

The function $\Delta \in M_{12}$ is the unique normalized cusp form of the smallest weight for $\mathrm{SL}_2(\mathbb{Z})$. For any integer $k \in 2\mathbb{Z}$, a weakly holomorphic modular form of weight k on $\mathrm{SL}_2(\mathbb{Z})$ is a meromorphic modular form whose poles (if any) are at $i\infty$. The function

$$j(\tau) = \frac{E_4^3(\tau)}{\Delta(\tau)} = q^{-1} + 744 + \sum_{n=1}^{\infty} c(n) q^n$$

is a fundamental weakly holomorphic modular form of weight 0. In 1918, the "circle method" was invented by G. H. Hardy and S. Ramanujan [7] to derive the well known asymptotic formula for the partition function $p(n)$:

$$(1) \qquad p(n) \sim \frac{e^{\pi \sqrt{2n/3}}}{4\sqrt{3}\, n} \quad \text{as } n \to \infty.$$

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Using the circle method, H. Petersson [8] and later H. Rademacher [9] independently derived the asymptotic formula for the Fourier coefficients $c(n)$ of the j-function:

$$c(n) \sim \frac{e^{4\pi\sqrt{n}}}{\sqrt{2}\, n^{3/4}} \quad \text{as } n \to \infty. \tag{2}$$

H. Rademacher and H. S. Zuckerman [10, 11, 12] subsequently obtained exact formulas for the coefficients of generic weakly holomorphic modular forms of negative weight.

In a recent article [4], M. Dewar and M. R. Murty gave a new proof of (1) without using the circle method. Subsequently in [5], they derived an asymptotic formula for the Fourier coefficients of any weakly holomorphic modular form of integral weight for $\mathrm{SL}_2(\mathbb{Z})$ by using the asymptotic formula (1) derived in [4] and without using the circle method.

Inspired by the method of Dewar and Murty, in this article we first derive an asymptotic formula for the Fourier coefficients of a certain class of weakly holomorphic Jacobi forms. Using this asymptotic formula we estimate the growth of the Fourier coefficients of two important weak Jacobi forms of index 1. Secondly, we derive an asymptotic formula for the Fourier coefficients of a certain class of weakly holomorphic modular forms which includes the functions θ^k/η^l for all integers $k, l \geq 1$, where θ is the weight 1/2 modular form and η is the Dedekind eta function. Since we apply the method of Dewar and Murty, our proof does not use the circle method.

We now describe our results more precisely. Let $k \geq 4$ be an even integer and m be any positive integer. We define the Jacobi–Eisenstein series of weight k and index m as

$$E_{k,m}(\tau, z) = \frac{1}{2} \sum_{\substack{c,d\in\mathbb{Z} \\ (c,d)=1}} \sum_{\lambda\in\mathbb{Z}} (c\tau + d)^{-k} e^{2\pi i m(\lambda^2 \frac{a\tau+b}{c\tau+d} + 2\lambda \frac{z}{c\tau+d} - \frac{cz^2}{c\tau+d})},$$

where $a, b \in \mathbb{Z}$ are chosen so that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Let

$$\theta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2}$$

be the usual theta function, which is a modular form of weight 1/2 for $\Gamma_0(4)$. We prove the following theorem.

Theorem 1.1. *Let*

$$f(\tau) = \sum_{n\geq 0} a_f(n) q^n$$

be any q-series with non-negative real coefficients $a_f(n)$ such that

$$a_f(n) \sim \frac{c_f e^{A\sqrt{n}}}{n^{\alpha}}$$

for positive real numbers c_f, A and α. For any positive integers k and m with $k \geq 4$ even, define the complex numbers $b_{fE_{k,m}}(N,r)$ by the Fourier series

$$fE_{k,m}(\tau,z) = f(\tau)E_{k,m}(\tau,z) = \sum_{\substack{N\geq 0,\, r\in\mathbb{Z}\\ N\equiv -r^2 \,(\mathrm{mod}\, 4m)}} b_{fE_{k,m}}(N,r)e^{2\pi i\frac{N+r^2}{4m}\tau}e^{2\pi irz}.$$

Also, for any positive integer k define the real numbers $a_{f\theta^k}(n)$ by the Fourier series

$$f\theta^k(\tau) = f(\tau)\theta^k(\tau) = \sum_{n\geq 0} a_{f\theta^k}(n)q^n, \qquad q = e^{2\pi i\tau}.$$

Then

(i) $$b_{fE_{k,m}}(N,r) \sim c_f i^k\left(\frac{4\pi}{A}\right)^{k-1/2}(2m)^{-1/2}(4m)^{\alpha-k/2+1/4}\frac{e^{\frac{A}{2\sqrt{m}}\sqrt{N}}}{N^{\alpha-k/2+1/4}}$$

for any sequence $\{(N,r)\in\mathbb{Z}\times\mathbb{Z} : N\equiv -r^2 \pmod{4m}\}$ as $N\to\infty$, and

(ii) $$a_{f\theta^k}(n) \sim c_f\left(\frac{2\pi}{A}\right)^{k/2}\frac{e^{A\sqrt{n}}}{n^{\alpha-k/4}} \quad \text{as } n\to\infty.$$

We give two interesting applications of Theorem 1.1. The first one estimates the growth of the Fourier coefficients of the weak Jacobi forms

$$\varphi_{0,1} = \frac{E_4^2E_{4,1} - E_6E_{6,1}}{144\Delta} \quad \text{and} \quad \varphi_{-2,1} = \frac{E_6E_{4,1} - E_4E_{6,1}}{144\Delta}.$$

These are weak Jacobi forms of index 1 and non-positive weight. They generate the ring of weak Jacobi forms of even weight freely over the ring of modular forms on $\mathrm{SL}_2(\mathbb{Z})$ [3, §4.3].

COROLLARY 1.2. *For $k = 0, -2$, let*

$$\varphi_{k,1}(\tau,z) = \sum_{\substack{N\geq -1,\, r\in\mathbb{Z}\\ N\equiv -r^2 \,(\mathrm{mod}\, 4),\, N+r^2\geq 0}} b_k(N,r)e^{2\pi i\frac{N+r^2}{4}\tau}e^{2\pi irz}.$$

Then

$$b_k(N,r) = o\left(\frac{e^{2\pi\sqrt{N}}}{N^{1-k/2}}\right)$$

for any sequence $\{(N,r)\in\mathbb{Z}\times\mathbb{Z} : N\equiv -r^2 \pmod 4\}$ as $N\to\infty$.

REMARK 1.3. Note that the Fourier coefficients of any weak Jacobi form which is not a Jacobi form have exponential growth, and hence the above corollary is not obvious.

Let $\eta(\tau) = q^{1/24}\prod_{n\geq 1}(1-q^n)$ denote the Dedekind eta function. Another consequence of our theorem is the following corollary.

