ACTA ARITHMETICA
121.3 (2006)

# On the number of $m$-term zero-sum subsequences

by

DAVID J. GRYNKIEWICZ (Pasadena, CA)

**1. Introduction.** A sequence $S$ of terms from an abelian group is *zero-sum* if the sum of the terms of $S$ is zero. In 1961 Erdős, Ginzburg and Ziv proved that any sequence of $2m-1$ terms from an abelian group of order $m$ contains an $m$-term zero-sum subsequence [10]. This sparked a flurry of generalizations, variations and extensions [1], [3], [7], [8], [11], [13]–[18], [21], [25]–[27], [36]. Since a sequence from the cyclic group $\mathbb{Z}/m\mathbb{Z}$ consisting of only 0's and 1's has its $m$-term zero-sum subsequences in exact correspondence with its $m$-term monochromatic subsequences, the Erdős–Ginzburg–Ziv Theorem can be viewed as a generalization of the pigeonhole principle for $m$ pigeons and two boxes. In essence, the Erdős–Ginzburg–Ziv Theorem expresses the idea that often the best way to avoid zero-sums is to consider sequences with very few distinct terms.

For sequences whose length is greater than $2m-1$, a natural question to ask is *how many* $m$-term zero-sum subsequences one can expect. If the sequence $S$ has length $n$ and consists of at most two distinct terms, then there will be at least $\binom{\lceil \frac{n}{2} \rceil}{m} + \binom{\lfloor \frac{n}{2} \rfloor}{m}$ $m$-term monochromatic subsequences. Thus if the best way to avoid $m$-term zero-sum subsequences were still to use only two distinct residues from $\mathbb{Z}/m\mathbb{Z}$, then one would expect there to always be at least $\binom{\lceil \frac{n}{2} \rceil}{m} + \binom{\lfloor \frac{n}{2} \rfloor}{m}$ $m$-term zero-sum subsequences. This was conjectured by Bialostocki in 1989 [2] and later appeared in [5].

CONJECTURE 1.1. *If $S$ is a sequence of $n$ terms from the cyclic group $\mathbb{Z}/m\mathbb{Z}$, then $S$ has at least $\binom{\lceil \frac{n}{2} \rceil}{m} + \binom{\lfloor \frac{n}{2} \rfloor}{m}$ $m$-term zero-sum subsequences.*

A few years after the conjecture was made, Kisin verified it in the case $m = p^\alpha$ and $m = p^\alpha q$, where $p$ and $q$ are primes and $\alpha \geq 1$, and expressed reasons why the conjecture might fail for $m$ not of this form [30]. At the same time, Füredi and Kleitman showed that Conjecture 1.1 held for sufficiently large $n$ (of order $m^{6m}$), as well as for $m$ of the form $m = pq$, where $p$ and $q$ are distinct primes, and showed that $2\binom{\lfloor \frac{n}{2} \rfloor}{m} - m^2\binom{\lfloor \frac{n}{2} \rfloor - 1}{m-1}$ was a general lower

2000 *Mathematics Subject Classification*: Primary 11B75; Secondary 11B50.

bound on the number of $m$-term zero-sum subsequences [12]. Their results, contrary to those of Kisin, led them to strongly believe the conjecture of Bialostocki to be true for $n > 4m$. Unfortunately, the lower bound shown by Füredi and Kleitman, while being very nice asymptotically for large $n$ and fixed $m$, tells us very little for small $n$, particularly if $m$ is also large.

The aim of this paper is to give a proof, using some recently developed machinery from zero-sum Ramsey theory, of the following general bound on the number of $m$-term zero-sum subsequences.

THEOREM 1.1. *If $S$ is a sequence of $n$ terms from an abelian group $G$ of order $m \geq 30$, then $S$ contains at least*

$$\min\left\{\binom{\lceil\frac{n}{2}\rceil}{m} + \binom{\lfloor\frac{n}{2}\rfloor}{m},\ \binom{n-m}{\lceil\frac{2m-1}{3}\rceil}\right\}$$

*$m$-term zero-sum subsequences.*

Unlike the general bound of Füredi and Kleitman, the bound given by Theorem 1.1 is much more accurate for sequences of small length, and, as will be shown in Section 3, verifies Conjecture 1.1 for $n \leq 6\frac{1}{3}m$. Ironically, this confirms the conjecture of Bialostocki for those cases least thought to be true. Theorem 1.1 also gives a bound for more general abelian groups in addition to cyclic groups.

**2. Preliminaries.** Let $(G, +, 0)$ be an abelian group. If $A, B \subseteq G$, then their *sumset*, $A + B$, is the set of all possible pairwise sums, i.e. $\{a + b \mid a \in A,\, b \in B\}$. A set $A \subseteq G$ is *$H_a$-periodic* if it is the union of $H_a$-cosets for some subgroup $H_a$ of $G$ (note this definition allows $H_a$ to be trivial). We say that $A$ is *maximally* $H_a$-periodic if $A$ is $H_a$-periodic, and $H_a$ is the maximal subgroup for which $A$ is $H_a$-periodic; in this case, $H_a = \{x \in G \mid x+A = A\}$, and $H_a$ is sometimes referred to as the *stabilizer* of $A$. A set $A$ which is maximally $H_a$-periodic, with $H_a$ the trivial group, is *aperiodic*, and otherwise we refer to $A$ as *periodic*. An *$H_a$-hole* of $A$ (where the subgroup $H_a$ is usually understood) is an element $\alpha \in (A + H_a) \setminus A$. For notational convenience, we use $\phi_a : G \to G/H_a$ to denote the natural homomorphism. If $S$ is a sequence of elements from $G$, then an *$n$-set partition* of $S$ is a partition of the sequence $S$ into $n$ nonempty subsequences, $A_1, \ldots, A_n$, such that the terms in each subsequence $A_i$ are all distinct (thus allowing each subsequence $A_i$ to be considered a set). Also, $|S|$ denotes the cardinality of $S$, if $S$ is a set, and the length of $S$, if $S$ is a sequence. Finally, if $S'$ is a subsequence of $S$, then $S \setminus S'$ denotes the subsequence of $S$ obtained by deleting all terms in $S'$.

