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The sum of digits of $\lfloor n^c \rfloor$

by

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1. Introduction. In this work q denotes an integer ≥ 2 and c is a non-integer positive real number. We use the notation $\mathrm{e}(x)$ for the exponential function $e^{2\pi i x}$. If x is a real number then $\|x\|$ denotes the distance from x to the nearest integer and $\{x\}$ is the fractional part of x.

Every integer $n \geq 0$ has a unique representation in base q of the form

$$n = \sum_{j=0}^{\nu} n_j q^j, \qquad n_j \in \{0, 1, \ldots, q-1\},$$

with $n_\nu \neq 0$. The sum-of-digits function $\mathrm{s}_q(n)$ is defined by $\mathrm{s}_q(n) = \sum_{j=0}^{\nu} n_j$. Gelfond [10] showed in 1968 that if $q, m > 1$ and r, ℓ, a are integers with $(m, q-1) = 1$, then

$$\#\{n \leq x : n \equiv \ell \bmod r,\ \mathrm{s}_q(n) \equiv a \bmod m\} = \frac{x}{mr} + O(x^\lambda), \tag{1.1}$$

where $\lambda < 1$ is a positive constant depending only on q and m. If one replaces the arithmetic progression $\{n \geq 0 : n \equiv \ell \bmod r\}$ by another sequence, then the corresponding question is in general much harder to answer. A first result concerning almost primes (positive integers consisting of at most two prime factors) was obtained by Fouvry and Mauduit [9]. In particular, they gave a lower bound on the number of almost primes m such that $\mathrm{s}_q(m)$ lies in a fixed residue class. Recently, Mauduit and Rivat [18] showed that $(\mathrm{s}_q(p))$, where p ranges over all primes, is well distributed in arithmetic progressions. (Drmota, Mauduit, and Rivat [7] also showed a local limit theorem.) The treatment of the sequence $(\mathrm{s}_q(P(n)))_{n\in\mathbb{N}}$, where $P(n)$ is a polynomial with $P(\mathbb{N}) \subseteq \mathbb{N}$, seems to be even more complex. Dartyge and Tenenbaum showed in [2] that if $(m, q-1) = 1$, then

$$\#\{n \leq x : \mathrm{s}_q(P(n)) \equiv a \bmod m\} \geq C x^{\min(1, 2/d!)},$$

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where d is the degree of the polynomial P and C is a positive constant depending on P, q and m. Mauduit and Rivat solved the problem in the quadratic case for general $q \geq 2$:

Theorem A. *Let q and m be integers ≥ 2. Set $d = (q-1, m)$ and $Q(a,d) = \#\{0 \leq n < d : n^2 \equiv a \bmod d\}$. Then there exists a constant $\sigma_{q,m} > 0$ such that for all $a \in \mathbb{Z}$* (1),

$$\#\{n \leq x : \mathrm{s}_q(n^2) \equiv a \bmod m\} = \frac{x}{m}\, Q(a,d) + O_{q,m}(x^{1-\sigma_{q,m}}).$$

Furthermore, the sequence $(\alpha\, \mathrm{s}_q(n^2))_{n\in\mathbb{N}}$ is uniformly distributed modulo 1 if and only if α is irrational.

Recently, Drmota, Mauduit, and Rivat considered in [6] the sequence $(\mathrm{s}_q(P(n)))_{n\in\mathbb{N}}$ for sufficiently large prime bases q:

Theorem B. *Let $d \geq 2$ be an integer, $q \geq q_0(d)$ a sufficiently large prime number, $P \in \mathbb{Z}[X]$ of degree d such that $P(\mathbb{N}) \subset N$ for which the leading coefficient a_d is coprime to q, and $m \geq 1$ an integer. Then there exists $\sigma_{q,m} > 0$ such that for all integers a,*

$$\#\{n \leq x : \mathrm{s}_q(P(n)) \equiv a \bmod m\} = \frac{x}{m}\, Q(a,D) + O(x^{1-\sigma_{q,m}}),$$

where $D = (q-1, m)$ and $Q(a,D) = \#\{0 \leq n < D : P(n) \equiv a \bmod D\}$. Furthermore, the sequence $(\alpha\, \mathrm{s}_q(P(n)))_{n\in\mathbb{N}}$ is uniformly distributed modulo 1 if and only if α is irrational.

A related question is whether a Gelfond type result also holds true for the sequence $(\mathrm{s}_q(\lfloor n^c \rfloor))_{n\in\mathbb{N}}$, where c is a non-integer real number. This can be understood as an intermediate case between polynomials of different degree. Mauduit and Rivat gave a positive answer for $c \in (1, 4/3)$ in 1995 (see [15]) and for $c \in (1, 7/5)$ in 2005 (see [16]). They considered more generally q-multiplicative functions and used, among other tools, the double large sieve of Bombieri and Iwaniec to solve this problem. In particular, they showed the following result:

Theorem C. *Let $c \in (1, 7/5)$ and $q \geq 2$. If $(a,m) \in \mathbb{N} \times \mathbb{N}^*$, then*

$$\lim_{x\to\infty} \frac{1}{x} \#\{n \leq x : \mathrm{s}_q(\lfloor n^c \rfloor) \equiv a \bmod m\} = \frac{1}{m}. \tag{1.2}$$

Furthermore, the sequence $(\alpha\, \mathrm{s}_q(\lfloor n^c \rfloor))_{n\in\mathbb{N}}$ is uniformly distributed modulo 1 if and only if α is irrational.

As pointed out by Mauduit (see [14, Section II.4]), one can deduce from a result of Harman and Rivat [12] that (1.2) holds for almost all $c \in [1, 2)$.

(1) $f = O_r(g)$ means that there exists a constant κ (depending on r) such that $|f| \leq \kappa g$.

Indeed, if $\mathcal{A}$ is an infinite set of positive integers such that $\#\{n \le x : n \in \mathcal{A}\} \gg x$, then [12, Theorem 3] implies that for almost all $c \in (1,2)$,

$$\#\{n \le x : \lfloor n^c \rfloor \in \mathcal{A}\} = \gamma \sum_{\substack{n \le x^c \\ n \in \mathcal{A}}} n^{-1+\gamma} + o(x), \tag{1.3}$$

where $\gamma = 1/c$. Setting $\mathcal{A} = \{n \in \mathbb{N} : \mathrm{s}_q(n) \equiv a \bmod m\}$, a refined version of Gelfond's work (cf. (1.1)) implies that $\#\{n \le x : n \in \mathcal{A}\} \gg x$. Elementary discrete Fourier analysis and partial summation (similar to Section 6.1 below) allow one to evaluate the sum occurring in (1.3) and we finally deduce that (1.2) holds true for almost all $c \in (1,2)$ and for every integer $q \ge 2$ and $(a,m) \in \mathbb{N} \times \mathbb{N}^*$.

This leads to the following conjecture which can be found in [14, Conjecture 1]:

Conjecture 1 (Mauduit). *For almost all $c > 1$ we have, for every integer q and m greater than 1 and $0 \le a < m$,*

$$\lim_{x\to\infty} \frac{1}{x} \#\{n \le x : \mathrm{s}_q(\lfloor n^c \rfloor) \equiv a \bmod m\} = \frac{1}{m}. \tag{1.4}$$

Other interesting questions deal with the asymptotic behavior of the sum-of-digits function of $\lfloor n^c \rfloor$. Using a method of Bassily and Kátai [1], it is relatively easy to show that $\mathrm{s}_q(\lfloor n^c \rfloor)$ satisfies a central limit theorem. More precisely, we have

$$\frac{1}{x} \#\Big\{n \le x : \mathrm{s}_q(\lfloor n^c \rfloor) \le c\mu_q \log_q x + y\sqrt{\sigma_q^2 c \log_q x}\Big\} = \Phi(y) + o(1), \tag{1.5}$$

where

$$\mu_q := \frac{q-1}{2}, \qquad \sigma_q^2 := \frac{q^2-1}{12},$$

and $\Phi(y)$ denotes the normal distribution function (see [5]).

2. Main results. The main objective of this paper is to enlarge the range of possible real numbers c in Theorem C for which we can show uniform distribution results (Corollaries 1 and 2). We are able to deal with all positive real numbers c which are not integers but we restrict ourselves to bases q which are not too small. It turns out that the case $c \in \mathbb{N}$ is of completely different nature. This makes it eventually impossible to treat general numbers c with the methods presented in this paper. In Section 5 we will provide a precise analysis of this problem and discuss the differences of our method from the methods used in [6] (see Remark 7).

Furthermore, we show a local limit theorem (Corollary 3) which generalizes (1.5).

In our main theorem we study the exponential sum $\sum_n \mathrm{e}(\alpha\, \mathrm{s}_q(\lfloor n^c \rfloor))$:

Theorem 1. *Let $c > 0$ be a non-integer real number and let $\alpha \in \mathbb{R}$. Then there exists a constant $q_0(c)$ such that for all $q \geq q_0(c)$ we have* (2)

$$\sum_{1 \leq n \leq x} \mathrm{e}(\alpha\, \mathrm{s}_q(\lfloor n^c \rfloor)) \ll_{c,q} (\log x) x^{1-\sigma_{c,q}\|(q-1)\alpha\|^2}, \tag{2.1}$$

where $\sigma_{c,q} > 0$ is a computable positive constant. In the case $0 < c < 1$ we have $q_0(c) = 2$ and the exponent on the right hand side of (2.1) *can be replaced by $1 - \sigma_{c,q}\|\alpha\|^2$.*

Remark 1. It follows from our proof that an admissible value of $q_0(c)$ is explicitly computable and that this value is bounded by Kc^{c^4}, where K is an absolute constant. We use different methods to show the result for different values of c in order to optimize $q_0(c)$ (see Sections 4 and 5 and the end of Section 6). If $1 < c < 7/5$, then [16, Theorem 1] and partial summation ensure that we can choose $q_0(c) = 2$. The case $0 < c < 1$ can be regarded trivial but for completeness we give a short proof in Section 6.

