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Primitive lattice points in a thin strip along the boundary of a large convex planar domain

by

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1. Introduction. Let $\mathcal{D}$ be a convex planar domain containing the origin in its interior whose boundary $\partial\mathcal{D}$ is of class C^4 (with respect to the arclength) and has finite nonvanishing curvature throughout. Let F denote the distance function of $\mathcal{D}$, i.e.,

$$F(\mathbf{u}) = \inf\{\tau > 0 : \mathbf{u}/\tau \in \mathcal{D}\} \quad (\mathbf{u} \in \mathbb{R}^2).$$

A point of the standard lattice $\mathbb{Z}^2$ is called *primitive* if its coordinates are relatively prime (*visible from the origin* in geometric terms). For a large real variable x, define $B_{\mathcal{D}}(x)$ as the number of primitive lattice points of $\mathbb{Z}_*^2 := \mathbb{Z}^2 \setminus \{(0,0)\}$ in the "blown up" domain $\sqrt{x}\,\mathcal{D}$. When counting lattice points, we shall throughout use the convention that points on the boundary of any two- or three-dimensional domain are counted with weight $1/2$, thus

$$\begin{aligned} B_{\mathcal{D}}(x) = \#\{\mathbf{m} = (m_1, m_2) \in \mathbb{Z}_*^2 : F^2(\mathbf{m}) < x,\ \gcd(m_1, m_2) = 1\} \\ + \tfrac{1}{2}\#\{\mathbf{m} = (m_1, m_2) \in \mathbb{Z}_*^2 : F^2(\mathbf{m}) = x,\ \gcd(m_1, m_2) = 1\}. \end{aligned}$$

The corresponding generating function (Dirichlet series) is obviously $Z_{\mathcal{D}}(s)/\zeta(2s)$ where $Z_{\mathcal{D}}(s)$ is the Hlawka zeta-function ([1]) of the domain $\mathcal{D}$. With Perron's formula in the back of the mind, it is clear that any unconditional asymptotic formula for $B_{\mathcal{D}}(x)$ depends on our knowledge about zero-free regions of the Riemann zeta-function. The sharpest result available to date reads thus

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The research for this paper was essentially done in 1996 (!) while E. Krätzel was visiting professor at the university of Vienna. For some reasons the completion of the manuscript was delayed until now.

([1]) This is a generalization of the Epstein zeta-function for the case that $\mathcal{D}$ is an origin-centered ellipse. See Hlawka [4].

$$B_{\mathcal{D}}(x) = \frac{6}{\pi^2}\text{area}(\mathcal{D})x + O(x^{1/2}\exp(-c(\log x)^{3/5}(\log\log x)^{-1/5})), \tag{1.1}$$

as was observed by Huxley & Nowak [7]. Improvements are only possible under the assumption of the Riemann Hypothesis (RH). In this direction W. Müller [13] obtained the conditional estimate

$$B_{\mathcal{D}}(x) = \frac{6}{\pi^2}\text{area}(\mathcal{D})x + O(x^{9/22}), \tag{1.2}$$

thereby refining the bound $O(x^{5/12+\varepsilon})$ of [7]. If $\mathcal{D}$ is a circle, the error term can be improved to $O(x^{11/30+\varepsilon})$, again under RH (see Zhai & Cao [15]).

The task of the present paper is the asymptotic evaluation of the quantity $B_{\mathcal{D}}(x+h) - B_{\mathcal{D}}(x)$ where h is another large real parameter but of smaller order than x. In geometric terms, we count the primitive lattice points in a strip (along the boundary of a "blown up" domain) whose width is of order $hx^{-1/2}$, thus (in the nontrivial case) less than unity ([2]). The question is for which range of h it can be guaranteed that

$$B_{\mathcal{D}}(x+h) - B_{\mathcal{D}}(x) \sim \frac{6}{\pi^2}\text{area}(\mathcal{D})h. \tag{1.3}$$

It is immediate from (1.1) and (1.2) that (1.3) is true unconditionally for h greater than $x^{1/2}\exp(-c'(\log x)^{3/5}(\log\log x)^{-1/5})$ and, under RH, for $h \ge x^{9/22}\lambda(x)$, $\lambda(x)$ tending to ∞ with x, otherwise arbitrary ($9/22=0.40909\ldots$).