We begin by stating Kneser's Theorem [31], [28], [32], [29], [34], [23]. The case with $m$ prime is known as the Cauchy–Davenport Theorem [9].

KNESER'S THEOREM. *Let $G$ be an abelian group, and let $A_1, \dots, A_n$ be a collection of finite, nonempty subsets of $G$. If $\sum_{i=1}^{n} A_i$ is maximally $H_a$-periodic, then*

$$\Big|\sum_{i=1}^{n} \phi_a(A_i)\Big| \geq \sum_{i=1}^{n} |\phi_a(A_i)| - n + 1.$$

Note that if $A$ is maximally $H_a$-periodic, then $\phi_a(A)$ is aperiodic. Also, observe that if $A+B$ is maximally $H_a$-periodic and $\varrho = |A+H_a| - |A| + |B+H_a| - |B|$ is the number of holes in $A$ and $B$, then Kneser's Theorem implies $|A+B| \geq |A|+|B|-|H_a|+\varrho$. Consequently, if either $A$ or $B$ contains a unique element from some $H_a$-coset, then $|A+B| \geq |A|+|B|-1$. More generally, if $\varrho = \sum_{i=1}^{n} |H_a+A_i| - |A_i|$ is the total number of holes in the $A_i$, then

$$\Big|\sum_{i=1}^{n} A_i\Big| \geq \sum_{i=1}^{n} |A_i| - (n-1)|H_a| + \varrho.$$

Hence, if $|\sum_{i=1}^{n} A_i| < \sum_{i=1}^{n} |A_i| - n + 1$, then $\varrho < (n-1)(|H_a|-1)$.

The following characterizes when a sufficiently compressed $n$-set partition exists [20], [4].

PROPOSITION 2.1. *Let $n_1$ and $n_0$ be positive integers with $n_0 \leq n_1$. A sequence $S$ of terms from $G$ has an $n_1$-set partition $A = A_1, \dots, A_{n_1}$ with $|A_i| = 1$ for $i > n_0$ (and $||A_i| - |A_j|| \leq 1$ for $i, j \leq n_0$) if and only if $|S| \geq n_1$, and for every nonempty subset $X \subseteq G$ with $|X| \leq (|S| - n_1 - 1)/n_0 + 1$ there are at most $n_1 + (|X|-1)n_0$ terms of $S$ from $X$. In particular, $S$ has an $n_1$-set partition if and only if $|S| \geq n_1$ and the multiplicity of every term of $S$ is at most $n_1$.*

The next simple proposition can often be quite useful when dealing with $n$-set partitions [4].

PROPOSITION 2.2. *Let $S$ be a finite sequence of elements from an abelian group $G$, and let $A = A_1, \dots, A_n$ be an $n$-set partition of $S$, where $|\sum_{i=1}^{n} A_i| = r$, and $\max_i\{|A_i|\} = s$.*

- (i) *There exists a subsequence $S'$ of $S$ and an $n'$-set partition $A' = A_{i_1}, \dots, A_{i_{n'}}$ of $S'$, which is a subsequence of the $n$-set partition $A = A_1, \dots, A_n$, such that $n' \leq r - s + 1$ and $|\sum_{j=1}^{n'} A_{i_j}| = r$.*
- (ii) *There exists a subsequence $S'$ of $S$ of length at most $n + r - 1$, and an $n$-set partition $A' = A'_1, \dots, A'_n$ of $S'$, where $A'_i \subseteq A_i$ for $i = 1, \dots, n$, such that $|\sum_{i=1}^{n} A'_i| = r$.*

The following theorem [20], [22] is a recent generalization of results of Mann [33], Olson [35], Bollobás and Leader [6], and Hamidoune [24].

Theorem 2.1. *Let $S'$ be a subsequence of a finite sequence $S$ of terms from an abelian group $G$, let $A = A_1, \ldots, A_n$ be an $n$-set partition of $S'$, and let $a_i \in A_i$ for $i \in \{1, \ldots, n\}$. Then there exists an $n$-set partition $A' = A'_1, \ldots, A'_n$ of a subsequence $S''$ of $S$ with sumset $H_a$-periodic, $|S'| = |S''|$, $\sum_{i=1}^n A_i \subseteq \sum_{i=1}^n A'_i$, $a_i \in A'_i$ for $i \in \{1, \ldots, n\}$, and*

$$\Big|\sum_{i=1}^{n} A'_i\Big| \geq (E(A', H_a) + (N(A', H_a) - 1)n + 1)|H_a|,$$

*where*

$$N(A', H_a) = \frac{1}{|H_a|}\Big|\bigcap_{i=1}^{n}(A'_i + H_a)\Big|,$$
$$E(A', H_a) = \sum_{j=1}^{n}\Big(|A'_j| - \Big|A'_j \cap \bigcap_{i=1}^{n}(A'_i + H_a)\Big|\Big).$$

*Furthermore, if $H_a$ is nontrivial, then $\phi_a(x) \in \phi_a(A'_i)$ for every $i \in \{1, \ldots, n\}$ and every $x \in S \setminus S''$.*

Note that Theorem 2.1 implies $|\sum_{i=1}^n A'_i| \geq \min\{|G|, |S'| - n + 1\}$ unless $N(A', H_a) > 0$ and $H_a$ is a proper, nontrivial subgroup. Let $\varrho = Nn|H_a| - |S'| + e$, where $N = N(A', H_a)$ and $e = E(A', H_a)$, be the number of $H_a$-holes contained among the sets $A'_j \cap \bigcap_{i=1}^n (A'_i + H_a)$, $j = 1, \ldots, n$. Also observe that if Theorem 2.1 does not hold with $H_a$ trivial, then $(e + (N-1)n + 1)|H_a| \leq |S'| - n$, implying $Nn|H_a| - |S'| \leq n(|H_a| - 1) - |H_a| - e|H_a|$, which from the previous sentences implies

$$\varrho < (n - 1 - e)(|H_a| - 1) \leq (n - 1)(|H_a| - 1),$$

mirroring the bound obtained from Kneser's Theorem discussed earlier.