Corollary 1. *Let $c > 0$ be a non-integer real number. There exists a constant $q_0(c) \geq 2$ such that for all $q \geq q_0(c)$ the following holds: If $(a, m) \in \mathbb{N} \times \mathbb{N}^*$, then there exists a constant $\sigma_{q,m,c} > 0$ such that*

$$\#\{n \leq x : \mathrm{s}_q(\lfloor n^c \rfloor) \equiv a \bmod m\} = \frac{x}{m} + O_{c,q,m}(x^{1-\sigma_{q,m,c}}).$$

Remark 2. Corollary 1 does not solve Conjecture 1 entirely, but it leads us to conjecture that (1.4) is valid for every $c > 1$ ($c \notin \mathbb{N}$). If $c > 1$ is an integer, then elementary arithmetic calculations may yield a different asymptotic formula which depends on a, m and q (cf. [6] and Theorem A).

Corollary 2. *Let $c > 0$ be a non-integer real number. There exists a constant $q_0(c)$ such that for all $q \geq q_0(c)$ the sequence $(\alpha\, \mathrm{s}_q(\lfloor n^c \rfloor))_{n \in \mathbb{N}}$ is uniformly distributed modulo 1 if and only if α is irrational.*

Corollary 3. *Let $c > 0$ be a non-integer real number. There exists a constant $q_0(c) \geq 2$ such that for all $q \geq q_0(c)$ the following holds: Uniformly for all integers $k \geq 0$ we have*

$$\frac{1}{x}\#\{n \leq x : \mathrm{s}_q(\lfloor n^c \rfloor) = k\} = \frac{1}{\sqrt{2\pi\sigma_q^2 c \log_q x}}\left(e^{-\Delta_k^2/2} + O_{c,q}\left(\frac{(\log\log x)^7}{(\log x)^{1/2}}\right)\right),$$

where $\Delta_k = \dfrac{k - \mu_q c \log_q x}{\sqrt{\sigma_q^2 c \log_q x}}$.

The main idea of showing Theorem 1 is to divide the proof up into a Fourier theory part and an exponential sums part (where no sum-of-digits function occurs). In Section 3 we state some known results on the discrete

(2) The symbol $f \ll_r g$ means that $f = O_r(g)$.

Fourier transform of the sum-of-digits function. In the two subsequent sections we discuss the sum $\sum_n \mathrm{e}(\beta\lfloor n^c\rfloor)$ (we present a method which works for $1 < c < 2$ in Section 4 and a general method in Section 5). In Section 6 we finally prove Theorem 1. Section 7 is devoted to the proofs of Corollaries 1 and 2. In the last section, we give a proof of Corollary 3.

3. Fourier transform of $\alpha\,\mathrm{s}_q(\cdot)$**.** Let $q \geq 2$, $\alpha \in \mathbb{R}$ and $\lambda \in \mathbb{N}$. The discrete Fourier transform $F_\lambda(\cdot, \alpha)$ of the function $u \mapsto \mathrm{e}(\alpha\,\mathrm{s}_q(\cdot))$ is defined for all $h \in \mathbb{Z}$ by

$$F_\lambda(h, \alpha) = \frac{1}{q^\lambda} \sum_{0 \leq u < q^\lambda} \mathrm{e}(\alpha\,\mathrm{s}_q(u) - huq^{-\lambda}).$$

This function is periodic with period q^λ in the first component and can be represented by a trigonometric product. Indeed, we have

$$|F_\lambda(h, \alpha)| = q^{-\lambda} \prod_{1 \leq j \leq \lambda} \varphi_q(\alpha - hq^{-j}),$$

where φ_q is defined by

$$\varphi_q(t) = \begin{cases} |\sin \pi q t| / |\sin \pi t| & \text{if } t \in \mathbb{R} \setminus \mathbb{Z}, \\ q & \text{if } t \in \mathbb{Z}. \end{cases} \tag{3.1}$$

Next, we state upper bounds of the L^1 and L^∞ norm of F_λ which are of particular importance for the proof of our main theorem. For a thorough analysis of φ_q and F_λ see [18, 17].

Lemma 1. *Let $q \geq 2$, $\alpha \in \mathbb{R}$, $h \in \mathbb{Z}$, $\lambda \geq 1$ and*

$$\sigma_q = \frac{\pi^2}{12 \log q}\left(1 - \frac{2}{q+1}\right).$$

Then

$$|F_\lambda(h, \alpha)| \leq e^{\pi^2/48} q^{-\sigma_q \|(q-1)\alpha\|^2 \lambda}.$$

Proof. This is Lemma 9 of [17]. ∎

Lemma 2. *For $q \geq 3$, $\alpha \in \mathbb{R}$ and $\lambda \geq 1$ we have*

$$\sum_{0 \leq h < q^\lambda} |F_\lambda(h, \alpha)| \leq q^{\eta_q \lambda},$$

where η_q is defined by

$$q^{\eta_q} = \max_{t \in \mathbb{R}} \left(\frac{1}{q} \sum_{0 \leq r < q} \varphi_q\left(t + \frac{r}{q}\right)\right).$$

Proof. This is (a part of) Lemma 16 of [18]. ∎

REMARK 3. The Cauchy–Schwarz inequality (together with [18, Lemma 13]) implies that $\eta_q \le 1/2$. Mauduit and Rivat showed in [18, Lemma 14] that

$$q^{\eta_q} = \frac{1}{q}\sum_{r=0}^{q-1} \frac{1}{\sin\frac{\pi}{q}\left(\frac{1}{2}+r\right)} \le \frac{2}{q\sin\frac{\pi}{2q}} + \frac{2}{\pi}\log\frac{2q}{\pi}.$$

This implies for example that $\eta_q \le (\log\log q)/(\log q)$ for $q \ge 15$ and hence η_q is arbitrarily small if q is large enough. Finally, we want to remark that the case $q = 2$ is treated in [18, Lemma 18].

4. Exponential sums for $1 < c < 2$. In this section we treat the exponential sum $\sum_n \mathrm{e}(\beta\lfloor n^c\rfloor)$ for $1 < c < 2$.

PROPOSITION 1. *Let $1 < c < 2$ and $x, \nu \in \mathbb{N}$ with $q^{\nu-1} < x \le q^\nu$. Furthermore, let $\beta \in \mathbb{R}\setminus\mathbb{Z}$. Then*

$$\sum_{q^{\nu-1}<n\le x} \mathrm{e}(\beta\lfloor n^c\rfloor) \ll_{c,q} \nu q^{\nu(1-(2-c)/3)} + \frac{1}{\|\beta\|} q^{\nu(1-c)}.$$

The method of proving this proposition is based on work of Mauduit and Rivat [16] and uses the fact that an integer m has the form $m = \lfloor n^c\rfloor$ if and only if

$$\lfloor -m^\gamma\rfloor - \lfloor -(m+1)^\gamma\rfloor = 1,$$

where $\gamma = 1/c$. If we set $\Psi(u) = u - \lfloor u\rfloor - 1/2$, then we obtain

$$\begin{aligned}(4.1)\qquad \sum_{q^{\nu-1}<n\le x} \mathrm{e}(\beta\lfloor n^c\rfloor) &= \sum_{q^{(\nu-1)c}<m\le x^c} \mathrm{e}(\beta m)(\lfloor -m^\gamma\rfloor - \lfloor -(m+1)^\gamma\rfloor)\\ &= \sum_{q^{(\nu-1)c}<m\le x^c} \mathrm{e}(\beta m)((m+1)^\gamma - m^\gamma)\\ &\quad + \sum_{q^{(\nu-1)c}<m\le x^c} \mathrm{e}(\beta m)(\Psi(-(m+1)^\gamma) - \Psi(-m^\gamma)).\end{aligned}$$

The first sum on the right hand side of (4.1) can be estimated by partial summation (see Lemma 3). To treat the second sum we follow the proof of [16, Lemma 3]. This leads us to consider the double sum

$$(4.2)\qquad S(K,\tilde{M},u) = \sum_{K<|k|\le 2K} \Big|\sum_{M<m\le\tilde{M}} f(m)\mathrm{e}(k(m+u)^\gamma)\Big|,$$

where $f(m) = \mathrm{e}(\beta m)$. The main difference from the cited lemma of Mauduit and Rivat is that they have to deal with q-multiplicative functions $f(m)$

instead of $\mathrm{e}(\beta m)$. Using van der Corput's method of estimating exponential sums finally enables us to obtain the desired result (see Lemma 4).

Lemma 3. *Let $c > 1$ and $\gamma = 1/c$. Furthermore, let $x, \nu \in \mathbb{N}$ with $q^{\nu-1} < x \le q^{\nu}$ and $\beta \in \mathbb{R} \setminus \mathbb{Z}$. Then*

$$\sum_{q^{(\nu-1)c} < m \le x^c} \mathrm{e}(\beta m)((m+1)^{\gamma} - m^{\gamma}) \tag{4.3}$$
$$\le \frac{\gamma}{|\sin \pi\beta|} q^{(\nu-1)(1-c)}(2 - q^{1-c}) + \frac{1}{4} \ll_{c,q} \frac{1}{\|\beta\|} q^{\nu(c-1)} + 1.$$

Proof. Let S be the sum in question. First we recall a result of [16, Lemma 2]. If $\theta \in [0,1]$, then

$$\sum_{m \ge 1} |(m+1)^{\theta} - m^{\theta} - \theta m^{\theta-1}| \le \frac{1}{4}.$$

Using this fact, we obtain

$$|S| \le \Big| \sum_{q^{(\nu-1)c} < m \le x^c} \gamma m^{\gamma-1} \mathrm{e}(\beta m) \Big| + \frac{1}{4}.$$

Partial summation yields

$$\sum_{q^{(\nu-1)c} < m \le x^c} \gamma m^{\gamma-1} \mathrm{e}(\beta m) = \gamma x^{c(\gamma-1)} \sum_{q^{(\nu-1)c} < m \le x^c} \mathrm{e}(\beta m)$$
$$- \gamma(\gamma-1) \int_{q^{(\nu-1)c}}^{x^c} \sum_{q^{(\nu-1)c} < m \le u} \mathrm{e}(\beta m) u^{\gamma-2}\, du.$$

Since $\beta \notin \mathbb{Z}$, for all $q^{(\nu-1)c} < u \le x^c$ we have

$$\Big| \sum_{q^{(\nu-1)c} < m \le u} \mathrm{e}(\beta m) \Big| \le \frac{1}{|\sin \pi\beta|}.$$