We shall establish a result which is considerably sharper and independent of any unproven hypothesis.

THEOREM 1. *The asymptotics* (1.3) *is true for* $h \ge x^{11/29}\lambda(x)\log x$, *with any* $\lambda(x)$ *tending to* ∞ *with* x ($11/29 = 0.37931\ldots$).

In the proof we shall have to consider the number of *all* lattice points (except the origin) in $\sqrt{x}\,\mathcal{D}$, i.e.,

$$A_{\mathcal{D}}(x) = \#\{\mathbf{n} \in \mathbb{Z}^2 : 0 < F(\mathbf{n}) < \sqrt{x}\} + \tfrac{1}{2}\#\{\mathbf{n} \in \mathbb{Z}^2 : F(\mathbf{n}) = \sqrt{x}\}.$$

Furthermore, we are lead in a natural way to the enumeration of the lattice points in a certain (nonconvex) *three-dimensional* domain, namely to evaluate

$$\begin{aligned} A^{(3)}(X) = \#\{&(n_1,n_2,n_3) \in \mathbb{Z}_*^2 \times \mathbb{N}^* : F(n_1,n_2)\,n_3 < X\} \\ &+ \tfrac{1}{2}\,\#\{(n_1,n_2,n_3) \in \mathbb{Z}_*^2 \times \mathbb{N}^* : F(n_1,n_2)n_3 = X\} \end{aligned}$$

where $\mathbb{Z}_*^2 = \mathbb{Z}^2 - \{(0,0)\}$ and $\mathbb{N}^*$ is the set of positive integers. For this quantity we shall develop an asymptotic formula which might be of some interest for itself.

([2]) Actually, for the assertion of Theorem 1 to be true, the width of this strip may be as small as $x^{-7/58}(\log x)\,\lambda(x)$, $\lambda(x) \to \infty$.

THEOREM 2. *For large X,*

$$A^{(3)}(X)=\frac{\pi^2}{6}\operatorname{area}(\mathcal{D})X^2+CX+O(X^{22/29}\log X),$$

where

$$C=-\operatorname{area}(\mathcal{D})F_0+\int_{F_0}^{\infty}\frac{P(t^2)}{t^2}\,dt,$$

with

$$F_0:=\min_{\mathbf{n}\in\mathbb{Z}^2-\{(0,0)\}}F(\mathbf{n}),\qquad P(u):=A_{\mathcal{D}}(u)-\operatorname{area}(\mathcal{D})u.$$

REMARKS. 1. For the constant C we can give the alternative representation $C=Z_{\mathcal{D}}(1/2)$ where $Z_{\mathcal{D}}$ is the Hlawka zeta-function of the convex set $\mathcal{D}$. (This is immediate, e.g., from the unnumbered formula below (3.5) in Huxley & Nowak [7].) Consequently, the main term can be written in the lucid form

$$\frac{\pi^2}{6}\operatorname{area}(\mathcal{D})X^2+CX=\sum_{s_0=1,1/2}\operatorname{Res}_{s=s_0}\left(\zeta(2s)Z_{\mathcal{D}}(s)\frac{X^{2s}}{s}\right).$$

2. If $\mathcal{D}$ is a circle, much sharper estimates are true. In fact, for this case ([3])

$$A^{(3)}(X)=\sum_{\substack{m,k\in\mathbb{N}^*\\ m^2k\le X^2}}r(k)=4\sideset{}{'}\sum_{l\le X^2}\Big(\sum_{\substack{uvw^2=l\\ u\equiv1\ (\mathrm{mod}\ 4)}}1-\sum_{\substack{uvw^2=l\\ u\equiv3\ (\mathrm{mod}\ 4)}}1\Big),\tag{1.4}$$

where u,v,w ranges over positive integers and $r(k)$ denotes as usual the number of ways to write $k\in\mathbb{N}^*$ as a sum of two squares. Clearly this is quite closely related to the three-dimensional asymmetric divisor function $d(1,1,2;k)$. For the latter, Liu [11] recently established an asymptotic formula of the shape

$$\sum_{k\le x}d(1,1,2;k)=\text{main terms}+O(x^{29/80+\varepsilon}).$$

Applying the corresponding argument to (1.4), one obtains Theorem 2 with the better error term $O(X^{29/40+\varepsilon})$ and consequently the validity of (1.3) for any $h\ge x^{29/80+\varepsilon}$ $(29/80=0.3625)$.