We will need the following draining theorem for $n$-set partitions [19].

Theorem 2.2. *Let $S$ be a finite sequence of elements from an abelian group $G$. If $S$ has an $n$-set partition, $A = A_1, \ldots, A_n$, such that*

$$\Big|\sum_{i=1}^{n} A_i\Big| \geq \sum_{i=1}^{n} |A_i| - n + 1,$$

*then there exists a subsequence $S'$ of $S$, with length $|S'| \leq \max\{|S| - n + 1, 2n\}$, and with an $n$-set partition, $A' = A'_1, \ldots, A'_n$, such that $|\sum_{i=1}^n A'_i| \geq \sum_{i=1}^n |A_i| - n + 1$. Furthermore, if $|\,|A_i| - |A_j|\,| \leq 1$ for all $i$ and $j$, or if $|A_i| \geq 3$ for all $i$, then $A'_i \subseteq A_i$.*

Finally, we conclude with the following well known and basic theorem bounding the real roots of a polynomial with real coefficients.

Theorem 2.3. *Let $P(x)$ be a polynomial with real coefficients and positive leading coefficient, and let $a$ be a real number. If $a > 0$, and all nonzero terms of $P(x)/(x-a)$, including remainder (computed by polynomial division), are positive, then $a$ is an upper bound for all real roots of $P(x)$.*

*Proof.* Let $P(x) = Q(x)(x-a) + r$, with $r \in \mathbb{R}$. Since all nonzero terms of $P(x)/(x-a)$, including remainder (computed by polynomial division), are positive, it follows that $r \geq 0$ and $Q(x) > 0$ for all real $x > 0$. Thus, since for $x > a > 0$ we have $x - a > 0$, it follows that $P(x) = Q(x)(x-a) + r > 0$ for $x > a$. ∎

**3. The proof.** In view of the results of Kisin [30] mentioned in the introduction, it follows that Conjecture 1.1 is known for $m < 30$, as well as for $m = 2^5 = 32$, $m = 5 \cdot 7 = 35$, and $m = 2 \cdot 19 = 38$. We begin by proving several lemmas relating the sizes of two different binomial coefficients. Note that in view of the first sentence of this section, Lemma 3.2 below and Theorem 1.1 together imply Conjecture 1.1 for $n \leq 6\frac{1}{3}m$. Both Lemmas 3.1 and 3.2 are straightforward computations, best done with machine assistance, but for the benefit of the reader we include many of the details.

Lemma 3.1. *If $m \geq 30$ and $n$ are integers with $2m-1 \leq n \leq 3m + \lceil\frac{2m-1}{3}\rceil - 2$, then*

$$\binom{n-m}{\lceil\frac{m}{2}\rceil} > 2\binom{\lceil\frac{n}{2}\rceil}{m}.$$

*Proof.* Let

$$\begin{aligned} R(n,m) &= \binom{n-m}{\lceil\frac{m}{2}\rceil} \Big/ 2\binom{\frac{n+1}{2}}{m} \\ &= \frac{(n-m)\cdots(n-m-\lceil\frac{m}{2}\rceil+1)(m)\cdots(\lceil\frac{m}{2}\rceil+1)}{2(\frac{n+1}{2})\cdots(\frac{n+1}{2}-m+1)}. \end{aligned}$$

Since $\binom{\frac{n+1}{2}}{m} \geq \binom{\lceil\frac{n}{2}\rceil}{m}$, it suffices to show $R(n,m) > 1$. We begin by showing that $R(n,m) \geq R(n+2,m)$.

Let

$$\begin{aligned} Q(n,m) &= \frac{(n-m-\frac{m+1}{2}+2)(n-m-\frac{m+1}{2}+1)(\frac{n+1}{2}+1)}{(n-m+2)(n-m+1)(\frac{n+1}{2}-m+1)} \\ &\leq R(n,m)/R(n+2,m). \end{aligned}$$

To show $R(n,m) \geq R(n+2,m)$, we will show that $Q(n,m) \geq 1$, i.e. (by multiplying out the denominator, expanding and collecting terms) that

$$4(m-1)n^2 - (11m^2 - 12m + 17)n + (8m^3 - 9m^2 + 16m - 15) \geq 0.$$

This will occur if both roots of the above polynomial are imaginary, which by the quadratic formula occurs when

$$m^4 - \frac{8}{7}m^3 - \frac{118}{7}m^2 - \frac{88}{7}m - 7 > 0. \tag{1}$$

However, Theorem 2.3 shows that the roots of the polynomial $m^4 - \frac{8}{7}m^3 - \frac{118}{7}m^2 - \frac{88}{7}m - 7$ are bounded from above by 6. Consequently, (1) holds for $m \geq 7$, and we can assume $R(n, m) \geq R(n + 2, m)$.

Since $R(n, m) \geq R(n + 2, m)$, it suffices to show $R(3\frac{2}{3}m + b, m) > 1$ for $b = -2 + (\lceil \frac{2m-1}{3} \rceil - \frac{2}{3}m)$ and $b = -3 + (\lceil \frac{2m-1}{3} \rceil - \frac{2}{3}m)$. Note $b \in \{-\frac{5}{3}, -\frac{6}{3}, -\frac{7}{3}, -\frac{8}{3}, -\frac{9}{3}, -\frac{10}{3}\}$. Let $S(m) = R(3\frac{2}{3}m + b, m)$. Next we show that $S(m + 6) \geq S(m)$. Note that computing $S(m)$ for each $m \in \{30, \ldots, 35\}$ and both possible values for $b$ shows that $S(m) > 1$ for those $m$. Hence the proof will be complete once we have shown that $S(m + 6) \geq S(m)$.