We get (note that $x \le q^{\nu}$)

$$S \le \frac{\gamma}{|\sin \pi\beta|} \Big(q^{(\nu-1)c(\gamma-1)} - \int_{q^{(\nu-1)c}}^{x^c} (\gamma-1) u^{\gamma-2}\, du \Big) + \frac{1}{4}$$
$$\le \frac{\gamma}{|\sin \pi\beta|} q^{(\nu-1)(1-c)}(2 - q^{1-c}) + \frac{1}{4},$$

and the result follows. ∎

Lemma 4. *Let $c \in (1,2)$ and $\beta \in \mathbb{R}$. Furthermore, let x and ν be integers with $q^{\nu-1} < x \le q^{\nu}$. Then*

$$\sum_{q^{(\nu-1)c} < m \le x^c} \mathrm{e}(\beta m)(\Psi(-(m+1)^{\gamma}) - \Psi(-m^{\gamma})) \ll_q \nu q^{\nu(1-(2-c)/3)}.$$

Proof. We can write

$$\sum_{q^{(\nu-1)c}<n\le x^c} \mathrm{e}(\beta m)(\Psi(-(m+1)^\gamma)-\Psi(-m^\gamma))$$
$$= \sum_{0\le j<c\log q/\log 2} \sum_{\substack{q^{(\nu-1)c}2^j<n\le q^{(\nu-1)c}2^{j+1}\\ q^{(\nu-1)c}<n\le x}} \mathrm{e}(\beta m)(\Psi(-(m+1)^\gamma)-\Psi(-m^\gamma))$$
$$\ll_q \max_{q^{(\nu-1)c}\le M\le q^{\nu c}} \max_{M<M'\le 2M} \sum_{M<n\le M'} \mathrm{e}(\beta m)(\Psi(-(m+1)^\gamma)-\Psi(-m^\gamma)).$$

In order to prove Lemma 4, it suffices to show that for $M>q^{(\nu-1)c}$ we have

$$S_M := \Big|\sum_{M<m\le M'} \mathrm{e}(\beta m)(\Psi(-(m+1)^\gamma)-\Psi(-m^\gamma))\Big| \ll (\log M)M^{\gamma(1-(2-c)/3)}. \tag{4.4}$$

The next steps are very similar to the proof of [16, Lemma 3]. Thus, we give only a rough outline. We begin by approximating Ψ by trigonometric polynomials. Let $K\ge 1$ be an integer. Then it follows from a theorem of Vaaler [22, Theorem 18] that there exist coefficients $a_K(k)$ with $0\le a_K(k)\le 1$ such that the trigonometric polynomials

$$\Psi_K(t) = -\frac{1}{2i\pi}\sum_{1\le|k|\le K}\frac{a_K(k)}{k}\mathrm{e}(kt)$$

and

$$\kappa_K(t) = \sum_{|k|\le K}\Big(1-\frac{|k|}{K+1}\Big)\mathrm{e}(kt) \tag{4.5}$$

satisfy

$$|\Psi(t)-\Psi_K(t)| \le \frac{1}{2K+2}\kappa_K(t).$$

Note that $\kappa_K(t)$ is the periodic and positive Fejér kernel and that

$$\frac{1}{2K+2}\sum_{M\le m\le 2M}\kappa_K(m^\theta) \ll_\theta K^{-1}M + K^{1/2}M^{\theta/2} + K^{-1/2}M^{1-\theta/2} \tag{4.6}$$

for every $0<\theta<1$ and for every $M\ge 1$ (this is [16, Lemma 5] and can be shown by using [11, Theorem 2.2]). We set $K_0 := \lfloor M^{1-\gamma(1-\delta)}\rfloor$, where $\delta>0$ will be chosen later on, and obtain

$$S_M \le \Big|\sum_{M<m\le M'}\mathrm{e}(\beta m)(\Psi_{K_0}(-(m+1)^\gamma)-\Psi_{K_0}(-m^\gamma))\Big| + \frac{1}{2K+2}\sum_{M<m\le M'}\kappa_{K_0}(-(m+1)^\gamma) + \sum_{M<m\le M'}\kappa_{K_0}(-m^\gamma).$$

The last two sums can be handled by (4.6). This yields

$$S_M \le \Big| \sum_{M<m\le M'} \mathrm{e}(\beta m)(\Psi_{K_0}(-(m+1)^\gamma) - \Psi_{K_0}(-m^\gamma))\Big| + K_0^{-1}M + K_0^{1/2}M^{\gamma/2} + K_0^{-1/2}M^{1-\gamma/2}.$$

For our special choice of K_0 we have

$$K_0^{1/2}M^{\gamma/2} = M^{1/2+\gamma\delta/2} \ge M^{1/2-\gamma\delta/2} = K_0^{-1/2}M^{1-\gamma/2}.$$

Thus

$$S_M \ll \Big| \sum_{M<m\le M'} \mathrm{e}(\beta m)(\Psi_{K_0}(-(m+1)^\gamma) - \Psi_{K_0}(-m^\gamma))\Big| + M^{\gamma(1-\delta)} + M^{1/2+\gamma\delta/2}. \tag{4.7}$$

Next we treat the sum that arises in (4.7). Replacing Ψ_{K_0} by its expression and following exactly the same steps as in [16, Section 2.3], we obtain

$$\sum_{M<m\le M'} \mathrm{e}(\beta m)(\Psi_{K_0}(-(m+1)^\gamma) - \Psi_{K_0}(-m^\gamma)) \ll (\log K_0) \max_{0<K\le K_0} \max_{u\in\{0,1\}} \max_{\tilde{M}\in[M,2M]} \min(M^{1-\gamma}, K^{-1}) S(K,\tilde{M},u), \tag{4.8}$$

where $S(K,\tilde{M},u)$ is defined by (4.2). In the interval $[M,2M]$ considered we have the estimate

$$|k|M^{\gamma-2} \ll \left|\frac{d^2(\beta y + k(y+u)^\gamma)}{dy^2}\right| \ll |k|M^{\gamma-2}.$$

It follows from [11, Theorem 2.2] that

$$S(K,\tilde{M},u) \ll \sum_{K<k\le 2K} (k^{1/2}M^{\gamma/2} + k^{-1/2}M^{1-\gamma/2}) \ll K^{3/2}M^{\gamma/2} + K^{1/2}M^{1-\gamma/2}.$$

If $K \le M^{1-\gamma}$ we have

$$M^{\gamma-1}S(K,\tilde{M},u) \ll K^{3/2}M^{3\gamma/2-1} + K^{1/2}M^{\gamma/2} \ll M^{1/2},$$

whereas

$$K^{-1}S(K,\tilde{M},u) \ll K^{1/2}M^{\gamma/2} + K^{-1/2}M^{1-\gamma/2} \ll K^{1/2}M^{\gamma/2} + M^{1/2}$$

if $K > M^{1-\gamma}$. With (4.8) and the definition of K_0 we get

$$\sum_{M<m\le M'} \mathrm{e}(\beta m)(\Psi_{K_0}(-(m+1)^\gamma) - \Psi_{K_0}(-m^\gamma)) \ll (\log K_0)(K_0^{1/2}M^{\gamma/2} + M^{1/2}) \ll (\log M)M^{1/2+\gamma\delta/2}.$$

Finally (see (4.7)),

$$S_M \ll (\log M)(M^{\gamma(1-\delta)} + M^{1/2+\gamma\delta/2}).$$

Now we choose $\delta > 0$ such that the upper bound is as small as possible. This is apparently the case if $\delta = (2-c)/3$ and we are done. ∎

Proof of Proposition 1. The proposition follows immediately from equation (4.1) and the previous two lemmas. ∎

5. Exponential sums for $c > 1$, $c \notin \mathbb{N}$. In this section we give a non-trivial upper bound of the sum $\sum_n \mathrm{e}(\beta\lfloor n^c\rfloor)$ for all real numbers $c > 1$ which are different from an integer. If $1 < c < 19/11$, then it turns out that the method based on Mauduit and Rivat's work gives a better result (see Remark 4).

If $\|\beta\|$ is relatively small, then the estimation of $\sum_n \mathrm{e}(\beta\lfloor n^c\rfloor)$ can be reduced to a similar problem where $\mathrm{e}(\beta\lfloor n^c\rfloor)$ is replaced by $\mathrm{e}(\beta n^c)$. This leads to a simple application of the Kusmin–Landau Theorem. In the other case, we enhance a method used by Deshouillers to obtain a non-trivial upper bound.

PROPOSITION 2. *Let c be a real number > 1 and x and ν be integers such that $q^{\nu-1} < x \le q^\nu$. Furthermore, let $\beta \in \mathbb{R}$ with $0 < \|\beta\| < \frac{1}{2c}q^{\nu(1-c)}$. Then*

$$\sum_{q^{\nu-1}<n\le x} \mathrm{e}(\beta\lfloor n^c\rfloor) \ll_{c,q} \frac{1}{\|\beta\|}q^{\nu(1-c)} + q^{\nu(2-c)}. \tag{5.1}$$

Proof. Let S be the sum in (5.1). Without loss of generality, we can assume that $0 < \beta < \frac{1}{2c}q^{\nu(1-c)}$. Since

$$\mathrm{e}(\beta\lfloor n^c\rfloor) = \mathrm{e}(\beta n^c)\mathrm{e}(-\beta\{n^c\}) = \mathrm{e}(\beta n^c)(1 + O(\beta)),$$

we obtain

$$|S| = \Big|\sum_{q^{\nu-1}<n\le x} \mathrm{e}(\beta n^c)\mathrm{e}(-\beta\{n^c\})\Big| \ll \Big|\sum_{q^{\nu-1}<n\le x} \mathrm{e}(\beta n^c)\Big| + \frac{1}{2c}q^{\nu(2-c)}.$$

Thus, it suffices to consider the last sum. If we set $f(y) = \beta y^c$, then we have for $y \in [q^{\nu-1}, q^\nu]$ the estimate

$$c\beta q^{(\nu-1)(c-1)} \le |f'(y)| \le c\beta q^{\nu(c-1)} \le 1/2.$$

Furthermore, $f''(y) \ne 0$ on the interval considered. Hence, we can use [11, Theorem 2.1] (the Kusmin–Landau Theorem) to get

$$\sum_{q^{\nu-1}<n\le x} \mathrm{e}(\beta n^c) \ll_{c,q} \frac{1}{\beta}q^{\nu(1-c)}.$$

This proves the desired result. ∎

In order to state the next proposition, we define the constant $\rho = \rho(c)$ by

$$\rho := \max(\rho_1, \rho_2, \rho_3, \rho_4), \tag{5.2}$$

where

$$\rho_1 = \frac{\lfloor c \rfloor + 1 - c}{2^{\lfloor c \rfloor + 1} - 1}, \quad \rho_3 = \left(3\left\lfloor c + \frac{301}{300}\right\rfloor^2 \log\left(125\left\lfloor c + \frac{301}{300}\right\rfloor\right)\right)^{-1},$$

$$\rho_2 = \frac{\lfloor c \rfloor + 2 - c}{2^{\lfloor c \rfloor + 2} - 1}, \quad \rho_4 = 2^{-18}\left(c + \frac{1}{2^{18}c^2}\right)^{-2}.$$

See Figure 1 for the terms considered in the definition of ρ in the interval $[1, 4]$ and Figure 2 in the interval $[9, 12]$. If $c < 12 - 1365/(121 \log 1375) \approx 10.4388$, then ρ_1 and ρ_2 contribute to the size of ρ. If otherwise $c > 12 - 1365/(121 \log 1375)$ then $\rho = \rho_3$ until ρ_4 is significant.