2. Deduction of Theorem 1 from Theorem 2. We shall employ the usual technique for the investigation of the average order of arithmetic

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([3]) $\sum'$ means throughout that terms corresponding to the upper limit(s) of summation are weighted with the factor 1/2.

functions in short intervals. (For a textbook reference, see e.g. Krätzel [8], p. 288.) We assume throughout that $h \le x$, otherwise Theorem 1 is trivial. By a usual device (cf. formula (1.4) in [7] ([4])),

$$B_{\mathcal{D}}(x+h) - B_{\mathcal{D}}(x) = \sum_{m\in\mathbb{N}^*} \mu(m)\left(A_{\mathcal{D}}\left(\frac{x+h}{m^2}\right) - A_{\mathcal{D}}\left(\frac{x}{m^2}\right)\right) = S_1 + O(S_2) \tag{2.1}$$

where $\mu(\cdot)$ is the Möbius function,

$$S_1 := \sum_{m\le x^\delta} \mu(m)\left(A_{\mathcal{D}}\left(\frac{x+h}{m^2}\right) - A_{\mathcal{D}}\left(\frac{x}{m^2}\right)\right),$$
$$S_2 := \sum_{m> x^\delta} \left(A_{\mathcal{D}}\left(\frac{x+h}{m^2}\right) - A_{\mathcal{D}}\left(\frac{x}{m^2}\right)\right),$$

$\delta > 0$ a suitably small fixed number. By the classic van der Corput's lattice point estimate,

$$S_1 = \sum_{m\le x^\delta} \mu(m)\left(\mathrm{area}(\mathcal{D})\frac{h}{m^2} + O(x^{1/3}m^{-2/3})\right) = \frac{6}{\pi^2}\,\mathrm{area}(\mathcal{D})h + O(hx^{-\delta}) + O(x^{(1+\delta)/3}). \tag{2.2}$$

Further, since

$$\sum_m A_{\mathcal{D}}\left(\frac{x}{m^2}\right) = \sum_m \sideset{}{'}\sum_{\substack{\mathbf{n}\in\mathbb{Z}_*^2\\ F(\mathbf{n})^2\le x/m^2}} 1 = \sideset{}{'}\sum_{\substack{(m,\mathbf{n})\in\mathbb{N}^*\times\mathbb{Z}_*^2\\ mF(\mathbf{n})\le\sqrt{x}}} 1 = A^{(3)}(\sqrt{x}),$$

it follows that

$$S_2 = A^{(3)}(\sqrt{x+h}) - A^{(3)}(\sqrt{x}) - S_3,$$
$$S_3 := \sum_{m\le x^\delta} \left(A_{\mathcal{D}}\left(\frac{x+h}{m^2}\right) - A_{\mathcal{D}}\left(\frac{x}{m^2}\right)\right).$$

As a consequence of Theorem 2,

$$A^{(3)}(\sqrt{x+h}) - A^{(3)}(\sqrt{x}) = \frac{\pi^2}{6}\,\mathrm{area}(\mathcal{D})h + O(hx^{-1/2}) + O(x^{11/29}\log x).$$

Repeating the argument used for S_1, we further get

([4]) Our convention concerning the weight of boundary lattice points does not affect the validity of this identity.

$$
(2.3) \qquad S_3 = \sum_{m \le x^\delta} \left(\operatorname{area}(\mathcal{D}) \frac{h}{m^2} + O(x^{1/3} m^{-2/3}) \right)
$$
$$
= \frac{\pi^2}{6} \operatorname{area}(\mathcal{D}) h + O(h x^{-\delta}) + O(x^{(1+\delta)/3}).
$$

Collecting all partial results finally gives

$$
\begin{aligned}
B_{\mathcal{D}}(x+h) - B_{\mathcal{D}}(x) &= \frac{6}{\pi^2} \operatorname{area}(\mathcal{D}) h + O(h x^{-\delta}) + O(x^{11/29} \log x) \\
&\sim \frac{6}{\pi^2} \operatorname{area}(\mathcal{D}) h
\end{aligned}
$$

for $h \ge x^{11/29} (\log x)\, \lambda(x)$, as asserted by Theorem 1.