Corollary 1.4. *Let k and l be positive integers. Assume that*

$$\frac{\theta^k}{\eta^l}(\tau) = \sum_{n\geq 0} a_{k,l}(n) q^{n-l/24}.$$

Then the coefficients $a_{k,l}(n)$ satisfy the asymptotic formula

$$a_{k,l}(n) \sim \frac{1}{\sqrt{2}}\left(\frac{l}{24}\right)^{(l+1)/24}\left(\frac{\sqrt{6}}{l}\right)^{k/2}\frac{e^{\pi\sqrt{2ln/3}}}{n^{(l-k+3)/4}}.$$

This article is organized as follows. In the next section, we define Jacobi forms, weak Jacobi forms and weakly holomorphic Jacobi forms and recall the theta decomposition at infinity of a Jacobi form. Then we recall the Fourier expansion of the Jacobi–Eisenstien series and establish some estimates for the coefficients, which is crucial in proving Theorem 1.1(i). We divide Section 3 into two subsections. In Subsection 3.1 we obtain the asymptotic formula for the coefficients $b_{fE_{k,m}}(N,r)$, whereas the asymptotic formula for the coefficients $a_{f\theta^k}(n)$ is established in the second subsection. In Section 4, we obtain our applications by proving Corollaries 1.2 and 1.4.

2. Preliminaries. For any $z \in \mathbb{C}$ and any real number c we denote $e^{2\pi i z/c}$ by $e_c(z)$. If $c = 1$, we simply write $e(z)$ instead of $e_1(z)$. Let k be any integer and m be any positive integer. Following [3, §4.1], we define Jacobi forms, weak Jacobi forms and weakly holomorphic Jacobi forms.

Definition 2.1. A holomorphic function ϕ from $\mathcal{H}\times\mathbb{C}$ to $\mathbb{C}$ is said to be a *weakly holomorphic Jacobi form* of weight k and index m if it satisfies the following conditions:

(i) For any $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ we have

$$\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e\left(\frac{mz^2}{c\tau+d}\right)\phi(\tau,z).$$

(ii) For any $(\lambda,\mu) \in \mathbb{Z}\times\mathbb{Z}$ we have

$$\phi(\tau, z+\lambda\tau+\mu) = e\big(-m(\lambda^2\tau+2\lambda z)\big)\phi(\tau,z).$$

(iii) The function φ has a Fourier expansion of the form

$$\phi(\tau,z) = \sum_{\substack{N,r\in\mathbb{Z} \\ N\geq N_0,\, N\equiv -r^2 \,(\mathrm{mod}\, 4m)}} b_\phi(N,r) e\left(\frac{N+r^2}{4m}\tau + rz\right)$$

for some $N_0 \in \mathbb{Z}$.

If ϕ also satisfies the condition $b_\phi(N,r) = 0$ unless $N + r^2 \geq 0$ then it is called a *weak Jacobi form* of weight k and index m. Further, if ϕ satisfies the even stronger condition that $b_\phi(N,r) = 0$ unless $N \geq 0$ then it is called a *Jacobi form* of weight k and index m.

Let m be any positive integer. For any $\mu \pmod{2m}$ we have the following Jacobi theta functions:

$$\theta_{m,\mu}(\tau,z) = \sum_{\substack{r\in\mathbb{Z} \\ r\equiv\mu \,(\mathrm{mod}\, 2m)}} e\left(\frac{r^2}{4m}\tau + rz\right). \tag{3}$$

As an application of the Poisson summation formula we obtain

$$\begin{aligned} &\theta_{m,\mu}(\tau+1,z) = e_{4m}(\mu^2)\theta_{m,\mu}(\tau,z), \\ &\theta_{m,\mu}\left(-\frac{1}{\tau},\frac{z}{\tau}\right) = \sqrt{\frac{\tau}{2mi}}\, e\left(\frac{mz^2}{\tau}\right) \sum_{\nu \,(\mathrm{mod}\, 2m)} e_{2m}(-\mu\nu)\theta_{m,\nu}(\tau,z). \end{aligned} \tag{4}$$

Property (ii) in the above definition of Jacobi form implies that $b_\phi(N,r) = b_\phi(N,r')$ whenever $r \equiv r' \pmod{2m}$. In particular any Jacobi form ϕ has the following theta decomposition [6, §5]:

$$\phi(\tau,z) = \sum_{\mu=0}^{2m-1} h_\mu(\tau)\theta_{m,\mu}(\tau,z), \tag{5}$$

where

$$h_\mu(\tau) = \sum_{N\geq 0,\, N\equiv -\mu^2 \,(\mathrm{mod}\, 4m)} b_\phi(N,\mu) e\left(\frac{N}{4m}\tau\right).$$

Since any Jacobi form has property (i) and $\theta_{m,\mu}$ satisfies (4), h_μ ($\mu = 0,1,\ldots,2m-1$) has the following transformation properties:

$$\begin{aligned} &h_\mu(\tau+1) = e_{4m}(-\mu^2)h_\mu(\tau), \\ &h_\mu\left(\frac{-1}{\tau}\right) = \frac{\tau^k}{\sqrt{2m\tau/i}} \sum_{\nu=0}^{2m-1} e_{2m}(\mu\nu)h_\nu(\tau). \end{aligned} \tag{6}$$

Let $k \geq 4$ be an even integer and $m \geq 1$ be any integer. As in the theory of modular forms, we obtain our first examples of Jacobi forms by constructing Eisenstein series. The Jacobi–Eisenstein series $E_{k,m}(\tau,z)$ of weight k and index m is defined before the statement of Theorem 1.1, and it has the following Fourier series expansion:

$$E_{k,m}(\tau,z) = \sum_{\substack{N,r\in\mathbb{Z} \\ N\geq 0,\, N\equiv -r^2 \,(\mathrm{mod}\, 4m)}} e_{k,m}(N,r) e\left(\frac{N+r^2}{4m}\tau + rz\right)$$

where $e_{k,m}(N,r)$ for $N = 0$ equals 1 if $r \equiv 0 \pmod{2m}$ and 0 otherwise, while for $N \geq 1$ we have

$$e_{k.m}(N,r) = (-1)^{k/2} \frac{\pi^{k-1/2}}{2^{k-2}\Gamma(k-1/2)\zeta(k-1)m^{k-1}} N^{k-3/2} \sum_{a=1}^{\infty} \frac{N_a(Q_{N,r})}{a^{k-1}}$$

where

$$N_a(Q_{N,r}) = \#\left\{\lambda \pmod a : m\lambda^2 + r\lambda + \frac{N+r^2}{4m} \equiv 0 \pmod a\right\}.$$

Note that for any pair $(N,r) \in \mathbb{N} \times \mathbb{Z}$ such that $N \equiv -r^2 \pmod{4m}$, $(-1)^{k/2} e_{k,m}(N,r)$ is always positive and $e_{k,m}(0,r)$ is either 1 or 0 (for more details of the definition and Fourier expansion of $E_{k,m}(\tau,z)$ see [6, §2]).