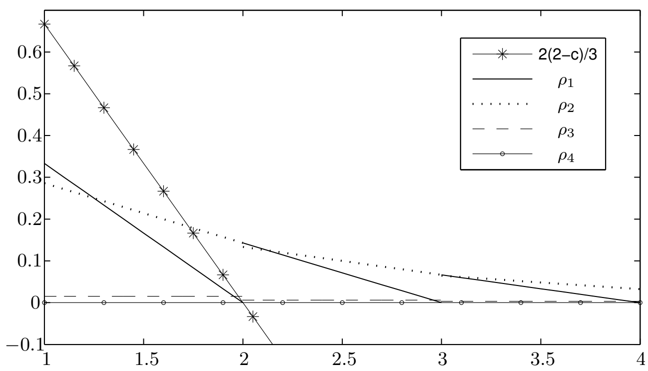

Fig. 1. ρ in the interval $[1, 4]$

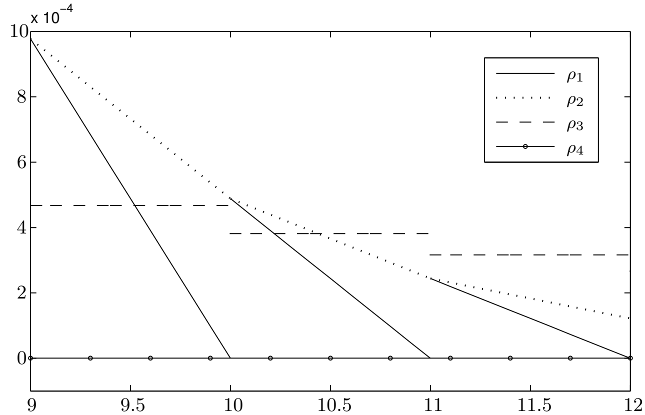

Fig. 2. ρ in the interval $[9, 12]$

Proposition 3. *Let $c > 1$ be a real number. Furthermore, let x and ν be integers with $q^{\nu-1} < x \le q^{\nu}$ and $\beta \in \mathbb{R}$ be such that $\|\beta\| \ge \frac{1}{2c} q^{\nu(1-c)}$. Then*

$$\sum_{q^{\nu-1}<n\le x} \mathrm{e}(\beta \lfloor n^c \rfloor) \ll_{c,q} \nu q^{\nu(1-\rho/2)}, \tag{5.3}$$

where ρ is defined by (5.2).

Remark 4. If $1 < c < 19/11$, then Proposition 1 implies Proposition 3. Indeed, $2(2-c)/3$ is greater than ρ in this case (see Figure 1) and the method of Mauduit and Rivat gives a better upper bound.

Remark 5. Let $c > 1$ be a non-integer real number and x and ν be integers with $q^{\nu-1} < x \le q^{\nu}$. If we set $\tilde{\rho} := \max(2(2-c)/3, \rho)$, then Proposition 1 together with Propositions 2 and 3 implies

$$\sum_{q^{\nu-1}<n\le x} \mathrm{e}(\beta \lfloor n^c \rfloor) \ll_{c,q} \nu q^{\nu(1-\tilde{\rho}/2)} + \frac{1}{\|\beta\|} q^{\nu(1-c)}$$

for every $\beta \in \mathbb{R} \setminus \mathbb{Z}$.

Remark 6. As already pointed out, the method of this section goes back to Deshouillers [3]. He showed that if $c > 12$ ($c \notin \mathbb{N}$) and $\|\beta\|$ is not too small, then the sum (5.3) is of order $O(x^{1-\rho})$, where $\rho = (6c^2(\log c + 14))^{-1}$. We improve this result by enhancing two main tools of his method. On the one hand, we use van der Corput's method for exponential sums with small c and a refined version of Vinogradov's method for exponential sums with large c (see Lemma 5). On the other hand, we employ Vaaler's method of approximate functions with bounded variation.

Remark 7. The method presented in this section cannot be applied for $c \in \mathbb{N}$. Note that Lemma 5 is false for integer exponents (take for example $\xi = 1$). The main difference for $c \in \mathbb{N}$ is that the mth derivative of x^c is zero if $m \ge c+1$ (cf. (5.5)). This makes it impossible to use van der Corput's and Vinogradov's method for exponential sums (even for $\xi < 1$). To prove Theorem B, Drmota et al. (see [6]) use a van der Corput-type inequality, which leads them to study sums of the form

$$\sum_{n} \mathrm{e}(\alpha\, \mathrm{s}_q(P(n+r)) - \alpha\, \mathrm{s}_q(P(n))),$$

where P is a polynomial of degree d. If r is small (compared to n), then in "most" of the cases the higher placed digits of $P(n+r)$ are the same as those of $P(n)$. Using this fact, the authors of [6] are able to apply Fourier-analytic tools in order to succeed. However, in doing so, they have to deal with congruence conditions that seem to be difficult to handle if one replaces $P(n)$ by $\lfloor n^c \rfloor$ for a non-integer valued positive real number c.

LEMMA 5. *Let $c>1$ be a non-integer real number and define ρ by (5.2). Furthermore, let x and ν be integers satisfying $q^{\nu-1} < x \le q^{\nu}$ and let $\xi \in \mathbb{R}$ be such that $\frac{1}{2c} q^{\nu(1-c)} \le |\xi| \le q^{(\nu-1)\rho}$. Then*

$$\sum_{q^{\nu-1}<n\le x} \mathrm{e}(\xi n^c) \ll_{c,q} q^{\nu(1-\rho)}.$$

Proof. We can write

$$\sum_{q^{\nu-1}<n\le x} \mathrm{e}(\xi n^c) = \sum_{0\le j<\frac{\log q}{\log 2}} \sum_{\substack{q^{\nu-1}2^j<n\le q^{\nu-1}2^{j+1}\\ q^{\nu-1}<n\le x}} \mathrm{e}(\xi n^c)$$

$$\ll_q \max_{q^{\nu-1}\le M\le q^{\nu}} \max_{M<M'\le 2M} \sum_{M<n\le M'} \mathrm{e}(\xi n^c).$$

Since for any $q^{\nu-1} \le M \le q^{\nu}$ we have

$$\frac{1}{2c} q^{1-c} M^{1-c} \le \frac{1}{2c} q^{\nu(1-c)} \le |\xi| \le q^{(\nu-1)\rho} \le M^{\rho},$$

it suffices to show that for $M \ge 1$, $M < M' \le 2M$ and $\frac{1}{2c} q^{1-c} M^{1-c} \le |\xi| \le M^{\rho}$ we have

$$\sum_{M<n\le M'} \mathrm{e}(\xi n^c) \ll_{c,q} M^{1-\rho}. \tag{5.4}$$

We set $f(y) = \xi y^s$. Then we derive, for every $m \ge 1$,

$$\left|\frac{y^m}{m!} f^{(m)}(y)\right| = |\xi| \left|\binom{c}{m}\right| y^c.$$

A short calculation shows that

$$\frac{\|c\|}{2m^{c+1}} \le \left|\binom{c}{m}\right| \le c^m.$$

Hence, there exists a constant $A = A(c,q) > 1$ such that

$$A^{-m} F \le \left|\frac{y^m}{m!} f^{(m)}(y)\right| \le A^m F \tag{5.5}$$

for every $y \in [M, 2M]$ and $m \ge 1$, where $F = |\xi| M^c$. In order to get a manageable notation, we set $\ell = (\log|\xi|)/(\log M)$. Then we have

$$M \ll \frac{1}{2c} q^{1-c} M^{1-c} M^c \le |\xi| M^c = F = M^{\ell+c} \le M^{\rho+c}.$$

We can apply [11, Theorem 2.9] (a van der Corput estimate) and deduce that for every $r \ge 0$,

$$\sum_{M<n\le M'} \mathrm{e}(\xi n^c) \ll_{c,q,r} F^{\frac{1}{2^{r+2}-2}} M^{1-\frac{r+2}{2^{r+2}-2}} = M^{1-\frac{r+2-\ell-c}{2^{r+2}-2}}. \tag{5.6}$$

Let us fix c. Then we find that ρ is equal to one of the four possible choices ρ_1, ρ_2, ρ_3 or ρ_4 (see (5.2)). Recall that ρ can be equal to ρ_3 or ρ_4 only if $c \geq 12 - 1365/(121 \log 1375)$.