3. Proof of Theorem 2. We start with one more convention: For integers $a < b$ and arbitrary f (defined on the integers from a to b), we write

$$
\sum_{a \le n \le b}'' f(n) = \frac{f(a) + f(b)}{2} + \sum_{a < n < b} f(n).
$$

To prepare the proof of Theorem 2, we split up (writing $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}_*^2$ for short)

$$
\begin{aligned}
(3.1) \qquad A^{(3)}(X) &= \sum_{\substack{F(\mathbf{n}) n_3 \le X \\ n_3 \le X^{5/8}}}' 1 + \sum_{\substack{F(\mathbf{n}) n_3 \le X \\ F(\mathbf{n}) \le X^{3/8}}}' 1 - \sum_{\substack{n_3 \le X^{5/8} \\ F(\mathbf{n}) \le X^{3/8}}}' 1 + O(1) \\
&= \sum_{n \le X^{5/8}}' \left(\operatorname{area}(\mathcal{D}) \frac{X^2}{n^2} + P\left(\frac{X^2}{n^2} \right) \right) \\
&\quad + \sum_{F(\mathbf{n}) \le X^{3/8}}' \left(\frac{X}{F(\mathbf{n})} + O(1) \right) \\
&\quad - (X^{5/8} + O(1))(\operatorname{area}(\mathcal{D}) X^{3/4} + P(X^{3/4})).
\end{aligned}
$$

We may evaluate directly some terms of this expression. Since

$$
\sum_{n > X^{5/8}} \frac{1}{n^2} = X^{-5/8} + O(X^{-5/4}),
$$

it is clear that

$$
(3.2) \qquad \sum_{n \le X^{5/8}}' \operatorname{area}(\mathcal{D}) \frac{X^2}{n^2} = \frac{\pi^2}{6} \operatorname{area}(\mathcal{D}) X^2 - \operatorname{area}(\mathcal{D}) X^{11/8} + O(X^{3/4}).
$$

Using Stieltjes integral notation, we see that

$$\sum_{0<F(\mathbf{n})\le X^{3/8}}' \frac{1}{F(\mathbf{n})} = \int\limits_{F_0-}^{X^{3/8}} \frac{1}{u}\, d(\operatorname{area}(\mathcal{D})u^2 + P(u^2))$$

$$= 2\operatorname{area}(\mathcal{D})(X^{3/8}-F_0) + \frac{1}{u}P(u^2)\Big|_{F_0-}^{X^{3/8}}$$

$$+ \int\limits_{F_0}^{\infty} \frac{1}{u^2}P(u^2)\,du - \int\limits_{X^{3/8}}^{\infty} \frac{1}{u^2}P(u^2)\,du.$$

The very last integral is $O(X^{-3/8})$. This follows by substituting $u^2=v$, then splitting up the range of integration into dyadic subintervals $[M,2M]$, applying the second mean-value theorem on each subinterval, and taking into account that (as is well known; cf., e.g., Hlawka [3])

$$\int\limits_0^V P(v)\,dv = -V + \int\limits_0^V (P(v)+1)\,dv = -V + O(V^{3/4}) = O(V).$$

Consequently,

$$\sum_{F(\mathbf{n})\le X^{3/8}}' \frac{X}{F(\mathbf{n})} = 2\operatorname{area}(\mathcal{D})X^{11/8} + X^{5/8}P(X^{3/4}) + CX + O(X^{5/8}), \tag{3.3}$$

with C as defined in Theorem 2. Using (3.2) and (3.3) to simplify (3.1), we arrive at

$$A^{(3)}(X) = \frac{\pi^2}{6}\operatorname{area}(\mathcal{D})X^2 + CX + \sum_{n\le X^{5/8}}' P\left(\frac{X^2}{n^2}\right) + O(X^{3/4}). \tag{3.4}$$

The next step is to express the two-dimensional lattice rest $P(t^2)$ by (a variant of) fractional part sums. According to our conventions, we put

$$\psi(w) = \begin{cases} w-[w]-1/2 & \text{for } w\notin\mathbb{Z},\\ 0 & \text{for } w\in\mathbb{Z}.\end{cases}$$