Let

$$\begin{aligned} P(m) &= \frac{(\frac{8}{3}m+b+16)\cdots(\frac{8}{3}m+b+1)(m+6)\cdots(m+1)(\frac{5}{6}m+\frac{b+1}{2}+5)\cdots(\frac{5}{6}m+\frac{b+1}{2}+1)}{(\frac{11}{6}m+\frac{b+1}{2}+11)\cdots(\frac{11}{6}m+\frac{b+1}{2}+1)(\frac{m+1}{2}+3)\cdots(\frac{m+1}{2}+1)(\frac{13}{6}m+b+13)\cdots(\frac{13}{6}m+b+1)} \\ &\leq S(m+6)/S(m). \end{aligned}$$

To see that $S(m + 6) \geq S(m)$, we will show that $P(m) \geq 1$. By multiplying out denominators, bringing all terms to the left hand side, expanding and collecting terms, and rounding coefficients down, it follows that it suffices to show

$$\begin{aligned} &- 3 \cdot 10^{17} - 4 \cdot 10^{18}m - 3 \cdot 10^{19}m^2 - 2 \cdot 10^{20}m^3 - 4 \cdot 10^{20}m^4 \\ &- 7 \cdot 10^{20}m^5 - 2 \cdot 10^{21}m^6 - 2 \cdot 10^{21}m^7 - 2 \cdot 10^{21}m^8 - 2 \cdot 10^{21}m^9 \\ &- 8 \cdot 10^{20}m^{10} - 5 \cdot 10^{20}m^{11} - 2 \cdot 10^{20}m^{12} - 8 \cdot 10^{19}m^{13} - 3 \cdot 10^{19}m^{14} \\ &- 7 \cdot 10^{18}m^{15} - 2 \cdot 10^{18}m^{16} - 4 \cdot 10^{17}m^{17} - 6 \cdot 10^{16}m^{18} \\ &- 7 \cdot 10^{15}m^{19} - 7 \cdot 10^{14}m^{20} - 5 \cdot 10^{13}m^{21} - 2 \cdot 10^{12}m^{22} \\ &- 4 \cdot 10^{10}m^{23} + 7 \cdot 10^{8}m^{24} + 2 \cdot 10^{8}m^{25} + 10^{7}m^{26} + 3 \cdot 10^{5}m^{27} > 0 \end{aligned}$$

(the rounded polynomial just given is strictly less, for positive $m$, than the corresponding polynomial for each value of $b \in \{-\frac{5}{3}, -\frac{6}{3}, -\frac{7}{3}, -\frac{8}{3}, -\frac{9}{3}, -\frac{10}{3}\}$ obtained by algebraic manipulation). However, by Theorem 2.3, the roots of the displayed polynomial are all bounded from above by 23, implying that the inequality from the last sentence holds for $m \geq 24$, which completes the proof. ∎

Lemma 3.2. *If $m \geq 30$ and $n$ are integers either with $2m - 1 \leq n \leq 6\frac{1}{3}m$, $m \neq 32, 35, 38$, or else with $2m - 1 \leq n \leq 6\frac{1}{3}m - 6$, then*

$$\binom{n-m}{\lceil \frac{2m-1}{3} \rceil} > 2\binom{\lceil \frac{n}{2} \rceil}{m}.$$

*Proof.* Let

$$R(n,m) = \binom{n-m}{\lceil\frac{2m-1}{3}\rceil}\Big/2\binom{\frac{n+1}{2}}{m} = \frac{(n-m)\cdots(n-m-\lceil\frac{2m-1}{3}\rceil+1)(m)\cdots(\lceil\frac{2m-1}{3}\rceil+1)}{2(\frac{n+1}{2})\cdots(\frac{n+1}{2}-m+1)}.$$

Since $\binom{\frac{n+1}{2}}{m} \geq \binom{\lceil\frac{n}{2}\rceil}{m}$, it suffices to show $R(n,m) > 1$. We begin by showing that $R(n,m) \geq R(n+2,m)$.

Let

$$Q(n,m) = \frac{(n-m-\frac{2m+1}{3}+2)(n-m-\frac{2m+1}{3}+1)(\frac{n+1}{2}+1)}{(n-m+2)(n-m+1)(\frac{n+1}{2}-m+1)} \leq R(n,m)/R(n+2,m).$$

To show $R(n,m) \geq R(n+2,m)$, we will show that $Q(n,m) \geq 1$, i.e. (by multiplying out the denominator, expanding and collecting terms) that

$$3(m-1)n^2 - (10m^2 - 5m + 13)n + (9m^3 - 3m^2 + 6m - 12) \geq 0.$$

This will occur if both roots of the above polynomial are imaginary, which by the quadratic formula occurs when

$$m^4 - \frac{11}{2}m^3 - \frac{177}{8}m^2 - \frac{43}{4}m - \frac{25}{8} > 0. \tag{2}$$

However, by Theorem 2.3 the roots of the polynomial $m^4 - \frac{11}{2}m^3 - \frac{177}{8}m^2 - \frac{43}{4}m - \frac{25}{8}$ are bounded from above by 9. Consequently, (2) holds for $m \geq 10$, and we can assume $R(n,m) \geq R(n+2,m)$.

First assume that $n \leq 6\frac{1}{3}m$ with $m \neq 32, 35, 38$. Since $R(n,m) \geq R(n+2,m)$, it suffices to show $R(6\frac{1}{3}m+b,m) > 1$ for $b = \lfloor 6\frac{1}{3}m\rfloor - 6\frac{1}{3}m$ and $b = -1 + (\lfloor 6\frac{1}{3}m\rfloor - 6\frac{1}{3}m)$. Note $b \in \{0, -\frac{1}{3}, -\frac{2}{3}, -\frac{3}{3}, -\frac{4}{3}, -\frac{5}{3}\}$. Let $S(m) = R(6\frac{1}{3}m+b,m)$. Next we show that $S(m+6) \geq S(m)$ for $m \geq 43$. Note that computing $S(m)$ for each $m \leq 48$, $m \neq 32, 35, 38$, and both possible values for $b$, shows that $S(m) > 1$ for such $m$. Hence the first part of the lemma will be complete once we have shown that $S(m+6) \geq S(m)$ for $m \geq 43$.