First, we assume that $\rho = \rho_1 = (\lfloor c\rfloor + 1 - c)/(2^{\lfloor c\rfloor+1} - 1)$. Using inequality (5.6) with $r = \lfloor c\rfloor - 1$, we obtain

$$\sum_{M<n\leq M'} \mathrm{e}(\xi n^c) \ll_{c,q} M^{1-\frac{\lfloor c\rfloor+1-\ell-c}{2^{\lfloor c\rfloor+1}-2}} \ll_{c,q} M^{1-\rho_1}.$$

The last inequality follows from the fact that

$$\frac{\lfloor c\rfloor + 1 - \ell - c}{2^{\lfloor c\rfloor+1} - 2} \geq \frac{\lfloor c\rfloor + 1 - \rho_1 - c}{2^{\lfloor c\rfloor+1} - 2} = \rho_1. \tag{5.7}$$

Next we consider the case $\rho = \rho_2 = (\lfloor c\rfloor + 2 - c)/(2^{\lfloor c\rfloor+2} - 1)$. We apply inequality (5.6) with $r = \lfloor c\rfloor$ and obtain

$$\sum_{M<n\leq M'} \mathrm{e}(\xi n^c) \ll_{c,q} M^{1-\frac{\lfloor c\rfloor+2-\ell-c}{2^{\lfloor c\rfloor+2}-2}} \ll_{c,q} M^{1-\rho_2}.$$

The same calculation as above (see (5.7)) verifies the last inequality. Note that we cannot improve these estimates by employing (5.6) with other values of r. Indeed, it is easy to show that for $c > 1$,

$$\sup_{r\geq 0}\left(\frac{r+2-c}{2^{r+2}-1}\right) = \max\left(\frac{\lfloor c\rfloor + 1 - c}{2^{\lfloor c\rfloor+1} - 1}, \frac{\lfloor c\rfloor + 2 - c}{2^{\lfloor c\rfloor+2} - 1}\right).$$

If c is large (and ρ is small), then we use van der Corput's method in combination with Vinogradov's method. Let us assume that $\rho = \rho_3$. As already noticed, c must be larger than 10 in this case. For $\ell < 10 - c$ we use (5.6) with $r = \lfloor c + \ell\rfloor$ and obtain

$$\sum_{M<n\leq M'} \mathrm{e}(\xi n^c) \ll_{c,q} M^{1-\frac{1}{2^{\lfloor c+\ell\rfloor+2}-2}}.$$

Note that $\lfloor c + \ell\rfloor \leq 9$ and that we have, for $c > 10$,

$$\frac{1}{2^{11}-2} > 0.000488 > 0.000382 > \frac{1}{3\lfloor 10 + \frac{301}{300}\rfloor^2 \log\left(125\lfloor 10 + \frac{301}{300}\rfloor\right)} \geq \rho_3.$$

Hence, we get

$$\sum_{M<n\leq M'} \mathrm{e}(\xi n^c) \ll_{c,q} M^{1-\rho_3}.$$

If $10 - c \leq \ell \leq \rho$, then

$$M \leq M^{-\ell-c+\lfloor \ell+c+1\rfloor+1} = F^{-1} M^{\lfloor \ell+c+1\rfloor+1} \leq M^2,$$

and $\lfloor \ell+c+1\rfloor \geq 11$. This allows us to use a well-known result of Vinogradov

[23, Theorem 2a, p. 109]. We get

$$\sum_{M<n\le M'} \mathrm{e}(\xi n^c) \ll_{c,q} M^{1-\frac{1}{3\lfloor \ell+c+1\rfloor^2 \log(125\lfloor \ell+c+1\rfloor)}} \ll_{c,q} M^{1-\rho_3}$$

in this case too.

It remains to consider the case $\rho = \rho_4 = 2^{-18}(c+1/(2^{18}c^2))^{-2}$. Again, this is only possible if $c > 10 > 4$ and we employ (5.6) if $\ell < 4-c$ with $r = \lfloor \ell + c \rfloor$. We get

$$\sum_{M<n\le M'} \mathrm{e}(\xi n^c) \ll_{c,q} M^{1-\frac{1}{2^{\lfloor c+\ell\rfloor+2}-2}} \ll_{c,q} M^{1-\frac{1}{2^5-2}} \ll_{c,q} M^{1-\rho_4}.$$

On the contrary, if $4-c \le \ell \le \rho_4$, then we can write

$$M^4 = M^{4-c}M^c \le M^{\ell+c} = F \le M^{\rho_4+c}.$$

Using this fact and (5.5), we can employ [13, Theorem 8.25] (again a Vinogradov-type estimate) and obtain

$$\sum_{M<n\le M'} \mathrm{e}(\xi n^c) \ll_{c,q} M^{1-\frac{1}{2^{18}(\ell+c)^2}} \ll_{c,q} M^{1-\frac{1}{2^{18}(\rho_4+c)^2}} \ll_{c,q} M^{1-\rho_4}.$$

This finally shows (5.4) and finishes the proof of Lemma 5. ∎

Proof of Proposition 3. We can assume that $\frac{1}{2c}q^{\nu(1-c)} \le \beta \le 1/2$. Let k be a positive integer (which we choose later) and set

$$I_\ell := \left[\frac{\ell}{k}, \frac{\ell+1}{k}\right), \qquad \ell = 0, \dots, k-1.$$

We start with the following correlation:

$$\sum_{q^{\nu-1}<n\le x} \mathrm{e}(\beta\lfloor n^c\rfloor) = \sum_{0\le\ell<k} \sum_{\substack{q^{\nu-1}<n\le x\\ \{n^c\}\in I_\ell}} \mathrm{e}(\beta\lfloor n^c\rfloor).$$

If $\{n^c\} \in I_\ell$, then there exists a real number $0 \le \theta < 1$ such that

$$\mathrm{e}(\beta\lfloor n^c\rfloor) = \mathrm{e}\left(\beta n^c - \beta\frac{\ell}{k} - \beta\frac{\theta}{k}\right) = \mathrm{e}\left(\beta n^c - \beta\frac{l}{k}\right)\left(1 + O\left(\frac{1}{k}\right)\right).$$

Thus, we obtain

$$\left|\sum_{q^{\nu-1}<n\le x} \mathrm{e}(\beta\lfloor n^c\rfloor)\right| \ll \sum_{0\le\ell<k}\left|\sum_{\substack{q^{\nu-1}<n\le x\\ \{n^c\}\in I_\ell}} \mathrm{e}(\beta n^c)\right| + \frac{q^\nu}{k}. \tag{5.8}$$

If we set $f_\ell(x) := \mathbf{1}_{I_\ell}(\{x\})$, where $\mathbf{1}_A$ denotes the characteristic function of the set A, then inequality (5.8) reads as follows:

$$\left|\sum_{q^{\nu-1}<n\le x} \mathrm{e}(\beta\lfloor n^c\rfloor)\right| \ll \sum_{0\le\ell<k}\left|\sum_{q^{\nu-1}<n\le x} \mathrm{e}(\beta n^c) f_\ell(n^c)\right| + \frac{q^\nu}{k}. \tag{5.9}$$

Next, we approximate the function f_ℓ by trigonometric polynomials. Let $H \geq 1$ be an integer. Then there exist coefficients $a_H(h)$ with $|a_H(h)| \leq 2$, such that the trigonometric polynomial

$$f^*_{\ell,H}(t) = \frac{1}{k} + \frac{1}{2i\pi} \sum_{1\leq|h|\leq H} \frac{a_H(h)}{h} \mathrm{e}(ht)$$

verifies

$$|f_\ell(t) - f^*_{\ell,H}(t)| \leq \frac{1}{2H+2}\left(\kappa_H\left(t - \frac{\ell}{k}\right) + \kappa_H\left(t - \frac{\ell+1}{k}\right)\right), \tag{5.10}$$

where κ_H is the periodic Fejér kernel already defined by (4.5). Indeed, this can be deduced by another theorem of Vaaler [22, Theorem 19] (since the functions $f^*_{\ell,H}$ and $\kappa_H(t)$ are continuous, (5.10) follows from the cited theorem and a simple continuity argument even though f_ℓ does not satisfy Vaaler's normalizing condition). We obtain (the integer H will be chosen in the last step of the proof)

$$\Big| \sum_{q^{\nu-1}<n\leq x} \mathrm{e}(\beta n^c) f_\ell(n^c)\Big| \leq \Big| \sum_{q^{\nu-1}<n\leq x} \mathrm{e}(\beta n^c) f^*_{\ell,H}(n^c)\Big| + R(H), \tag{5.11}$$

where

$$R(H) := \frac{1}{2H+2} \sum_{q^{\nu-1}<n\leq x} \left(\kappa_H\left(n^c - \frac{\ell}{k}\right) + \kappa_H\left(n^c - \frac{\ell+1}{k}\right)\right).$$

The error term $R(H)$ can be estimated by

$$\frac{1}{2H+2} \sum_{q^{\nu-1}<n\leq x} \sum_{0\leq|h|\leq H} \left(1 - \frac{|h|}{H+1}\right)\left(1 + \mathrm{e}\left(-\frac{h}{k}\right)\right)\mathrm{e}\left(-\frac{h\ell}{k}\right)\mathrm{e}(hn^c)$$
$$\leq \frac{2}{2H+2} \sum_{0\leq|h|\leq H} \Big| \sum_{q^{\nu-1}<n\leq x} \mathrm{e}(hn^c)\Big|.$$

We distinguish the cases $h = 0$ and $h \neq 0$ and apply Lemma 5. This is admissible as long as $H \leq q^{(\nu-1)\rho}$, where ρ is defined by (5.2). We obtain

$$R(H) \ll_{c,q} \frac{q^\nu}{H} + q^{\nu(1-\rho)}.$$

Next, we use the definition of $f^*_{\ell,H}$ to deal with the first expression on the right hand side of (5.11). We can write

$$\begin{aligned}
\Big|\sum_{q^{\nu-1}<n\le x} &\mathrm{e}(\beta n^c) f^*_{\ell,H}(n^c)\Big| \\
&= \Big|\sum_{q^{\nu-1}<n\le x} \mathrm{e}(\beta n^c)\Big(\frac{1}{k} + \frac{1}{2i\pi}\sum_{1\le|h|\le H}\frac{a_H(h)}{h}\mathrm{e}(hn^c)\Big)\Big| \\
&\le \frac{1}{k}\Big|\sum_{q^{\nu-1}<n\le x}\mathrm{e}(\beta n^c)\Big| + \sum_{1\le|h|\le H}\frac{1}{h}\Big|\sum_{q^{\nu-1}<n\le x}\mathrm{e}((\beta+h)n^c)\Big|.
\end{aligned}$$