On the boundary $\partial\mathcal{D}$ there exist four points $P_1,\dots,P_4$ where the slope is equal to ±1. Drawing straight line segments from each of $P_1,\dots,P_4$ to the origin, we subdivide $\mathcal{D}$ into four domains $\mathcal{D}_1,\dots,\mathcal{D}_4$. To each of $t\mathcal{D}_1,\dots,t\mathcal{D}_4$ we apply a standard elementary lattice point counting argument (involving the Euler summation formula) to see that

$$P(t^2) = \sum_{r=1}^{4} \sum_{a_r t\le k\le b_r t} \psi\left(tf_r\left(\frac{k}{t}\right)\right) + O(1) \tag{3.5}$$

where each f_r satisfies either $F(u,f_r(u))=1$ or $F(f_r(u),u)=1$, and a_r, b_r

are constants depending only on $\mathcal{D}$ for which

$$f_r'(a_r) = 1, \qquad f_r'(b_r) = -1. \tag{3.6}$$

In view of (3.4) and (3.5), our task will be achieved if we can show that, for $r = 1, \ldots, 4$,

$$\sideset{}{'}\sum_{n \le X^{5/8}} \sum_{a_r X/n \le k \le b_r X/n} \psi\left(\frac{X}{n} f_r\left(\frac{kn}{X}\right)\right) \ll X^{22/29} \log X. \tag{3.7}$$

The next important step is to use a sharp result due to Vaaler [14] which relates the ψ-sum involved to exponential sums. (Cf. the exposition in the book of Graham & Kolesnik [2], p. 116) ([5]): For every positive integer D there exists a sequence $(\alpha_{h,D})_{h=1}^{D}$ contained in the interval $[0,1]$ such that for all real numbers w,

$$\left|\psi(w) + \frac{1}{2\pi i} \sum_{1 \le |h| \le D} \frac{\alpha_{|h|,D}}{h} e(hw)\right| \le \frac{1}{2D+2} \sum_{h=-D}^{D} \left(1 - \frac{|h|}{D+1}\right) e(hw),$$

with $e(u) = e^{2\pi i u}$ as usual. From this it is easy to see that there exists a complex-valued sequence $(\beta_{h,D})_{h=1}^{D}$ with

$$\beta_{h,D} \ll \frac{1}{h} \tag{3.8}$$

such that

$$\sideset{}{'}\sum_{n \le X^{5/8}} \sum_{a_r X/n \le k \le b_r X/n} \psi\left(\frac{X}{n} f_r\left(\frac{kn}{X}\right)\right) \ll \left| \sideset{}{'}\sum_{n \le X^{5/8}} \sum_{h=1}^{D} \beta_{h,D} E_h\left(\frac{X}{n}\right)\right| + \frac{X}{D} \sideset{}{'}\sum_{n \le X^{5/8}} \frac{1}{n} \tag{3.9}$$

with

$$E_h(t) = \sum_{a_r t \le k \le b_r t} e\left(-ht f_r\left(\frac{k}{t}\right)\right).$$

(Later on, D will be chosen depending on X but not on n.) Our next step is to submit this exponential sum to a sufficiently strong form of the van der Corput transform which we state as follows.

LEMMA. *Suppose that g is a real-valued function with four continuous derivatives on the interval $[A, B]$. Let L and T be real parameters not less than 2 such that $B - A \asymp L$,*

$$g^{(j)}(w) \ll TL^{1-j} \quad \textit{for } w \in [A,B],\ j = 1,2,3,4,$$

([5]) Properly speaking, both Vaaler and Graham–Kolesnik formulate the result for a ψ defined differently at the integers. What we need is immediate from this by a continuity argument or by direct evaluation.

and, *for some* $C^* > 0$,

$$g''(w) \geq C^* T L^{-1} \quad \textit{for } w \in [A, B].$$

Suppose further that $g'(A)$ *and* $g'(B)$ *are integers*, *and denote by* ϕ *the inverse function of* g'. *Then*

$$\sum_{A \leq k \leq B} e(g(k)) = e\left(\frac{1}{8}\right) \sideset{}{''}\sum_{g'(A) \leq m \leq g'(B)} \frac{e(g(\phi(m)) - m\phi(m))}{\sqrt{g''(\phi(m))}} + O(\log T),$$

with the O-*constant depending on* C^* *and on the constants implied in the order symbols in the suppositions.*