Let

$$P(m) = \frac{(\frac{16}{3}m+b+32)\cdots(\frac{16}{3}m+b+1)(m+6)\cdots(m+1)(\frac{13}{6}m+\frac{b+1}{2}+13)\cdots(\frac{13}{6}m+\frac{b+1}{2}+1)}{(\frac{19}{6}m+\frac{b+1}{2}+19)\cdots(\frac{19}{6}m+\frac{b+1}{2}+1)(\frac{2m+1}{3}+4)\cdots(\frac{2m+1}{3}+1)(\frac{14}{3}m+b+\frac{1}{3}+28)\cdots(\frac{14}{3}m+b+\frac{1}{3}+1)}.$$

Note that $P(m) \leq S(m+6)/S(m)$ for $m \geq 43$. To see that $S(m+6) \geq S(m)$, it suffices to show $P(m) \geq 1$. The proof proceeds as in the previous lemma. The case with $n \leq 6\frac{1}{3}m - 6$ is handled similarly. ∎

LEMMA 3.3. *Let $n$, $m$, and $x$ be positive integers. If $n \geq \frac{3}{2}m - 1$, then*

$$3^x \binom{n}{m} \geq \binom{n+x}{m}.$$

*Proof.* Observe that the following binomial identity holds:

$$\binom{n}{m} = \frac{n-m+1}{m}\binom{n}{m-1}. \tag{3}$$

Since $n \geq \frac{3}{2}m - 1$, (3) implies that $2\binom{n+x'}{m} \geq \binom{n+x'}{m-1}$ for $x' \geq 0$. Hence from the Pascal identity, it follows that

$$3\binom{n+x'}{m} \geq \binom{n+x'}{m} + \binom{n+x'}{m-1} = \binom{n+x'+1}{m}$$

for $x' \geq 0$. Iterating the above inequality for $x' = 0, \ldots, x-1$ yields $3^x\binom{n}{m} \geq \binom{n+x}{m}$. ■

We now proceed with the proof of Theorem 1.1, which will be divided into several steps. For our main method to work, we will need the existence of a sufficiently compressed $\lceil \frac{n}{2} \rceil$-set partition. Thus we will first handle several special and highly restrictive sequences $S$ which do not admit such a compressed set partition.

Let $\mathcal{Z}_m(S)$ denote the number of $m$-term zero-sum subsequences of $S$. Note that from the Erdős–Ginzburg–Ziv Theorem it follows trivially that $\mathcal{Z}_m(S) \geq n-2m+2$. Thus $\mathcal{Z}_m(S) \geq \binom{\lceil \frac{n}{2} \rceil}{m} + \binom{\lfloor \frac{n}{2} \rfloor}{m}$ for $n \leq 2m$. Consequently, inductively assume

$$\mathcal{Z}_m(S') \geq \min\left\{\binom{\lceil \frac{n}{2} \rceil}{m} + \binom{\lfloor \frac{n}{2} \rfloor}{m},\ \binom{n-m}{\lceil \frac{2m-1}{3} \rceil}\right\}$$

for any sequence $S'$ of $n'$ terms from an abelian group of order $m$ provided $n' < n$, and also assume that $n \geq 2m+1$. In view of the results of Kisin [30], we may assume that $m$ is composite.

STEP 1 ($S$ essentially monochromatic): Suppose that there is a term $x$ of $S$ with multiplicity at least $\lceil \frac{n}{2} \rceil$. Then there will be at least $\binom{\lceil \frac{n}{2} \rceil - 1}{m-1}$ $m$-term monochromatic (and hence also zero-sum) subsequences of $S$ that include $x$. By induction hypothesis there are at least $\min\left\{\binom{\lceil \frac{n-1}{2} \rceil}{m} + \binom{\lfloor \frac{n-1}{2} \rfloor}{m},\ \binom{n-m-1}{\lceil \frac{2m-1}{3} \rceil}\right\}$ $m$-term zero-sum subsequences that do not include $x$. Hence there are in total at least

$$\min\left\{\binom{\lceil \frac{n}{2} \rceil - 1}{m-1} + \binom{\lceil \frac{n-1}{2} \rceil}{m} + \binom{\lfloor \frac{n-1}{2} \rfloor}{m},\ \binom{\lceil \frac{n}{2} \rceil - 1}{m-1} + \binom{n-m-1}{\lceil \frac{2m-1}{3} \rceil}\right\}$$

$m$-term zero-sum subsequences. By the Pascal identity for binomial coefficients,

$$\binom{\lceil\frac{n}{2}\rceil-1}{m-1}+\binom{\lceil\frac{n-1}{2}\rceil}{m}+\binom{\lfloor\frac{n-1}{2}\rfloor}{m}=\binom{\lceil\frac{n}{2}\rceil-1}{m-1}+\binom{\lceil\frac{n}{2}\rceil-1}{m}+\binom{\lfloor\frac{n}{2}\rfloor}{m}$$
$$=\binom{\lceil\frac{n}{2}\rceil}{m}+\binom{\lfloor\frac{n}{2}\rfloor}{m}.$$

Thus the proof is complete unless

$$\binom{n-m-1}{\lceil\frac{2m-1}{3}\rceil}<\binom{\lceil\frac{n-1}{2}\rceil}{m}+\binom{\lfloor\frac{n-1}{2}\rfloor}{m}, \tag{4}$$

and

$$\binom{\lceil\frac{n}{2}\rceil-1}{m-1}+\binom{n-m-1}{\lceil\frac{2m-1}{3}\rceil}<\binom{n-m}{\lceil\frac{2m-1}{3}\rceil}.$$

From the above inequality and the Pascal identity, it follows that

$$\binom{\lceil\frac{n}{2}\rceil-1}{m-1}<\binom{n-m-1}{\lceil\frac{2m-1}{3}\rceil-1}. \tag{5}$$

From (4) and Lemma 3.2, it follows that $n-1>6\frac{1}{3}m-6$. Applying the binomial identity given in (3) to (5) yields

$$\binom{\lceil\frac{n}{2}\rceil-1}{m-1}<\frac{\lceil\frac{2m-1}{3}\rceil}{n-m-\lceil\frac{2m-1}{3}\rceil}\binom{n-m-1}{\lceil\frac{2m-1}{3}\rceil}, \tag{6}$$

and

$$\binom{\lceil\frac{n}{2}\rceil-1}{m}<\frac{\lceil\frac{n}{2}\rceil-m}{m}\cdot\frac{\lceil\frac{2m-1}{3}\rceil}{n-m-\lceil\frac{2m-1}{3}\rceil}\binom{n-m-1}{\lceil\frac{2m-1}{3}\rceil}. \tag{7}$$