Applying Lemma 5 again (if $H \le q^{(\nu-1)\rho}$), we see that this is bounded by

$$\frac{q^{\nu(1-\rho)}}{k} + q^{\nu(1-\rho)}\log H.$$

We obtain

$$\sum_{q^{\nu-1}<n\le x}\mathrm{e}(\beta n^c) f_\ell(n^c) \ll_{c,q} \frac{q^\nu}{H} + q^{\nu(1-\rho)}\log H.$$

Together with inequality (5.9) this yields

$$\sum_{q^{\nu-1}<n\le x}\mathrm{e}(\beta\lfloor n^c\rfloor) \ll_{c,q} \frac{kq^\nu}{H} + kq^{\nu(1-\rho)}\log H + \frac{q^\nu}{k}.$$

If we set $k = \lfloor q^{(\nu\rho)/2}\rfloor$ and $H = \lfloor q^{(\nu-1)\rho}\rfloor$ (which actually shows that we were allowed to use Lemma 5), we finally obtain

$$\sum_{q^{\nu-1}<n\le x}\mathrm{e}(\beta\lfloor n^c\rfloor) \ll_{c,q} \nu q^{\nu(1-\rho/2)}. \ \blacksquare$$

6. Proof of Theorem 1. In this section we prove Theorem 1. First we briefly treat the (trivial) case $0 < c < 1$. The second part of the proof deals with the case $c > 1$ ($c \notin \mathbb{N}$) and it is based on methods coming from harmonic analysis (Sections 3) and on exponential sum estimates (Sections 4 and 5).

6.1. Case $0 < c < 1$. We set $\gamma = 1/c$ and $a_m := \#\{n \le x : \lfloor n^c\rfloor = m\}$. Then we can write

$$\sum_{1\le n\le x}\mathrm{e}(\alpha\,\mathrm{s}_q(\lfloor n^c\rfloor)) = \sum_{1\le m\le x^c}\mathrm{e}(\alpha\,\mathrm{s}_q(m))a_m.$$

For $m = \lfloor x^c\rfloor$ we observe that $a_m = x - (\lfloor x^c\rfloor)^\gamma + O(1) = O(x^{1-c})$, and for $m < \lfloor x^c\rfloor$ that $a_m = (m+1)^\gamma - m^\gamma + O(1) = \gamma m^{\gamma-1} + O(m^{\gamma-2}+1)$. Since

$$\sum_{1\le m\le x^c}(m^{\gamma-2}+1) \ll_c x^{1-c} + x^c,$$

we obtain

$$\sum_{1\le n\le x} \mathrm{e}(\alpha\,\mathrm{s}_q(\lfloor n^c\rfloor)) \ll_c \sum_{1\le m\le x^c} \mathrm{e}(\alpha\,\mathrm{s}_q(m))m^{\gamma-1} + x^{1-c} + x^c.$$

By partial summation we can write the last sum as

$$\begin{aligned}\sum_{1\le m\le x^c} &\mathrm{e}(\alpha\,\mathrm{s}_q(m))m^{\gamma-1}\\ &= x^{1-c}\sum_{1\le m\le x^c} \mathrm{e}(\alpha\,\mathrm{s}_q(m)) - (\gamma-1)\int_1^{x^c} \sum_{1\le m<u} \mathrm{e}(\alpha\,\mathrm{s}_q(m))u^{\gamma-2}\,du.\end{aligned}$$

Thus, we get

$$\sum_{1\le n\le x} \mathrm{e}(\alpha\,\mathrm{s}_q(\lfloor n^c\rfloor)) \ll_c x^{1-c} \max_{1\le N\le x^c}\Big|\sum_{1\le m\le N} \mathrm{e}(\alpha\,\mathrm{s}_q(m))\Big| + x^{1-c} + x^c. \tag{6.1}$$

Since $\mathrm{s}_q(a+bq^j) = \mathrm{s}_q(a)+\mathrm{s}_q(b)$ for $a<q^j$, a simple calculation shows that

$$\Big|\sum_{0\le m<N} \mathrm{e}(\alpha\,\mathrm{s}_q(m))\Big| \ll_q N^{\log_q \varphi_q(\alpha)}, \tag{6.2}$$

where φ_q is defined by (3.1) (see for example [15, Section 3]). By local expansion we have

$$\varphi_q(t) \le q^{1-\sigma_q'\|t\|^2},$$

where σ_q' is a positive computable constant only depending on q (see for example [17, Lemmas 3 and 5]). Together with (6.1) and (6.2) this implies Theorem 1 for $0<c<1$.

6.2. Case $c>1$. For the following part we assume that x and ν are integers such that $q^{\nu-1}<x\le q^\nu$. We set

$$S := \sum_{q^{\nu-1}<n\le x} \mathrm{e}(\alpha\,\mathrm{s}_q(\lfloor n^c\rfloor)),$$

and use the abbreviation

$$\lambda := \lfloor \nu c\rfloor + 1. \tag{6.3}$$

Then we can write

$$\begin{aligned} S &= \sum_{0\le u<q^\lambda}\ \sum_{q^{\nu-1}<n\le x} \mathrm{e}(\alpha\,\mathrm{s}_q(u))\cdot\frac{1}{q^\lambda}\sum_{0\le h<q^\lambda} \mathrm{e}\bigg(\frac{h(\lfloor n^c\rfloor-u)}{q^\lambda}\bigg)\\ &= \sum_{0\le h<q^\lambda}\frac{1}{q^\lambda}\sum_{0\le u<q^\lambda} \mathrm{e}(\alpha\,\mathrm{s}_q(u)-huq^{-\lambda})\sum_{q^{\nu-1}<n\le x} \mathrm{e}\bigg(\frac{h\lfloor n^c\rfloor}{q^\lambda}\bigg).\end{aligned}$$

Using the notation of the Fourier transform, we have

$$|S| \leq \sum_{0 \leq h < q^\lambda} |F_\lambda(h, \alpha)| \cdot \Bigg| \sum_{q^{\nu-1} < n \leq x} \mathrm{e}\left(\frac{h\lfloor n^c \rfloor}{q^\lambda}\right)\Bigg|. \tag{6.4}$$

It follows from Lemma 1 that the contribution of the term where $h = 0$ is bounded above by

$$|F_\lambda(0, \alpha)| q^\nu \ll q^{\nu - \sigma_q \|(q-1)\alpha\|^2 \lambda}.$$

If $0 < h < q^\lambda$, Remark 5 implies

$$\sum_{q^{\nu-1} < n \leq x} \mathrm{e}\left(\frac{h\lfloor n^c \rfloor}{q^\lambda}\right) \ll_{c,q} \nu q^{\nu(1-\tilde{\rho}/2)} + \frac{q^\nu}{\min(h, q^\lambda - h)},$$

where $\tilde{\rho} := \max(2(2-c)/3, \rho)$. We obtain (using Lemmas 1 and 2)

$$\sum_{0 \leq h < q^\lambda} |F_\lambda(h, \alpha)| \left(\nu q^{\nu(1-\tilde{\rho}/2)} + \frac{q^\nu}{\min(h, q^\lambda - h)}\right)$$
$$\ll \nu q^{\nu(1-\tilde{\rho}/2)+\lambda\eta_q} + \log(q^\lambda) q^{\nu - \sigma_q \|(q-1)\alpha\|^2 \lambda}.$$

Thus, we can bound the sum S by

$$S \ll_{c,q} \nu\big(q^{\nu(1-\sigma_q \|(q-1)\alpha\|^2 c)} + q^{\nu(1-\tilde{\rho}/2 + c\eta_q)}\big).$$

If q is large enough (larger than some constant $q_0(c)$), then it follows from Remark 3 that

$$\tilde{\rho}/2 - c\eta_q > 0. \tag{6.5}$$

Setting $\sigma_{c,q} = \min(\sigma_q c, \tilde{\rho}/2 - \eta_q c)$, we have, for every $q \geq q_0(c)$,

$$\sum_{q^{\nu-1} < n \leq x} \mathrm{e}(\alpha\, \mathrm{s}_q(\lfloor n^c \rfloor)) \ll_{c,q} \nu q^{\nu(1-\sigma_{c,q}\|(q-1)\alpha\|^2)}.$$

Theorem 1 is a direct consequence of this fact. Let ν_0 be the integer such that $q^{\nu_0 - 1} < x \leq q^{\nu_0}$. Then we can write

$$\sum_{1 \leq n \leq x} \mathrm{e}(\alpha\, \mathrm{s}_q(\lfloor n^c \rfloor))$$
$$= \sum_{0 \leq \nu < \nu_0} \sum_{q^{\nu-1} < n \leq q^\nu} \mathrm{e}(\alpha\, \mathrm{s}_q(\lfloor n^c \rfloor)) + \sum_{q^{\nu_0-1} < n \leq x} \mathrm{e}(\alpha\, \mathrm{s}_q(\lfloor n^c \rfloor))$$
$$\ll_{c,q} \sum_{0 \leq \nu \leq \nu_0} \nu q^{\nu(1-\sigma_{c,q}\|(q-1)\alpha\|^2)} \ll_{c,q} \nu_0 q^{\nu_0(1-\sigma_{c,q}\|(q-1)\alpha\|^2)}.$$

Since $\nu_0 \leq \lfloor \log x / \log q + 1 \rfloor$, we obtain

$$\sum_{1 \leq n \leq x} \mathrm{e}(\alpha\, \mathrm{s}_q(\lfloor n^c \rfloor)) \ll_{c,q} (\log x) x^{1-\sigma_{c,q}\|(q-1)\alpha\|^2}.$$

Finally, note that we can see from (6.5) that the constant $\tilde{\rho}$ determines the size of an admissible (and computable) value $q_0(c)$. Inequality (6.5) is satisfied if $\log\log q/\log q < \tilde{\rho}/(2c)$. This implies for example that such an admissible value is given by Kc^{c^4}, where K is an absolute constant. ■

7. Proof of Corollaries 1 and 2. In order to show Corollary 1 we need information on the distribution of $\lfloor n^c \rfloor$ in arithmetic progressions. For $1 < c < 2$ this has been studied for example in [4] (see also [21, 24]), and for $c > 12$ (not an integer) in [4]. For the convenience of the reader we state and prove the following lemma which holds true for all non-integral reals $c > 1$. It confirms the already known result for $1 < c < 2$ and slightly improves the known results in the other cases. Note that a shorter proof can be obtained by using Proposition 3 directly. However, the exponent $1 - \rho$ in (7.1) has then to be replaced by $1 - \rho/2$.