The restriction of any Jacobi form $\phi(\tau,z)$ to $z=0$ gives a modular form of the same weight. Since $E_{k,m}(\tau,0)$ is a modular form of weight k and $e_{k,m}(N,r) = e_{k,m}(N,r')$ for $r \equiv r' \pmod{2m}$, there exist positive constants $C'_{k,m}$ and $C_{k,m}$ depending on k,m such that

$$C'_{k,m} \le (-1)^{k/2} e_{k,m}(N,r) \le C_{k,m} N^{k-1} \tag{7}$$

for any pair $(N,r) \in \mathbb{N} \times \mathbb{Z}$ with $N \equiv -r^2 \pmod{4m}$.

3. Proof of Theorem 1.1

3.1. Proof of Theorem 1.1(i). To derive the asymptotic formula (i) for the coefficients of the function $fE_{k,m}(\tau,z)$, we extend the method of Dewar and Murty [5] to the case of Jacobi forms, which presents technical difficulties due to the nature of Jacobi forms. We construct a function $\phi_{k,m}(\tau,z)$ with positive Fourier coefficients by removing some terms from the Fourier expansion of $E_{k,m}(\tau,z)$ and derive an asymptotic formula for the coefficients of $f\phi_{k,m}(\tau,z)$ by considering the lim sup and the lim inf cases separately. We obtain the asymptotic formula for the coefficients of the function $fE_{k,m}(\tau,z)$ by handling suitably the combination of $f\phi_{k,m}(\tau,z)$ and $f\theta_{m,0}(\tau,z)$, where $\theta_{m,0}$ is one of the Jacobi theta series occurring in (3). The proof of [5, Theorem 3] relies on the asymptotic for $E_k(q)$ as $q \to 1$ which Dewar and Murty deduce by using the transformation properties of $E_k(\tau)$, whereas to prove our theorem we use the transformation properties of the $2m$ functions $h_\mu(\tau)$ which appear in the theta decomposition (5) of $E_{k,m}(\tau,z)$ and hence deal with a certain combination of the functions $h_\mu(\tau)$.

Let us set

$$\phi_{k,m}(\tau,z) = (-1)^{k/2}\left(E_{k,m}(\tau,z) - \sum_{r\in 2m\mathbb{Z}} e\left(\frac{r^2}{4m}\tau + rz\right)\right). \tag{8}$$

Then

$$\phi_{k,m}(\tau,z) = \sum_{\substack{N,r\in\mathbb{Z} \\ N\ge 1,\, N\equiv -r^2 \pmod{4m}}} b_{\phi_{k,m}}(N,r)\, e\left(\frac{N+r^2}{4m}\tau + rz\right),$$

where

$$b_{\phi_{k,m}}(N,r) = (-1)^{k/2} e_{k,m}(N,r),$$

and therefore $b_{\phi_{k,m}}(N,r)$ is always positive. Suppose we have the following lemma.

Lemma 3.1. *For*

$$\begin{aligned} f\phi_{k,m}(\tau,z) &= f(\tau)\phi_{k,m}(\tau,z) \\ &= \sum_{\substack{N,r\in\mathbb{Z} \\ N\geq 1,\, N\equiv -r^2 \pmod{4m}}} b_{f\phi_{k,m}}(N,r)e\biggl(\frac{N+r^2}{4m}\tau + rz\biggr), \end{aligned}$$

we have

$$b_{f\phi_{k,m}}(N,r) \sim c_f\biggl(\frac{4\pi}{A}\biggr)^{k-1/2}(2m)^{-1/2}(4m)^{\alpha-k/2+1/4}\frac{e^{\frac{A}{2\sqrt{m}}\sqrt{N}}}{N^{\alpha-k/2+1/4}} \tag{9}$$

for any sequence $\{(N,r)\in\mathbb{Z}\times\mathbb{Z} : N\equiv -r^2 \pmod{4m}\}$ *as* $N\to\infty$.

Then we derive the asymptotic formula for the coefficients $b_{fE_{k,m}}(N,r)$ of the function $fE_{k,m}(\tau,z)$. Using (8), we have

$$f(\tau)E_{k,m}(\tau,z) = (-1)^{k/2}f(\tau)\phi_{k,m}(\tau,z) + f(\tau)\theta_{m,0}(\tau,z), \tag{10}$$

where

$$\theta_{m,0}(\tau,z) = \sum_{r\in 2m\mathbb{Z}} e\biggl(\frac{r^2}{4m}\tau + rz\biggr)$$

is one of the Jacobi theta functions defined by (3). We have

$$f(\tau)\theta_{m,0}(\tau,z) = \sum_{N\in 4m\mathbb{Z},\, r\in 2m\mathbb{Z}} b_{f\theta_{m,0}}(N,r)e\biggl(\frac{N+r^2}{4m}\tau\biggr)e(rz),$$

where

$$b_{f\theta_{m,0}}(N,r) = a_f(N/4m).$$

Therefore

$$b_{f\theta_{m,0}}(N,r) \sim c_f(4m)^{\alpha}\frac{e^{\frac{A}{2\sqrt{m}}\sqrt{N}}}{N^{\alpha}} = o\biggl(\frac{e^{\frac{A}{2\sqrt{m}}\sqrt{N}}}{N^{\alpha-k/2+1/4}}\biggr) \tag{11}$$

for any sequence $\{(N,r)\in 4m\mathbb{Z}\times 2m\mathbb{Z}\}$ as $N\to\infty$.

From (10), we have

$$b_{fE_{k,m}}(N,r) \sim i^k b_{f\phi_{k,m}}(N,r) + b_{f\theta_{m,0}}(N,r).$$

Using (9) and (11) in the above equation, we get

$$b_{fE_{k,m}}(N,r) \sim c_f i^k\biggl(\frac{4\pi}{A}\biggr)^{k-1/2}(2m)^{-1/2}(4m)^{\alpha-k/2+1/4}\frac{e^{\frac{A}{2\sqrt{m}}\sqrt{N}}}{N^{\alpha-k/2+1/4}}$$

for any sequence $\{(N,r)\in\mathbb{Z}\times\mathbb{Z} : N\equiv -r^2 \pmod{4m}\}$ as $N\to\infty$.