If $n$ is odd, then (4) implies $\binom{n-m-1}{\lceil\frac{2m-1}{3}\rceil}<\binom{\lceil\frac{n}{2}\rceil-1}{m}+\binom{\lceil\frac{n}{2}\rceil-1}{m}$, and if $n$ is even, then (4) and the Pascal identity imply

$$\binom{n-m-1}{\lceil\frac{2m-1}{3}\rceil}<\binom{\lceil\frac{n}{2}\rceil}{m}+\binom{\lceil\frac{n}{2}\rceil-1}{m}=\binom{\lceil\frac{n}{2}\rceil-1}{m}+\binom{\lceil\frac{n}{2}\rceil-1}{m-1}+\binom{\lceil\frac{n}{2}\rceil-1}{m}.$$

Hence from (6) and (7), it follows that

$$\binom{n-m-1}{\lceil\frac{2m-1}{3}\rceil}$$
$$<\left(2\cdot\frac{\lceil\frac{n}{2}\rceil-m}{m}\cdot\frac{\lceil\frac{2m-1}{3}\rceil}{n-m-\lceil\frac{2m-1}{3}\rceil}+\frac{\lceil\frac{2m-1}{3}\rceil}{n-m-\lceil\frac{2m-1}{3}\rceil}\right)\binom{n-m-1}{\lceil\frac{2m-1}{3}\rceil},$$

which in turn implies that

$$1<\frac{2(\frac{n+1}{2}-m)\cdot\frac{2m+1}{3}+m\cdot\frac{2m+1}{3}}{m\cdot(n-m-\frac{2m+1}{3})}.$$

From the above inequality, it follows that $(m-1)n < 3m^2+2m+1$, implying $n < 3m+5+\frac{6}{m-1}$, which contradicts $n-1 > 6\frac{1}{3}m-6$ and $m \geq 30$. So we may assume that the multiplicity of every term $x$ of $S$ is at most $\lceil \frac{n}{2} \rceil - 1$.

Step 2 ($S$ essentially dichromatic): Suppose that every term of $S$, with at most $\max\{m-\frac{m}{p},\, \lfloor \frac{2m-4}{3} \rfloor\}$ exceptions if $n \geq 3m+\lceil \frac{2m-1}{3} \rceil - 1$, and with at most $m-\frac{m}{p}$ exceptions if $n \leq 3m+\lceil \frac{2m-1}{3} \rceil - 2$, is equal to one of two elements $x, y \in G$, where $p$ is the smallest prime divisor of $m$. Let $n_x$ and $n_y$ denote the respective multiplicities of $x$ and $y$ in $S$. Rearrange the terms of $S$ so that all the terms equal to $x$ precede all the terms equal to $y$, which in turn precede all terms equal to neither $x$ nor $y$, and let $x_1, \ldots, x_n$ be the resulting sequence. For $i \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$, let $A_i = \{x_i,\, x_{i+\lceil \frac{n}{2} \rceil}\}$, and if $n$ is odd, then let $A_{\lceil \frac{n}{2} \rceil} = \{x_{\lceil \frac{n}{2} \rceil}\}$. Then in view of Step 1, $A = A_1, \ldots, A_{\lceil \frac{n}{2} \rceil}$ is an $\lceil \frac{n}{2} \rceil$-set partition of $S$ such that either $x \in A_i$ or $y \in A_i$ for every set $A_i$.

There are $\binom{\lfloor \frac{n}{2} \rfloor}{m}$ ways to choose $m$ sets $A_i$ from $A$ all with $|A_i| = 2$, and (in case $n$ odd) there are $\binom{\lceil \frac{n}{2} \rceil - 1}{m-1}$ ways to choose $m$ sets $A_i$ from $A$ that include the set $A_{\lceil \frac{n}{2} \rceil}$ of cardinality one. Consequently, if we can show that any such selection $A_{i_1}, \ldots, A_{i_m}$ has a set $A_{i_k}$ such that $0 \in z + \sum_{j=1,\, j\neq k}^{m} A_{i_j}$ for every $z \in A_{i_k}$ (in which case we will say that the selection $A_{i_1}, \ldots, A_{i_m}$ is *good*), then there will be (in case $n$ even) at least $2\binom{\lfloor \frac{n}{2} \rfloor}{m} = \binom{\lfloor \frac{n}{2} \rfloor}{m} + \binom{\lceil \frac{n}{2} \rceil}{m}$ $m$-term zero-sum subsequences, and (in case $n$ odd), in view of the Pascal identity, at least

$$2\binom{\lfloor \frac{n}{2} \rfloor}{m}+\binom{\lceil \frac{n}{2} \rceil - 1}{m-1} = \binom{\lfloor \frac{n}{2} \rfloor}{m}+\binom{\lceil \frac{n}{2} \rceil - 1}{m}+\binom{\lceil \frac{n}{2} \rceil - 1}{m-1} = \binom{\lfloor \frac{n}{2} \rfloor}{m}+\binom{\lceil \frac{n}{2} \rceil}{m}$$

$m$-term zero-sum subsequences, whence the proof is complete. We proceed to show this is the case, except for a highly restrictive sequence that we handle separately afterwards.