LEMMA 6. *Let $c > 1$ be a non-integer real number and let $(a, d) \in \mathbb{N} \times \mathbb{N}^*$. Then*

$$\#\{n \le x : \lfloor n^c \rfloor \equiv a \bmod d\} = \frac{x}{d} + O_{c,d}((\log x) x^{1-\rho}), \tag{7.1}$$

where ρ is defined by (5.2).

Proof. We begin with the following observation: The integer n satisfies $\lfloor n^c \rfloor \equiv a \bmod d$ if and only if $a/d \le \{n^c/d\} < (a+1)/d$. In order to prove the lemma, it suffices to show that the discrepancy D of (n^c/d), where n ranges from 1 to x, can be bounded by $D \ll_{c,d} (\log x) x^{-\rho}$. We use the Erdős–Turán inequality (see for example [19, Lemma 1] or [8, Theorem 1.21]) saying that

$$D \le \frac{1}{H+1} + \sum_{h=1}^{H} \frac{1}{h} \bigg| \frac{1}{x} \sum_{1 \le n \le x} \mathrm{e}\Big(\frac{h}{d} n^c\Big) \bigg|,$$

where the integer $H > 0$ can be chosen arbitrarily. Let ν_0 be the smallest positive integer such that $1/d \ge \frac{1}{2c} 2^{\nu_0(1-c)}$ and let λ be defined by $2^{\lambda-1} < x \le 2^{\lambda}$. Lemma 5 implies

$$\begin{aligned}\bigg| \sum_{1 \le n \le x} \mathrm{e}\Big(\frac{h}{d} n^c\Big) \bigg| &\le 2^{\nu_0 - 1} + \sum_{\nu_0 \le \nu \le \lambda} \sum_{\substack{2^{\nu-1} < n \le 2^{\nu} \\ n \le x}} \mathrm{e}\Big(\frac{h}{d} n^c\Big) \\ &\ll_{c,d} \sum_{\nu=\nu_0}^{\lambda} 2^{\nu(1-\rho)} \ll_{c,d} x^{1-\rho},\end{aligned}$$

where ρ is defined by (5.2). If we set $H := \lfloor 2^{(\lambda-1)\rho} \rfloor$, then the Erdős–Turán inequality yields

$$D \ll 2^{(1-\lambda)\rho} + \frac{1}{x} \sum_{h=1}^{\lfloor 2^{(\lambda-1)\rho} \rfloor} \frac{1}{h} \Big| \sum_{1 \le n \le x} \mathrm{e}\Big(\frac{h}{d} n^c\Big) \Big|$$
$$\ll_{c,d} 2^{(1-\lambda)\rho} + \log(2^{(\lambda-1)\rho}) x^{-\rho} \ll_{c,d} (\log x) x^{-\rho}.$$

As indicated above, this shows the desired result. ∎

Proof of Corollary 1. We can write

$$\#\{n \le x : \mathrm{s}_q(\lfloor n^c \rfloor) \equiv a \bmod m\} = \sum_{n \le x} \frac{1}{m} \sum_{0 \le \ell < m} \mathrm{e}\Big(\ell \, \frac{\mathrm{s}_q(\lfloor n^c \rfloor) - a}{m}\Big).$$

Let us first consider the case $0 < c < 1$. The main term comes from $\ell = 0$ and equals x/m. Due to Theorem 1 there exists a constant $\sigma'_{c,q,\ell/m}$ for every $1 \le \ell < m$ such that

$$\sum_{n \le x} \mathrm{e}\Big(\frac{\ell}{m} \, \mathrm{s}_q(\lfloor n^c \rfloor)\Big) \ll_{c,q} (\log x) x^{1 - \sigma'_{c,q,\ell/m}}.$$

The result follows by setting $\sigma_{c,q,m} = \min_{1 \le \ell < m}(\sigma'_{c,q,\ell/m})$. If $c > 1$, then we put $d = (m, q-1)$, $m' = m/d$, $J = \{km' : 0 \le k < d\}$ and $J' = \{0, \ldots, m-1\} \setminus J = \{km' + r : 0 \le k < d,\ 1 \le r < m'\}$. For $\ell = km' \in J$ we have

$$\mathrm{e}\Big(\frac{\ell}{m} \, \mathrm{s}_q(\lfloor n^c \rfloor)\Big) = \mathrm{e}\Big(\frac{k}{d} \, \mathrm{s}_q(\lfloor n^c \rfloor)\Big) = \mathrm{e}\Big(\frac{k}{d} \lfloor n^c \rfloor\Big).$$

Hence, applying Lemma 6 yields

$$\frac{1}{m} \sum_{\ell \in J} \sum_{n \le x} \mathrm{e}\Big(\ell \, \frac{\mathrm{s}_q(\lfloor n^c \rfloor) - a}{m}\Big) = \frac{d}{m} \sum_{\substack{n \le x \\ \lfloor n^c \rfloor \equiv a \bmod d}} 1 \tag{7.2}$$
$$= \frac{x}{m} + O_{c,d}((\log x) x^{1-\rho}).$$

If $J' = \emptyset$, Lemma 6 already implies Corollary 1 (we can choose $\sigma_{c,q,m} = (9/10)\rho$). If $J' \neq \emptyset$, we set $q' = (q-1)/d$. Since $(q', m') = 1$, we obtain, for $\ell = km' + r \in J'$,

$$\frac{(q-1)\ell}{m} = \frac{dq'(km' + r)}{dm'} = q'k + \frac{q'r}{m'} \notin \mathbb{Z}.$$

Theorem 1 implies that there exists a constant $\sigma'_{c,q,\ell/m}$ for every $\ell \in J'$ such that

$$\sum_{n \le x} \mathrm{e}\Big(\frac{\ell}{m} \, \mathrm{s}_q(\lfloor n^c \rfloor)\Big) \ll_{c,q,m} (\log x) x^{1 - \sigma'_{c,q,\ell/m}}.$$

Put

$$\sigma_{q,m,c} = \frac{9}{10} \min(\min_{\ell \in J'}(\sigma'_{c,q,\ell/m}), \rho) > 0.$$

Together with (7.2) this proves Corollary 1. ∎

Proof of Corollary 2. If $\alpha \in \mathbb{Q}$, then the sequence $(\alpha\, \mathrm{s}_q(\lfloor n^c \rfloor))_{n\in\mathbb{N}}$ takes modulo 1 only a finite number of values and is therefore not uniformly distributed modulo 1. If $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, then Theorem 1 and Weyl's criterion (see e.g. [8, Theorem 1.19]) imply the result. ∎

8. Proof of Corollary 3. In this section we show Corollary 3. The proof is very similar to that of [7, Theorem 1.1], where a local limit theorem for the sum-of-digits function of primes is shown. A similar method is used in [20, Section 6] for the proof of a local limit theorem in the Gaussian integers. Thus, we only give a rough outline and refer at appropriate places to [7].

The starting point of our considerations is the equality

$$\#\{n \le x : \mathrm{s}_q(\lfloor n^c \rfloor) = k\} = \int_0^1 S(\alpha)\mathrm{e}(-\alpha k)\, d\alpha,$$

where $S(\alpha) := \sum_{n\le x} \mathrm{e}(\alpha\, \mathrm{s}_q(\lfloor n^c \rfloor))$. Set $I(x,k,c) := \{0 \le n \le x : \lfloor n^c \rfloor \equiv k \bmod q-1\}$. With $S_k(\alpha) := \sum_{n\in I(x,k,c)} \mathrm{e}(\alpha\, \mathrm{s}_q(\lfloor n^c \rfloor))$, we have (see [7, Section 5.1])

$$\int_0^1 S(\alpha)\mathrm{e}(-\alpha k)\, d\alpha = (q-1) \int_{-1/(2(q-1))}^{1/(2(q-1))} S_k(\alpha)\mathrm{e}(-\alpha k)\, d\alpha.$$

The last integral is split up into two different domains:

$$\int_{-1/(2(q-1))}^{1/(2(q-1))} = \int_{|\alpha|\le(\log\log x)(\log x)^{-1/2}} + \int_{(\log\log x)(\log x)^{-1/2}<|\alpha|\le 1/(2(q-1))}. \tag{8.1}$$

The second integral (where α is large) can be bounded above using Theorem 1 (combined with discrete Fourier analysis). We obtain

$$\int S_k(\alpha)\mathrm{e}(-\alpha k)\, d\alpha \ll_{c,q} (\log x) x^{1-\sigma_{c,q}(q-1)^2(\log\log x)^2(\log x)^{-1}} \ll_{c,q} \frac{x}{\log x}.$$

Here we used the fact that the estimate in Theorem 1 is uniform in α. To calculate the first integral in (8.1), we set $R(x,k,c) = \#I(x,k,c)$. Note that Lemma 6 implies $R(x,k,c) = x/(q-1) + O_{c,q}((\log x)x^{1-\rho})$. Because of this fact, the following proposition implies Corollary 3 (see [7, Section 5.1]).

Proposition 4. *Let $q \ge 2$. Then for every non-negative integer k we have*

$$\sum_{n\in I(x,k,c)} \mathrm{e}(\alpha\, \mathrm{s}_q(\lfloor n^c \rfloor)) = R(x,k,c)\mathrm{e}(\alpha\mu_q c \log_q x) \times (e^{-2\pi^2\alpha^2\sigma_q^2 c\log_q x} + O_{c,q}(|\alpha|(\log\log x)^5)) \tag{8.2}$$

uniformly for real α with $|\alpha| \le (\log\log x)(\log x)^{-1/2}$.