Now we prove Lemma 3.1. The key step is to approximate a certain exponential term by using the following real valued functions $F(x)$, $G(x)$, and then use the modularity of the $2m$ functions $h_\mu(\tau)$: Consider the following functions:

$$F(x) := \sqrt{1-x} : (0,1) \to \mathbb{R}_{>0}, \tag{12}$$

$$G(x) := \frac{1}{(1-x)^\alpha} : (0,1) \to \mathbb{R}_{>0}, \tag{13}$$

where α is any positive real number. For $0 < x \le \delta < 1$, Taylor's theorem gives

$$\begin{aligned} &1 - \frac{x}{2} - \frac{\delta^2}{8(1-\delta)^{3/2}} \le F(x) \le 1 - \frac{x}{2}, \\ &1 \le G(x) \le 1 + \alpha \frac{\delta}{(1-\delta)^{\alpha+1}}. \end{aligned} \tag{14}$$

Proof of Lemma 3.1. We have

$$b_{f\phi_{k,m}}(N,r) = \sum_{N' \ge 1,\, N' \equiv -r^2 \pmod{4m}}^{N} a_f\left(\frac{N-N'}{4m}\right) b_{\phi_{k,m}}(N',r).$$

In order to derive the asymptotic formula for the coefficients $b_{f\phi_{k,m}}(N,r)$ of the function $f\phi_{k,m}(\tau,z)$ we prove

$$\limsup_{\substack{(N,r)\in\mathbb{Z}\times\mathbb{Z},\, N\equiv -r^2 \pmod{4m} \\ N\to\infty}} \frac{b_{f\phi_{k,m}}(N,r)}{c_f(\frac{4\pi}{A})^{k-1/2}(2m)^{-1/2}(4m)^{\alpha-k/2+1/4}\frac{e^{\frac{A}{2\sqrt{m}}\sqrt{N}}}{N^{\alpha-k/2+1/4}}} \le 1 \tag{15}$$

and

$$\liminf_{\substack{(N,r)\in\mathbb{Z}\times\mathbb{Z},\, N\equiv -r^2 \pmod{4m} \\ N\to\infty}} \frac{b_{f\phi_{k,m}}(N,r)}{c_f\left(\frac{4\pi}{A}\right)^{k-1/2}(2m)^{-1/2}(4m)^{\alpha-k/2+1/4}\frac{e^{\frac{A}{2\sqrt{m}}\sqrt{N}}}{N^{\alpha-k/2+1/4}}} \ge 1. \tag{16}$$

For the proof of (15), choose any $0 < \delta < 1$ and write

$$\begin{aligned} b_{f\phi_{k,m}}(N,r) = &\sum_{N'=1,\, N'\equiv -r^2 \pmod{4m}}^{\lfloor \delta N \rfloor} a_f\left(\frac{N-N'}{4m}\right) b_{\phi_{k,m}}(N',r) \\ &+ \sum_{N'=\lfloor \delta N\rfloor+1,\, N'\equiv -r^2 \pmod{4m}}^{N} a_f\left(\frac{N-N'}{4m}\right) b_{\phi_{k,m}}(N',r). \end{aligned} \tag{17}$$

Consider the sum

$$S_\delta(N,r) := \sum_{N'=1,\, N'\equiv -r^2 \pmod{4m}}^{\lfloor \delta N \rfloor} a_f\left(\frac{N-N'}{4m}\right) b_{\phi_{k,m}}(N',r).$$

Using the given asymptotic formula for the coefficients of $f(\tau)$, for any $\epsilon > 0$

we have

$$a_f\left(\frac{N-N'}{4m}\right) < (1+\epsilon)c_f\frac{e^{A\sqrt{\frac{N-N'}{4m}}}}{\left(\frac{N-N'}{4m}\right)^{\alpha}}.$$

Therefore,

$$S_\delta(N,r) < (1+\epsilon)\frac{c_f}{(N/4m)^\alpha} \times \sum_{N'=1,\, N'\equiv -r^2 \,(\mathrm{mod}\, 4m)}^{\lfloor \delta N\rfloor} \frac{1}{(1-N'/N)^\alpha} e^{\frac{A}{2\sqrt{m}}\sqrt{N}\sqrt{1-N'/N}} b_{\phi_{k,m}}(N',r).$$

Since $e^{\frac{A}{2\sqrt{m}}\sqrt{N}x}$ is an increasing function of x, using the right hand inequality of (14) for the function $F(x)$ (defined by (12)), we have

$$e^{\frac{A}{2\sqrt{m}}\sqrt{N}\sqrt{1-N'/N}} \le e^{\frac{A}{2\sqrt{m}}\sqrt{N}(1-N'/2N)}. \tag{18}$$

Since the function $G(x)$ defined in (13) is continuous, for any $\epsilon > 0$ we can fix $0 < \delta < 1$ such that for $0 < x \le \delta$ we have $G(x) \le 1+\epsilon$. Using (18) together with this observation, we have

$$\begin{aligned} S_\delta(N,r) &< (1+\epsilon)^2\frac{c_f}{(N/4m)^\alpha}e^{\frac{A}{2\sqrt{m}}\sqrt{N}} \sum_{N'=1,\, N'\equiv -r^2 \,(\mathrm{mod}\, 4m)}^{\lfloor \delta N\rfloor} b_{\phi_{k,m}}(N',r)e^{-\frac{A}{2\sqrt{m}}\frac{N'}{2\sqrt{N}}} \\ &\le (1+\epsilon)^2\frac{c_f}{(N/4m)^\alpha}e^{\frac{A}{2\sqrt{m}}\sqrt{N}} \sum_{N'=1,\, N'\equiv -r^2 \,(\mathrm{mod}\, 4m)}^{\infty} b_{\phi_{k,m}}(N',r)e^{-\frac{A}{2\sqrt{m}}\frac{N'}{2\sqrt{N}}}. \end{aligned}$$

Choose $\mu_r \in \{0,1,\dots,2m-1\}$ such that $\mu_r \equiv r \pmod{2m}$. Let

$$h_{\mu_r}(\tau) = \sum_{N'\ge 0,\, N'\equiv -\mu_r^2 \,(\mathrm{mod}\, 4m)} e_{k,m}(N',\mu_r)e\left(\frac{N'}{4m}\tau\right)$$

be the μ_rth component in the theta decomposition of the Eisenstein series $E_{k,m}$. Then

$$S_\delta(N,r) < (1+\epsilon)^2\frac{c_f}{(N/4m)^\alpha}e^{\frac{A}{2\sqrt{m}}\sqrt{N}}(-1)^{k/2}\left(h_{\mu_r}\left(i\frac{A}{2\pi}\sqrt{\frac{m}{N}}\right) - C_{\mu_r}\right), \tag{19}$$

where C_{μ_r} is either 1 or 0 depending on μ_r. Using (6) we have

$$\begin{aligned} h_{\mu_r}\left(i\frac{A}{2\pi}\sqrt{\frac{m}{N}}\right) &= h_{\mu_r}\left(\frac{-1}{i\frac{2\pi}{A}\sqrt{\frac{N}{m}}}\right) \\ &= \frac{i^k}{\sqrt{2m}}\left(\frac{2\pi}{A}\sqrt{\frac{N}{m}}\right)^{k-1/2}\sum_{\nu=0}^{2m-1} e_{2m}(\mu_r\nu)h_\nu\left(i\frac{2\pi}{A}\sqrt{\frac{N}{m}}\right). \end{aligned}$$