Remark. It appears that, until recently, this result was not available explicitly in the literature. There were versions which stated what is needed but imposed a complicated condition essentially meaning that g be algebraic. (See Krätzel [8], Theorem 2.11, which is based on ideas due to I. M. Vinogradov.) Graham & Kolesnik [2], Lemma 3.6, avoided this restriction but unfortunately produced an error term $O(\sqrt{T/L})$ (in our notation) which is too crude for the present purpose. However, it is easy to construct what we need from the ideas in Graham & Kolesnik [2]. The Lemma in its present form was verified by the second named author in 1996 and first published with a proof in Kühleitner [9] (with permission) and also in Kühleitner & Nowak [10]. The subject was taken up recently also by Liu [12].

We now use this Lemma to transform our exponential sums $E_h(t)$. In this application, $g(w) = -htf_r(w/t)$, thus the suppositions are satisfied with $L = t$ and $T = h$. (The lower bound for g'' follows from the condition that the curvature of $\partial\mathcal{D}$ does not vanish.) Furthermore, (3.6) ensures that $g'(a_r t) = -h$, $g'(b_r t) = h$, thus integer values. Consequently, the Lemma yields

$$\begin{aligned}(3.10) \qquad E_h(t) = \sideset{}{''}\sum_{-h \leq m \leq h} \left| \frac{h}{t} f_r''\left(\chi\left(-\frac{m}{h}\right)\right)\right|^{-1/2} e\left(-tH(m,h) + \frac{1}{8}\right)& \\ + O(\log(1+h)).&\end{aligned}$$

Here χ is the inverse function of f_r', and $H(m,h)$ is the so-called tac-function of the domain $\mathcal{D}$ ("Stützfunktion" according to Bonnesen & Fenchel [1]) which is defined by

$$H(m,h) = \sup_{(u,v) \in \mathcal{D}} (mu + hv) = \max_{F(u,v)=1} (mu + hv).$$

It is an easy exercise in classic analysis that the expression for $g(\phi(m)) - m\phi(m)$ which arises directly from the main term of the Lemma is equal to $-tH(m,h)$.

We now use (3.10) in (3.9) to obtain

$$(3.11)\quad \sum_{n\le X^{5/8}}{}' \sum_{a_rX/n\le k\le b_rX/n} \psi\left(\frac{X}{n} f_r\left(\frac{kn}{X}\right)\right) \ll X^{1/2}\left|\sum_{h=1}^{D} \frac{\beta_{h,D}}{\sqrt{h}} \sum_{-h\le m\le h}{}'' \left|f_r''\left(\chi\left(-\frac{m}{h}\right)\right)\right|^{-1/2} \mathcal{E}_{m,h}(X)\right| + O\left(\frac{X}{D}\log X\right) + O(X^{5/8}(\log D)^2)$$

with

$$\mathcal{E}_{m,h}(X) = \sum_{n\le X^{5/8}}{}' \frac{1}{\sqrt{n}} e\left(-\frac{X}{n} H(m,h)\right).$$

(In the estimation of the error terms (3.8) has been used.)

To estimate $\mathcal{E}_{m,h}(X)$ we employ the method of (classic) exponent pairs. According to Krätzel [8], p. 57, $(1/9, 13/18)$ is an exponent pair. By formula (3.3.4) in Graham & Kolesnik [2],

$$\sum_{N<n\le 2N} \frac{1}{\sqrt{n}} e\left(-\frac{X}{n} H(m,h)\right) \ll \left(\frac{XH(m,h)}{N^2}\right)^{1/9} N^{13/18-1/2} + \frac{N^{3/2}}{XH(m,h)}.$$

Splitting up the range $n \le X^{5/8}$ into dyadic subintervals thus yields

$$\mathcal{E}_{m,h}(X) \ll (XH(m,h))^{1/9}\log X + \frac{(X^{5/8})^{3/2}}{XH(m,h)} \ll (XH(m,h))^{1/9}\log X.$$

Now we recall that $H(m,h) \asymp \|(m,h)\|$ (the Euclidean norm). For $-h \le m \le h$ this implies further that $H(m,h) \asymp h$. Therefore, if we combine the last estimate with (3.11), and observe (3.8) and the fact that the factor involving f'' is bounded, we obtain

$$\sum_{n\le X^{5/8}}{}' \sum_{a_rX/n\le k\le b_rX/n} \psi\left(\frac{X}{n} f_r\left(\frac{kn}{X}\right)\right) \ll (XD)^{1/9+1/2}\log X + \frac{X}{D}\log X + X^{5/8}(\log D)^2.$$

Balancing out the terms gives $D = X^{7/29}$ and thus

$$\sum_{n\le X^{5/8}}{}' \sum_{a_rX/n\le k\le b_rX/n} \psi\left(\frac{X}{n} f_r\left(\frac{kn}{X}\right)\right) \ll X^{22/29}\log X.$$

But this proves (3.7) and thus the assertion of Theorem 2.