If the selection $A_{i_1}, \ldots, A_{i_m}$ contains the set $A_{\lceil \frac{n}{2} \rceil}$ and $n$ is odd, then let $A_{i_k} = A_{\lceil \frac{n}{2} \rceil}$, and otherwise let $A_{i_k}$ be a set $A_{i_j} = \{x, y\}$ (such a set exists, since at most $\max\{m-\frac{m}{p},\, \lfloor \frac{2m-4}{3} \rfloor\} < m$ terms of $S$ are equal to neither $x$ nor $y$). If

$$\Big|\sum_{\substack{j=1\\ j\neq k}}^{m} A_{i_j}\Big| \geq \sum_{\substack{j=1\\ j\neq k}}^{m} |A_j| - (m-1) + 1 = m,$$

then for each $z \in A_{i_k}$ we can select a term from each of the $A_{i_j}$, $j \neq k$, so that the sum of the $m-1$ terms so selected is the additive inverse of $z$, whence the selection $A_{i_1}, \ldots, A_{i_m}$ is good. Otherwise, from Kneser's Theorem it follows that $\sum_{j=1,\, j\neq k}^{m} A_{i_j}$ is maximally $H_a$-periodic, with $H_a$ of index $a$ and $1 < a < m$.

Suppose that $\phi_a(x) = \phi_a(y)$, i.e. $x$ and $y$ are from the same $H_a$-coset. Hence, since every set $A_{i_j}$ contains either $x$ or $y$, it contains a representative from the coset $x + H_a$. Since $\sum_{j=1, j\neq k}^{m} A_{i_j}$ is $H_a$-periodic, it follows that $0 \in H_a = mx + H_a \subset z + \sum_{j=1, j\neq k}^{m} A_{i_j}$ for $z \in A_{i_k} \subseteq \{x, y\}$, and the proof is again complete. So we may assume that $\phi_a(x) \neq \phi_a(y)$.

If there are at most $m - \frac{m}{p}$ terms of $S$ equal to neither $x$ nor $y$, then there must be at least $a-1$ sets $A_{i_j}$, $j \neq k$, with $A_{i_j} = \{x, y\}$, and hence, since $\phi_a(x) \neq \phi_a(y)$, at least $a-1$ sets $A_{i_j}$ with $|\phi_a(A_{i_j})| = 2$. On the other hand, if there are at most $\lfloor \frac{2m-4}{3} \rfloor$ terms of $S$ equal to neither $x$ nor $y$, then either there likewise must be at least $a-1$ sets $A_{i_j}$ with $|\phi_a(A_{i_j})| = 2$, or else $|H_a| = 2$, and there are at least $\frac{m}{2} + 2$ sets $A_{i_j}$ with $A_{i_j} \neq \{x, y\}$ and $A_{i_j}$ contained in an $H_a$-coset. If the former holds, then Kneser's Theorem yields

$$\Big|\sum_{\substack{j=1\\ j\neq k}}^{m} A_j\Big| \geq |H_a|\Big(\sum_{\substack{j=1\\ j\neq k}}^{m} |\phi_a(A_j)| - (m-1) + 1\Big) \geq m,$$

and the proof is again complete. Therefore we may instead assume the latter. Consequently, we can assume $n \geq 3m + \lceil \frac{2m-1}{3} \rceil - 1$, that $m$ is even, and that there are at least $m - \frac{m}{p} + 1 = \frac{m}{2} + 1$ terms $t$ of $S$ with $t \notin \{x, y\}$.

Suppose that $x - y$ generates a proper subgroup $H_b$ of index $b$. Since there are at most $\lfloor \frac{2m-4}{3} \rfloor$ terms of $S$ equal to neither $x$ nor $y$, and at least $\frac{m}{2} + 1$ sets $A_i$ with $A_i \neq \{x, y\}$ and $A_i$ an $H_a$-coset, we can re-index the sets $A_i$ so that $A_i = \{x, y\}$ for $i \leq \lfloor \frac{n}{2} \rfloor - \lfloor \frac{2m-4}{3} \rfloor$, and $A_i$ is an $H_a$-coset for $\lfloor \frac{n}{2} \rfloor - \lfloor \frac{2m-4}{3} \rfloor + 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - \lfloor \frac{2m-4}{3} \rfloor + \frac{m}{2} + 1$. Let $A_{i'_1}, \ldots, A_{i'_m}$ be a selection of $m$ sets $A_i$ all with

$$i \leq \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{2m-4}{3} \right\rfloor + \frac{m}{2} + 1 = \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{m-8}{6} \right\rfloor + 1.$$

If $A_{i'_j} = \{x, y\}$ for all $j$, then $\sum_{j=1}^{m/b-1} A_{i'_j}$ is an $H_b$-coset, whence there will be at least $2^{m - \frac{m}{b} + 1} \geq 2^{\frac{m}{2}}$ ways to select a term from each $A_{i'_j}$ and get an $m$-term zero-sum subsequence. Next suppose that at least one of the $A_{i'_j}$, say $A_{i'_1}$, is an $H_a$-coset. Since at most $\frac{m}{2} + 1$ of the $A_{i'_j}$ can be $H_a$-cosets, there are at least $\frac{m}{2} - 1$ indices $j$ with $A_{i'_j} = \{x, y\}$. Re-index so that $A_{i'_j} = \{x, y\}$ for $2 \leq i \leq \frac{m}{2}$. Hence $\sum_{j=1}^{m/2} A_{i'_j}$ is an $(H_a + H_b)$-coset. Thus, since every $A_{i'_j}$ contains either $x$ or $y$, it is contained in the same $(H_a + H_b)$-coset $x + H_a + H_b$, whence there will also be at least $2^{\frac{m}{2}}$ ways to select a term from each $A_{i'_j}$ and get an $m$-term zero-sum subsequence. Thus there are at least

$$2^{\frac{m}{2}} \binom{\lfloor \frac{n}{2} \rfloor - \lfloor \frac{m-8}{6} \rfloor + 1}{m} \tag{8}$$

$m$-term zero-sum subsequences. Since $n \geq 3m + \lceil \frac{2m-1}{3} \rceil - 1$, Lemma 3.3 shows that

$$3^x \binom{\lfloor \frac{n}{2} \rfloor - \lfloor \frac{m-8}{6} \rfloor + 1}{m} \geq \binom{\lfloor \frac{n}{2} \rfloor - \lfloor \frac{m-8}{6} \rfloor + 1 + x}{m}.$$