We also have

$$\lim_{N\to\infty} h_0\left(i\frac{2\pi}{A}\sqrt{\frac{N}{m}}\right) = 1, \qquad \lim_{N\to\infty} h_\nu\left(i\frac{2\pi}{A}\sqrt{\frac{N}{m}}\right) = 0, \quad \nu \neq 0.$$

Therefore for large enough N, we have

$$(-1)^{k/2}\left(h_{\mu_r}\left(i\frac{A}{2\pi}\sqrt{\frac{m}{N}}\right) - C_{\mu_r}\right) < (1+\epsilon)(2m)^{-1/2}\left(\frac{2\pi}{A}\sqrt{\frac{N}{m}}\right)^{k-1/2}.$$

Using the above equation in (19), we get

$$S_\delta(N,r) < (1+\epsilon)^3 c_f\left(\frac{4\pi}{A}\right)^{k-1/2}(2m)^{-1/2}(4m)^{\alpha-k/2+1/4}\frac{e^{\frac{A}{2\sqrt{m}}\sqrt{N}}}{N^{\alpha-k/2+1/4}}. \tag{20}$$

Since $b_{\phi_{k,m}}(N,r) = O(N^{k-1})$, we have

$$\begin{aligned}\sum_{\substack{N'=\lfloor\delta N\rfloor+1\\ N'\equiv -r^2 \pmod{4m}}}^{N} a_f\left(\frac{N-N'}{4m}\right)b_{\phi_{k,m}}(N',r) &= O\left(e^{A\sqrt{\frac{(1-\delta)N}{4m}}}N^k\right)\\ &= o\left(\frac{e^{\frac{A}{2\sqrt{m}}\sqrt{N}}}{N^{\alpha-k/2+1/4}}\right) \quad \text{as } N\to\infty.\end{aligned}$$

Using the above bound and (20) in (17), we get (15).

Now we prove (16). Since all the coefficients of $f\phi_{k,m}$ are non-negative, for any $0<\delta<1$ we have

$$\begin{aligned}&b_{f\phi_{k,m}}(N,r) \\ &\geq S_\delta(N,r) = \sum_{N'\geq 1,\, N'\equiv -r^2 \pmod{4m}}^{\lfloor\delta N\rfloor} a_f\left(\frac{N-N'}{4m}\right)b_{\phi_{k,m}}(N',r)\\ &> (1-\epsilon)\frac{c_f}{(N/4m)^\alpha}\sum_{N'=1,\, N'\equiv -r^2 \pmod{4m}}^{\lfloor\delta N\rfloor}\frac{1}{\left(1-\frac{N'}{N}\right)^\alpha}e^{\frac{A}{2\sqrt{m}}\sqrt{N}\sqrt{1-\frac{N'}{N}}}b_{\phi_{k,m}}(N',r).\end{aligned} \tag{21}$$

Using the fact that the function $e^{\frac{A}{2\sqrt{m}}\sqrt{N}x}$ is increasing together with the left hand inequality of (14) for the function $F(x)$, we have

$$\begin{aligned}S_\delta(N,r) > (1-\epsilon)&\frac{c_f}{(N/4m)^\alpha}e^{\frac{A}{2\sqrt{m}}\sqrt{N}\left(1-\frac{\delta^2}{8(1-\delta)^{3/2}}\right)}\\ &\times \sum_{N'=1,\, N'\equiv -r^2 \pmod{4m}}^{\lfloor\delta N\rfloor} b_{\phi_{k,m}}(N',r)e^{-\frac{A}{4\sqrt{mN}}N'}.\end{aligned} \tag{22}$$

Clearly

$$(23)\qquad \sum_{\substack{N'=1\\ N'\equiv -r^2 \,(\mathrm{mod}\, 4m)}}^{\lfloor \delta N\rfloor} b_{\phi_{k,m}}(N',r)e^{-\frac{A}{4\sqrt{mN}}N'}$$

$$= \sum_{\substack{N'=1\\ N'\equiv -r^2 \,(\mathrm{mod}\, 4m)}}^{\infty} b_{\phi_{k,m}}(N',r)e^{-\frac{A}{4\sqrt{mN}}N'} - \sum_{\substack{N'=\lfloor \delta N\rfloor+1\\ N'\equiv -r^2 \,(\mathrm{mod}\, 4m)}}^{\infty} b_{\phi_{k,m}}(N',r)e^{-\frac{A}{4\sqrt{mN}}N'}.$$

In order to handle the second sum on the right hand side of (23) we use the following lemma, which we prove at the end of this subsection.

LEMMA 3.2. *For $\delta = N^{-1/3}$, we have*

$$\lim_{\substack{(N,r)\in\mathbb{Z}\times\mathbb{Z},\, N\equiv -r^2 \,(\mathrm{mod}\, 4m)\\ N\to\infty}} \frac{\sum_{N'=\lfloor \delta N\rfloor+1,\, N'\equiv -r^2 \,(\mathrm{mod}\, 4m)}^{\infty} b_{\phi_{k,m}}(N',r)e^{-\frac{A}{4\sqrt{mN}}N'}}{\sum_{N'=1,\, N'\equiv -r^2 \,(\mathrm{mod}\, 4m)}^{\infty} b_{\phi_{k,m}}(N',r)e^{-\frac{A}{4\sqrt{mN}}N'}} = 0.$$

Using Lemma 3.2 in (23), for N large enough and $\delta = N^{-1/3}$, (22) gives us

$$S_\delta(N,r) > (1-\epsilon)^2 \frac{c_f}{(N/4m)^\alpha} e^{\frac{A}{2\sqrt{m}}\sqrt{N}(1-\frac{\delta^2}{8(1-\delta)^{3/2}})} \times \sum_{N'=1,\, N'\equiv -r^2 \,(\mathrm{mod}\, 4m)}^{\infty} b_{\phi_{k,m}}(N',r)e^{-\frac{A}{4\sqrt{mN}}N'}.$$

Now using the argument similar to the lim sup case for the series

$$\sum_{N'=1,\, N'\equiv -r^2 \,(\mathrm{mod}\, 4m)}^{\infty} b_{\phi_{k,m}}(N',r)e^{-\frac{A}{4\sqrt{mN}}N'},$$

we get

$$S_\delta(N,r) > (1-\epsilon)^3 \frac{c_f}{(N/4m)^\alpha} e^{A\sqrt{\frac{N}{4m}}(1-\frac{\delta^2}{8(1-\delta)^{3/2}})} (2m)^{-1/2} \left(\frac{4\pi}{A}\sqrt{\frac{N}{4m}}\right)^{k-1/2}.$$

For $\delta = N^{-1/3}$, we have

$$\lim_{N\to\infty} \frac{\sqrt{N}\,\delta^2}{(1-\delta)^{3/2}} = 0.$$

Using this together with (21), we get the lim inf inequality (16). Next we use the limits in (15) and (16) in order to get Lemma 3.1.