Proposition 4 can be translated into a probabilistic language. If we assume that every number in the set $I(x,k,c)$ is equally likely, then the function which assigns to each number its jth digit is a random variable. Hence, the sum-of-digits function $S_x(n) := \mathrm{s}_q(\lfloor n^c \rfloor)$ for $n \le x$ can also be interpreted as a random variable. Set $L = \log x^c$. Using this model, formula (8.2) is equivalent to the relation (set $\alpha = t/(2\pi\sigma_q L^{1/2})$)

$$\varphi_1(t) := \mathbb{E}\, e^{it(S_x - L\mu_q)/(L\sigma_q^2)^{1/2}} = e^{-t^2/2} + O_{c,q}\biggl(|t| \frac{(\log L)^5}{L^{1/2}}\biggr), \tag{8.3}$$

which is uniform for $|t| \le 2\pi\sigma_q L^{1/2}(\log\log x)(\log x)^{-1/2}$. Note that $\varphi_1(t)$ is the characteristic function of $(S_x - L\mu_q)/(L\sigma_q^2)^{1/2}$ and that (8.3) is a refined version of the central limit theorem (1.5).

In order to prove this, we approximate the sum-of-digits function with a sum of uniformly and independently distributed random variables (at the level of moments). The next lemma is the key to doing so. If $\sigma > 1$ (and x is large enough), we set

$$L' = \#\{j \in \mathbb{Z} : (\log L)^\sigma \le j \le L - (\log L)^\sigma\} = L - 2(\log L)^\sigma + O(1).$$

Lemma 7. *Let $1 \le d \le L'$ and $\sigma > 1$. Furthermore, let $j_1, \ldots, j_d$ and $l_1, \ldots, l_d$ be integers with*

$$(\log L)^\sigma \le j_1 < j_2 < \cdots < j_d \le L - (\log L)^\sigma$$

and $l_1, \ldots, l_d \in \{0, 1, \ldots, q-1\}$. Then uniformly (3)

$$\begin{aligned}\frac{1}{R(x,k,c)}\#\{n \in I(x,k,c) : \varepsilon_{j_1}(\lfloor n^c \rfloor) = l_1, \ldots, \varepsilon_{j_d}(\lfloor n^c \rfloor) = l_d\}\\ = q^{-d} + O_{c,q,\sigma}(L(4(\log L)^\sigma)^d e^{-c'(\log L)^\sigma}),\end{aligned}$$

where $c' = \min(1, 1/c)$.

For proving Lemma 7 we need the Erdős–Turán inequality, which leads to exponential sums of the form $\sum_n \mathrm{e}((A/Q)\lfloor n^c \rfloor)$:

Lemma 8. *Let $c > 0$ be a non-integer real number. Furthermore, let $A, Q \in \mathbb{Z}^+$ with $(A,Q) = 1$ and let $\sigma \in \mathbb{Z}^+$ be such that $1 < Q \le x^c e^{-(\log\log x^c)^\sigma}$. Then*

$$\sum_{1 \le n \le x} \mathrm{e}\biggl(\frac{A}{Q}\lfloor n^c \rfloor\biggr) \ll_{c,\sigma} (\log x) x e^{-c'(\log\log x^c)^\sigma},$$

where $c' = \min(1, 1/c)$.

---

(3) The notation $\varepsilon_j(m)$ means the jth digit of m.

Proof. Let S be the sum considered. We start the proof with the following estimate:

$$\left\| \frac{A}{Q} \right\|^{-1} \le Q \le x^c e^{-(\log\log x^c)^\sigma}. \tag{8.4}$$

If $0 < c < 1$, then we deduce (using the same calculations as in Section 6.1) that

$$\begin{aligned} S &\ll_c x^{1-c} \max_{1\le N\le x^c} \Bigl| \sum_{1\le m\le N} \mathrm{e}\Bigl(\frac{A}{Q} m\Bigr) \Bigr| + x^{1-c} + x^c \\ &\ll_c x^{1-c} \frac{1}{|\sin \pi \frac{A}{Q}|} + x^{1-c} + x^c. \end{aligned}$$

By (8.4), this leads to the desired result. Next, we treat the case $c > 1$. Let ν be the integer defined by $2^{\nu-1} < x \le 2^\nu$. If x is sufficiently large, then

$$\nu_0 := \nu - \Bigl\lfloor \frac{1}{c\log 2} (\log\log x^c)^\sigma \Bigr\rfloor$$

is positive. Remark 5 implies

$$\begin{aligned} S &\le 2^{\nu_0 - 1} + \sum_{\kappa=\nu_0}^{\nu} \sum_{\substack{2^{\kappa-1} < n \le 2^\kappa \\ n \le x}} \mathrm{e}\Bigl(\frac{A}{Q}\lfloor n^c \rfloor\Bigr) \\ &\ll_{c,\sigma} 2^{\nu_0-1} + \sum_{\kappa=\nu_0}^{\nu} \Bigl(\kappa 2^{\kappa(1-\tilde\rho/2)} + \frac{1}{\|A/Q\|} 2^{\kappa(1-c)}\Bigr). \end{aligned}$$

We finally obtain

$$\begin{aligned} S &\ll_{c,\sigma} 2^{\nu_0-1} + \nu q^{\nu(1-\tilde\rho/2)} + \nu x^c e^{-(\log\log x^c)^\sigma} 2^{\nu_0(1-c)} \\ &\ll_{c,\sigma} x e^{-(\log\log x^c)^\sigma/c} + (\log x) x e^{-(\log\log x^c)^\sigma/c}. \ \blacksquare \end{aligned}$$

Proof of Lemma 7. The proof of this lemma goes exactly as in [7, Section 4.2]. We just give a short outline. We have

$$\begin{aligned} &\#\{n \in I(x,k,c) : \varepsilon_{j_1}(\lfloor n^c \rfloor) = l_1, \ldots, \varepsilon_{j_d}(\lfloor n^c \rfloor) = l_d\} \\ &\qquad = \sum_{n\in I(x,k,c)} \prod_{i=1}^{d} \mathbf{1}_{[l_i/q,(l_i+1)/q)}\Bigl(\Bigl\{\frac{\lfloor n^c \rfloor}{q^{j_i+1}}\Bigr\}\Bigr), \end{aligned}$$

where $\mathbf{1}_A$ denotes the characteristic function of the set A. First, we approximate $\mathbf{1}_{[l/q,(l+1)/q)}(\{x\})$ with the function

$$f_{l,\Delta}(x) := \frac{1}{\Delta} \int_{-\Delta/2}^{\Delta/2} \mathbf{1}_{[l/q,(l+1)/q)}(\{x+z\})\, dz,$$

where $\Delta = e^{-(\log L)^\sigma}$. This approximation yields an error term that can be bounded above by using the Erdős–Turán inequality and Lemma 8 (cf. [7, Lemma 4.4]). With the help of the Fourier expansion of $f_{l,\Delta}(x)$, Lemma 8 finally implies the desired result (cf. [7, Lemma 4.5]). ∎

Proof of Proposition 4. First, we truncate the sum-of-digits function and approximate it appropriately. Let σ be a real number greater than 1 (which we choose at the end of the proof). Furthermore, let Z_j be a sequence of independent random variables with range $\{0, 1, \ldots, q-1\}$ and uniform probability distribution, and set

$$T_x := \sum_{(\log L)^\sigma \le j \le L-(\log L)^\sigma} \varepsilon_j(\lfloor n^c \rfloor), \qquad \overline{T}_x := \sum_{(\log L)^\sigma \le j \le L-(\log L)^\sigma} Z_j.$$

Define the random variables X and Y by $X := (T_x - L'\mu_Q)/(L'\sigma_Q^2)^{1/2}$ and $Y := (\overline{T}_x - L'\mu_Q)/(L'\sigma_Q^2)^{1/2}$, and let $\varphi_2(t)$ be the characteristic function of X and $\varphi_3(t)$ the characteristic function of Y. Then (see [7, Lemma 4.1])

$$|\varphi_1(t) - \varphi_2(t)| = O_q(|t|(\log L)^\sigma / L^{1/2}).$$

Furthermore, $\varphi_3(t)$ can be approximated by (see [7, Lemma 4.2])

$$\varphi_3(t) = e^{-t^2/2}(1 + O(t^4/L))$$

whenever $|t| \le L^{1/4}$. In what follows, we will show that $\overline{T}_x$ is a good approximation of the (truncated) sum-of-digits function. In order to prove (8.3), it suffices to show that uniformly for real t with $|t| \ll_{c,q} \log L$,

$$|\varphi_2(t) - \varphi_3(t)| = O_{c,q}(|t|/L).$$

Using Taylor's theorem we see that, for every even integer $D > 0$,

$$\begin{aligned} \mathbb{E}\, e^{itX} - \mathbb{E}\, e^{itY} &= \sum_{d<D} \frac{(it)^d}{d!}(\mathbb{E}\, X^d - \mathbb{E}\, Y^d) \\ &\quad + O\biggl(\frac{|t|^D}{D!}\bigl|\mathbb{E}\,|X|^D - \mathbb{E}\,|Y|^D\bigr| + 2\frac{|t|^D}{D!}\mathbb{E}\,|Y|^D\biggr) \\ &\ll |t| \max_{d\le D}(|\mathbb{E}\, X^d - \mathbb{E}\, Y^d|)e^{|t|} + \frac{|t|^D}{D!}\mathbb{E}\, Y^D. \end{aligned}$$

We have (cf. [7, Section 4.3])

$$\mathbb{E}\, Y^D \ll \frac{D!}{D^{D/2}e^{-D/2}D^{1/2}}$$

whenever $D = o((\log x)^{1/2})$. Recall that $|t| \ll_{c,q} \log L$. If we choose $D = \lfloor (\log L)^3 \rfloor$ (and assume without loss of generality that D is even), then

$$\frac{|t|^D}{D!}\mathbb{E}\, Y^D \ll_{c,q} |t|/L.$$

In order to complete the proof of Proposition 4, it remains to compare the moments of X and Y. Lemma 7 implies

$$|\mathbb{E}\, X^d - \mathbb{E}\, Y^d| \ll_{c,q,\sigma} \left(\frac{4q^2}{\sigma_q}\right)^d L^{1+d/2}(\log L)^{\sigma d} e^{-c'(\log L)^\sigma}.$$

If we choose $\sigma = 5$, we finally obtain

$$\max_{d\le D} |\mathbb{E}\, X^d - \mathbb{E}\, Y^d| \ll_{c,q} e^{-(\log L)^2},$$

which shows the desired result. ∎

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