4. Concluding remark. It is possible to improve slightly on the results of our theorems by means of M. Huxley's *discrete Hardy–Littlewood* *method* (as presented in his monograph [6]). An immediate possibility to do so is to use, instead of our classic exponent pair $(1/9, 13/18)$, Huxley's pair (according to his paper [5])

$$\left(\frac{187}{1692}+\varepsilon', \frac{305}{423}+\varepsilon'\right) = ABA\left(\frac{89}{570}+\varepsilon'', \frac{89}{570}+\frac{1}{2}+\varepsilon''\right)$$

($\varepsilon', \varepsilon''$ arbitrarily small positive numbers, A, B the usual exponent pair processes). Using this in the above calculation, one obtains Theorem 2 with $O(x^{2066/2725+\varepsilon})$, and thus Theorem 1 in the range $h \geq x^{\theta}$ for any $\theta > 1033/2725 = 0.37908\ldots$

Professor Huxley had the kindness to inform the authors that there are some other ways to apply his deep techniques: On the one hand, there is an unpublished refinement of the results of [5]. On the other hand, since (3.11) actually involves the average of $\mathcal{E}_{m,h}(X)$ with respect to h, some of his mean value estimates for exponential sums might be employed. However, these methods yield only small further improvements, at the cost of a lot of tough technical details. Therefore, it was decided not to pursue the matter further in the present paper.

The authors are glad to use this opportunity to express to Professor Huxley their most sincere gratitude for his valuable comments on the subject.

References

[1] T. Bonnesen und W. Fenchel, *Theorie der konvexen Körper*, Springer, Berlin, 1934.

[2] S. W. Graham and G. Kolesnik, *Van der Corput's Method of Exponential Sums*, Cambridge Univ. Press, 1991.

[3] E. Hlawka, *Über Integrale auf konvexen Körpern I*, Monatsh. Math. 54 (1950), 1–36.

[4] —, *Über die Zetafunktion konvexer Körper*, ibid., 100–107.

[5] M. N. Huxley, *Exponential sums and the Riemann zeta function. IV*, Proc. London Math. Soc. 66 (1993), 1–40.

[6] —, *Area, Lattice Points, and Exponential Sums*, Oxford Univ. Press, 1996.

[7] M. N. Huxley and W. G. Nowak, *Primitive lattice points in convex planar domains*, Acta Arith. 76 (1996), 271–283.

[8] E. Krätzel, *Lattice Points*, Deutscher Verlag Wiss., Berlin, 1988.

[9] M. Kühleitner, *On differences of two kth powers: an asymptotic formula for the mean-square of the error term*, J. Number Theory 76 (1999), 22–44.

[10] M. Kühleitner and W. G. Nowak, *The asymptotic behaviour of the mean-square of fractional part sums*, Proc. Edinburgh Math. Soc. 43 (2000), 309–323.

[11] H.-Q. Liu, *Divisor problems of 4 and 3 dimensions*, Acta Arith. 73 (1995), 249–269.

[12] —, *On a fundamental result in van der Corput's method of estimating exponential sums*, ibid. 90 (1999), 357–370.

[13] W. Müller, *Lattice points in convex planar domains: power moments with an application to primitive lattice points*, in: Proc. Number Theory Conf. 1996 held in Vienna, W. G. Nowak and J. Schoißengeier (eds.), Vienna, 1996, 189–199.

[14] J. D. Vaaler, *Some extremal problems in Fourier analysis*, Bull. Amer. Math. Soc. (2) 12 (1985), 183–216.

[15] W. G. Zhai and X. D. Cao, *On the number of coprime integer pairs within a circle*, Acta Arith. 90 (1999), 1–16.

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