Proof of Lemma 3.2. Let us set

$$F(N,r) := \frac{\sum_{N'\geq \lfloor \delta N\rfloor+1,\, N'\equiv -r^2 \,(\mathrm{mod}\, 4m)} b_{\phi_{k,m}}(N',r) e^{-\frac{A}{4\sqrt{mN}}N'}}{\sum_{N'\geq 1,\, N'\equiv -r^2 \,(\mathrm{mod}\, 4m)} b_{\phi_{k,m}}(N',r) e^{-\frac{A}{4\sqrt{mN}}N'}}.$$

Since $b_{\phi_{k,m}}(N,r) = (-1)^{k/2} e_{k,m}(N,r)$, using (7), we obtain

$$F(N,r) \leq \frac{C_{k,m}}{C'_{k,m}} \frac{\sum_{N'\geq \lfloor \delta N\rfloor+1,\, N'\equiv -r^2 \,(\mathrm{mod}\, 4m)} N'^k e^{-\frac{A}{4\sqrt{mN}}N'}}{\sum_{N'\geq 1,\, N'\equiv -r^2 \,(\mathrm{mod}\, 4m)} e^{-\frac{A}{4\sqrt{mN}}N'}}. \tag{24}$$

For all integers $j \geq 1$ and all real $0 < \beta < 1$, one has

$$j^k \leq \left(\frac{k}{\beta}\right)^k e^{\beta j}.$$

Therefore for any $0 < \beta < 1$, we have

$$\sum_{\substack{N'\geq \lfloor \delta N\rfloor+1\\ N'\equiv -r^2 \,(\mathrm{mod}\, 4m)}} N'^k e^{-\frac{A}{4\sqrt{mN}}N'} \leq \left(\frac{k}{\beta}\right)^k \sum_{\substack{N'\geq \lfloor \delta N\rfloor+1\\ N'\equiv -r^2 \,(\mathrm{mod}\, 4m)}} e^{(\beta-\frac{A}{4\sqrt{mN}})N'}.$$

Using the above inequality and then writing $N' = 4mn' - r^2$ in (24), we get

$$F(N,r) \leq \frac{C_{k,m}}{C'_{k,m}} \left(\frac{k}{\beta}\right)^k \frac{e^{-(\beta-\frac{A}{4\sqrt{mN}})r^2}}{e^{\frac{A}{4\sqrt{mN}}r^2}} \frac{\sum_{n'\geq \frac{\lfloor \delta N\rfloor+1+r^2}{4m}} e^{(\beta-\frac{A}{4\sqrt{mN}})4mn'}}{\sum_{n'\geq \frac{r^2+1}{4m}} e^{-\frac{A}{4\sqrt{mN}}4mn'}}.$$

If $\beta - \frac{A}{4\sqrt{mN}} < 0$ then

$$\begin{aligned} F(N,r) &\leq \frac{C_{k,m}}{C'_{k,m}} \left(\frac{k}{\beta}\right)^k \frac{e^{-(\beta-\frac{A}{4\sqrt{mN}})r^2}}{e^{\frac{A}{4\sqrt{mN}}r^2}} \frac{e^{4m(\beta-\frac{A}{4\sqrt{mN}})\frac{\lfloor \delta N\rfloor+1+r^2}{4m}}}{e^{-4m\frac{A}{4\sqrt{mN}}(\frac{r^2+1}{4m}+1)}} \frac{1-e^{-4m\frac{A}{4\sqrt{mN}}}}{1-e^{4m(\beta-\frac{A}{4\sqrt{mN}})}} \\ &\leq \frac{C_{k,m}}{C'_{k,m}} \left(\frac{k}{\beta}\right)^k \frac{e^{\beta(\lfloor \delta N\rfloor+1)} e^{-\frac{A}{4\sqrt{mN}}\lfloor \delta N\rfloor}}{e^{-4m\frac{A}{4\sqrt{mN}}}} \frac{1-e^{-4m\frac{A}{4\sqrt{mN}}}}{1-e^{4m(\beta-\frac{A}{4\sqrt{mN}})}}. \end{aligned}$$

Choosing $\beta = \frac{A}{4\sqrt{m}N^{3/2}}$ and $\delta = N^{-1/3}$, we get

$$\begin{aligned} &F(N,r) \\ &\leq \frac{C_{k,m}}{C'_{k,m}} \left(\frac{4\sqrt{m}k}{A}\right)^k \frac{N^{3k/2}}{e^{\frac{A}{4\sqrt{m}}N^{1/6}(1-1/N)}} e^{\frac{A}{4\sqrt{mN}}(1+4m+1/N)} \frac{1-e^{-4m\frac{A}{4\sqrt{mN}}}}{1-e^{-4m\frac{A}{4\sqrt{mN}}(1-1/N)}}. \end{aligned}$$

The right hand side above goes to 0 as $N \to \infty$.

3.2. Proof of Theorem 1.1(ii)**.** The proof of Theorem 1.1(ii) is quite similar to that of [5, Theorem 3], and so we give only a sketch of the proof, highlighting the main points and steps which are different from the proof of [5, Theorem 3]. Let us write

$$\theta^k(\tau) = \sum_{n\geq 0} r_k(n) q^n,$$

where

$$r_k(n) = \#\{(n_1, \dots, n_k) \in \mathbb{Z}^k : n_1^2 + \cdots + n_k^2 = n\}.$$

Since $\theta^{2k}(\tau)$ is a modular form of weight k, there exists a constant C_k such that

$$r_k(n) \leq r_{2k}(n) \leq C_k n^{k-1}, \qquad n \geq 1. \tag{25}$$

For any $n \geq 0$, we have

$$a_{f\theta^k}(n) = \sum_{j=0}^{n} a_f(n-j) r_k(j).$$

In order to get the asymptotic formula for $a_{f\theta^k}(n)$ we prove

$$\limsup_{n\to\infty} \frac{a_{f\theta^k}(n)}{c_f\left(\frac{2\pi}{A}\right)^{k/2} \frac{e^{A\sqrt{n}}}{n^{\alpha-k/4}}} \leq 1 \tag{26}$$

and

$$\liminf_{n\to\infty} \frac{a_{f\theta^k}(n)}{c_f\left(\frac{2\pi}{A}\right)^{k/2} \frac{e^{A\sqrt{n}}}{n^{\alpha-k/4}}} \geq 1. \tag{27}$$

First we prove (26). Using the asymptotic formula of $a_f(n)$, the inequality $e^{A\sqrt{n-j}} \leq e^{A\sqrt{n} - \frac{A}{2\sqrt{n}} j}$ and continuity of the function $G(x) = 1/(1-x)^\alpha$, for any $\epsilon > 0$ we can fix $0 < \delta < 1$ such that

$$\begin{aligned} S_\delta(n) := \sum_{j=0}^{\lfloor \delta n \rfloor} a_f(n-j) r_k(j) &< (1+\epsilon)^2 \frac{c_f}{n^\alpha} e^{A\sqrt{n}} \sum_{j=0}^{\lfloor \delta n \rfloor} r_k(j) e^{-\frac{A}{2\sqrt{n}} j} \\ &\leq (1+\epsilon)^2 \frac{c_f}{n^\alpha} e^{A\sqrt{n}} \sum_{j=0}^{\infty} r_k(j) e^{-\frac{A}{2\sqrt{n}} j} \leq (1+\epsilon)^2 \frac{c_f}{n^\alpha} e^{A\sqrt{n}} \theta^k\left(\frac{iA}{4\pi\sqrt{n}}\right). \end{aligned}$$

Using the transformation property of the θ function, we get

$$\theta^k\left(\frac{iA}{4\pi\sqrt{n}}\right) = \left(\frac{2\pi\sqrt{n}}{A}\right)^{k/2} \theta^k\left(\frac{\pi i\sqrt{n}}{A}\right).$$

Since $\lim_{n\to\infty} \theta^k(\pi i\sqrt{n}/A) = 1$, for n large enough we have

$$\theta^k\left(\frac{iA}{4\pi\sqrt{n}}\right) < (1+\epsilon)\left(\frac{2\pi\sqrt{n}}{A}\right)^{k/2}.$$

Therefore for n large enough we have

$$S_\delta(n,r) < (1+\epsilon)^3 c_f \left(\frac{2\pi}{A}\right)^{k/2} \frac{e^{A\sqrt{n}}}{n^{\alpha-k/4}}. \tag{28}$$

Using (25), we have

$$\sum_{j=\lfloor \delta n\rfloor+1}^{n} a_f(n-j)r_k(j) = O\big(e^{A\sqrt{(1-\delta)n}}n^k\big) = o\left(\frac{e^{A\sqrt{n}}}{n^{\alpha-k/4}}\right) \quad \text{as } n\to\infty.$$

Using the above bound together with (28), we get (26).

Now we prove (27). Choosing $0<\delta<1$, we have

$$a_{f\theta^k}(n) \ge S_\delta(n) := \sum_{j=0}^{\lfloor \delta n\rfloor} a_f(n-j)r_k(j).$$

Using the asymptotic formula for $a_f(n-j)$ and the inequality

$$e^{A\sqrt{n-j}} \ge e^{A\sqrt{n}(1-\frac{j}{2n}-\frac{\delta^2}{8(1-\delta)^{3/2}})},$$

for any $\epsilon>0$ we get

$$S_\delta(n) > (1-\epsilon)c_f\frac{e^{A\sqrt{n}}}{n^\alpha}e^{-A\sqrt{n}\frac{\delta^2}{8(1-\delta)^{3/2}}}\sum_{j=0}^{\lfloor \delta n\rfloor} r_k(j)e^{-\frac{A}{2\sqrt{n}}j}. \tag{29}$$

Now we write

$$\sum_{j=0}^{\lfloor \delta n\rfloor} r_k(j)e^{-\frac{A}{2\sqrt{n}}j} = \sum_{j=0}^{\infty} r_k(j)e^{-\frac{A}{2\sqrt{n}}j} - \sum_{j=\lfloor \delta n\rfloor+1}^{\infty} r_k(j)e^{-\frac{A}{2\sqrt{n}}j}. \tag{30}$$

We use the following lemma to finish the proof; the lemma will be proved afterwards.

Lemma 3.3. *For $\delta = n^{-1/3}$,*

$$\lim_{n\to\infty}\frac{\sum_{j=\lfloor \delta n\rfloor+1}^{\infty} r_k(j)e^{-\frac{A}{2\sqrt{n}}j}}{\sum_{j=0}^{\infty} r_k(j)e^{-\frac{A}{2\sqrt{n}}j}} = 0.$$

If we use Lemma 3.3 in (30), then (29) gives us

$$S_\delta(n) > (1-\epsilon)^2 c_f\frac{e^{A\sqrt{n}}}{n^\alpha}e^{-A\sqrt{n}\frac{\delta^2}{8(1-\delta)^{3/2}}}\sum_{j=0}^{\infty} r_k(j)e^{-\frac{A}{2\sqrt{n}}j}.$$

Now we deduce (27) by using the transformation property of θ^k as used in the lim sup case, and the fact that for $\delta=n^{-1/3}$ we have

$$\lim_{n\to\infty}\frac{\sqrt{n}\,\delta^2}{(1-\delta)^{3/2}} = 0.$$

Proof of Lemma 3.3. Using (25) and the fact that $r_k(j^2) \geq 1$ for any $j \geq 0$, we have

$$R(n) := \frac{\sum_{j=\lfloor \delta n \rfloor+1}^{\infty} r_k(j) e^{-\frac{A}{2\sqrt{n}} j}}{\sum_{j=0}^{\infty} r_k(j) e^{-\frac{A}{2\sqrt{n}} j}} \leq C_k \frac{\sum_{j=\lfloor \delta n \rfloor+1}^{\infty} j^k e^{-\frac{A}{2\sqrt{n}} j}}{\sum_{j=0}^{\infty} e^{-\frac{A}{2\sqrt{n}} j^2}}. \tag{31}$$

Since $e^{-\frac{A}{2\sqrt{n}} x^2}$ is a continuous decreasing positive function on $[0, \infty)$, comparing sum with integral we get

$$\sum_{j=0}^{\infty} e^{-\frac{A}{2\sqrt{n}} j^2} \geq \int_0^{\infty} e^{-\frac{A}{2\sqrt{n}} x^2} dx = \sqrt{\frac{\pi}{2A}} n^{1/4}. \tag{32}$$

Using the inequality $j^k \leq (k/\beta)^k e^{\beta j}$ for any real number $0 < \beta < 1$ such that $\beta - \frac{A}{2\sqrt{n}} < 0$ and (32) in (31), we have

$$R(n) \leq C_k \left(\frac{k}{\beta}\right)^k \sqrt{\frac{2A}{\pi}} n^{-1/4} \left(\frac{e^{(\beta - \frac{A}{2\sqrt{n}})(\lfloor \delta n \rfloor + 1)}}{1 - e^{\beta - \frac{A}{2\sqrt{n}}}}\right).$$

Choose $\beta = \frac{A}{2n^{3/2}}$ and $\delta = n^{-1/3}$; then we get

$$\begin{aligned} R(n) &\leq C_k \left(\frac{2k}{A}\right)^k \sqrt{\frac{2A}{\pi}} \left(\frac{n^{3k/2+1/4}}{e^{\frac{An^{1/6}}{2}(1-1/n)}}\right) \left(\frac{n^{-1/2}}{1 - e^{-\frac{A}{2\sqrt{n}}(1-1/n)}}\right) \\ &\to C_k \cdot \left(\frac{2k}{A}\right)^k \cdot \sqrt{\frac{2A}{\pi}} \cdot 0 \cdot \frac{A}{2} \end{aligned}$$

as $n \to \infty$. Thus $R(n)$ goes to 0 as $n \to \infty$.

4. Proof of the corollaries

4.1. Proof of Corollary 1.2. Set

$$\sum_{n=0}^{\infty} p^{(j)}(n) q^n = \prod_{n=0}^{\infty} (1 - q^n)^{-j}.$$

In [5], Dewar and Murty have the following asymptotic formula for $p^{(j)}(n)$:

$$p^{(j)}(n) \sim c_j n^{\alpha_j} e^{A_j \sqrt{n}}, \tag{33}$$

where

$$c_j = \frac{1}{\sqrt{2}} \left(\frac{j}{24}\right)^{(j+1)/4}, \quad \alpha_j = -\frac{j}{4} - \frac{3}{4}, \quad A_j = \pi \sqrt{\frac{2j}{3}}.$$

Let us set

$$f(\tau) := \frac{1}{144} q \Delta^{-1}(\tau) = \sum_{n=0}^{\infty} p^{(24)}(n) q^n.$$

Using (33), we have the following asymptotic formula for the coefficients $p^{(24)}(n)$ of the function $f(\tau)$:

$$p^{(24)}(n) \sim \frac{1}{144\sqrt{2}} \frac{e^{4\pi\sqrt{n}}}{n^{27/4}}.$$

Denote $f_k := (-1)^{k/2}(E_k - 1)$, $k = 4, 6, 8$. Set

$$ff_k(\tau) = \sum_{n=1}^{\infty} a_{ff_k}(n) q^n.$$

In the proof of [5, Theorem 3], Dewar and Murty have shown that the Fourier coefficients $a_{ff_k}(n)$ of the function ff_k satisfy the following asymptotic formula:

$$a_{ff_k}(n) \sim \frac{1}{144\sqrt{2}} \frac{e^{4\pi\sqrt{n}}}{n^{27/4-k/2}}.$$

Since the Fourier coefficients of f and f_k are non-negative, $a_{ff_k}(n) \geq 0$ for all $n \geq 1$. Also $27/4 - k/2 > 0$ for $k = 4, 6, 8$. Using Theorem 1.1(i) we get the following asymptotic formula for the coefficients $b_{ff_k E_{l,1}}(N, r)$ of the functions $ff_k E_{l,1}$, $l = 4, 6$:

$$b_{ff_k E_{l,1}}(N, r) \sim \frac{1}{144\sqrt{2}} i^l 4^{27/4-(k+l)/2} \frac{e^{2\pi\sqrt{n}}}{N^{27/4-k/2-l/2+1/4}}$$

for any sequence $\{(N, r) \in \mathbb{Z} \times \mathbb{Z} : N \equiv -r^2 \pmod 4\}$ as $N \to \infty$. Now

$$f E_k E_{l,1} = f((-1)^{k/2} f_k + 1) E_{l,1} = (-1)^{k/2} ff_k E_{l,1} + f E_{l,1}.$$

Therefore

$$\begin{aligned} b_{\frac{E_k E_{l,1}}{144\Delta}}(N, r) &\sim i^k b_{ff_k E_{l,1}}(N, r) + b_{fE_{l,1}}(N, r) \\ &\sim \frac{1}{144\sqrt{2}} i^{k+l} 4^{27/4-(k+l)/2} \frac{e^{2\pi\sqrt{N}}}{N^{27/4-k/2-l/2+1/4}} \\ &\quad + \frac{1}{144\sqrt{2}} i^l 4^{27/4-l/2} \frac{e^{2\pi\sqrt{N}}}{N^{27/4-l/2+1/4}} \\ &\sim \frac{1}{144\sqrt{2}} i^{k+l} 4^{27/4-(k+l)/2} \frac{e^{2\pi\sqrt{N}}}{N^{27/4-k/2-l/2+1/4}}. \end{aligned}$$

From the expressions for the weak Jacobi forms $\varphi_{0,1}$ and $\varphi_{-2,1}$ and the above asymptotic formula, we get

$$\lim_{\substack{(N,r)\in\mathbb{Z}\times\mathbb{Z},\, N\equiv -r^2 \pmod{4m} \\ N\to\infty}} \frac{b_k(N, r)}{e^{2\pi\sqrt{N}}/N^{1-k/2}} = 0.$$

4.2. Proof of Corollary 1.4. By (33), the coefficients of the function

$$f(\tau) = q^{l/24} \frac{1}{\eta^l}(\tau) = \sum_{n\geq 0} p^{(l)}(n) q^n$$

satisfy the following asymptotic formula:

$$p^{(l)}(n) \sim \frac{c_l e^{A\sqrt{n}}}{n^{\alpha}},$$

where

$$c_l = \frac{1}{\sqrt{2}}\left(\frac{l}{24}\right)^{(l+1)/24}, \quad A = \pi\sqrt{\frac{2l}{3}}, \quad \alpha = \frac{l+3}{4}.$$

Now applying Theorem 1.1(ii) in this situation we get the following asymptotic formula for the coefficients $a_{k,l}(n)$ of the function θ^k/η^l:

$$a_{k,l}(n) \sim \frac{1}{\sqrt{2}}\left(\frac{l}{24}\right)^{(l+1)/24}\left(\frac{\sqrt{6}}{l}\right)^{k/2}\frac{e^{\pi\sqrt{2ln/3}}}{n^{(l-k+3)/4}}.$$

5. Further remarks. In [1], Bringmann and Richter have provided exact formulas for the Fourier coefficients of harmonic Maass–Jacobi Poincaré series. In some cases the harmonic Maass–Jacobi Poincaré series are weak Jacobi forms. For example the weak Jacobi form $\varphi_{-2,1}$ is such a Poincaré series, which can be seen from Example 1 of [2]. In [2], the authors provide explicit formulas for the Fourier coefficients of holomorphic parts of harmonic Maass–Jacobi forms, and in particular, for the Fourier coefficients of weakly holomorphic Jacobi forms. These explicit formulas might give a different approach to obtain the asymptotic behavior of the Fourier coefficients of certain weak Jacobi forms and weakly holomorphic Jacobi forms.

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