Hence from (8) it follows that there are at least

$$\begin{aligned} 2^{\frac{m}{2}} \binom{\lfloor \frac{n}{2} \rfloor - \lfloor \frac{m-8}{6} \rfloor + 1}{m} &\geq 2 \cdot 4^{\lfloor \frac{m-2}{4} \rfloor} \binom{\lfloor \frac{n}{2} \rfloor - \lfloor \frac{m-8}{6} \rfloor + 1}{m} \\ &\geq 2 \cdot 3^{\lfloor \frac{m-2}{4} \rfloor} \binom{\lfloor \frac{n}{2} \rfloor - \lfloor \frac{m-8}{6} \rfloor + 1}{m} \\ &\geq 2 \binom{\lfloor \frac{n}{2} \rfloor - \lfloor \frac{m-8}{6} \rfloor + \lfloor \frac{m-2}{4} \rfloor + 1}{m} \\ &\geq 2 \binom{\lceil \frac{n}{2} \rceil}{m} \geq \binom{\lceil \frac{n}{2} \rceil}{m} + \binom{\lfloor \frac{n}{2} \rfloor}{m} \end{aligned}$$

$m$-term zero-sum subsequences, whence the proof is complete. So we may assume that $x - y$ generates $G$, implying $G$ is cyclic of order $m$.

Suppose $n_x \leq \lfloor \frac{n}{2} \rfloor - \frac{m}{2}$. Re-index the terms $x_i$ in the sequence $x_1, \ldots, x_n$ with $x_i \notin \{x, y\}$ (leaving unchanged the terms $x_i \in \{x, y\}$) so that all terms $x_i$ with $x_i \notin \{x, y, y + \frac{m}{2}\}$ occur in a consecutive block at the very end of the sequence. Then, since in a cyclic group there is a unique subgroup of order two, it follows that either every set $A_i$ will contain a representative from the common $H_a$-coset $y + H_a$, or else every set $A_i$ contained in an $H_{a'}$-coset with $|H_{a'}| = 2$ and $i \leq \lfloor \frac{n}{2} \rfloor$ must contain $x$. In the latter case, since $n_x \leq \lfloor \frac{n}{2} \rfloor - \frac{m}{2}$, there are at most $\lfloor \frac{2m-4}{3} \rfloor - \frac{m}{2} + 1 < \frac{m}{2} + 2$ sets $A_i$ contained in an $H_{a'}$-coset with $|H_{a'}| = 2$, which reduces to a case handled in the fifth paragraph of Step 2. Therefore we may assume the former case holds. From previous work, we know that any selection $A_{i_1}, \ldots, A_{i_m}$ is good unless $\sum_{j=1, j \neq k}^m A_{i_j}$ is maximally $H_{a'}$-periodic with $|H_{a'}| = 2$ and

$$\Big| \sum_{\substack{j=1 \\ j \neq k}}^m A_{i_j} \Big| < \sum_{\substack{j=1 \\ j \neq k}}^m |A_{i_j}| - (m-1) + 1.$$

However, since there is a unique subgroup $H_a$ of order two, it follows that $H_{a'} = H_a$. Hence, since every $A_i$ contains a representative from the common $H_a$-coset $y + H_a$, and $\sum_{j=1}^m A_{i_j}$ is $H_a$-periodic, it follows that $0 \in \sum_{j=1}^m A_{i_j}$. By the last displayed inequality, and since $|A_{i_j}| = 2$ for $j \neq i_k$, Proposition 2.2 shows that there exists $A_{i_l}$ with $l \neq k$ such that $|\sum_{j=1, j \neq l}^m A_{i_j}| = |\sum_{j=1}^m A_{i_j}|$, whence every $z \in \sum_{j=1}^m A_{i_j}$ can be represented in at least two different ways, including $0 \in \sum_{j=1}^m A_{i_j}$. Thus every selection $A_{i_1}, \ldots, A_{i_m}$ is good, completing the proof. So we may assume that $n_x \geq \lfloor \frac{n}{2} \rfloor - \frac{m}{2} + 1$.

Re-index the terms $x_i$ in the sequence $x_1, \dots, x_n$ with $x_i \notin \{x, y\}$ (leaving unchanged the terms $x_i \in \{x, y\}$) so that all terms $x_i$ with $x_i = x + \frac{m}{2}$ occur in a consecutive block at the very end of the sequence. Since $n_x \geq \lfloor \frac{n}{2} \rfloor - \frac{m}{2} + 1$, and there are at least $m - \frac{m}{p} + 1 = \frac{m}{2} + 1$ terms $t$ with $t \notin \{x, y\}$, it follows that $A_{n_x} = \{x, t\}$ with $t \notin \{x, y\}$. If $n$ is odd, then modify the definition of the set partition $A_1, \dots, A_{\lceil \frac{n}{2} \rceil}$ by swapping the term equal to $x$ in $A_{n_x}$ with the term equal to $y$ in $A_{\lceil \frac{n}{2} \rceil}$. The proof now proceeds as in the above paragraph with the roles of $x$ and $y$ interchanged, completing Step 2. So we may assume that given any two elements $x$, $y \in G$, then there are at least $m - \frac{m}{p} + 1$ terms of $S$ equal to neither $x$ nor $y$, and if $n \geq 3m + \lceil \frac{2m-1}{3} \rceil - 1$, then there are at least $\lfloor \frac{2m-1}{3} \rfloor$ terms of $S$ equal to neither $x$ nor $y$.

STEP 3 ($|S| \leq 3m + \lceil \frac{2m-1}{3} \rceil - 2$): Suppose that $n \leq 3m + \lceil \frac{2m-1}{3} \rceil - 2$. In view of Steps 1 and 2 and Proposition 2.1 applied with $n_1 = n - m + 1$ and $n_2 = \lfloor \frac{m}{2} \rfloor$, there exists an $(n - m + 1)$-set partition $P = P_1, \dots, P_{n-m+1}$ of $S$ with $|P_i| = 1$ for $i > \lfloor \frac{m}{2} \rfloor$. Let $P' = P_1, \dots, P_{\lfloor \frac{m}{2} \rfloor}$, and let $S'$ be the subsequence partitioned by the $\lfloor \frac{m}{2} \rfloor$-set partition $P'$. Apply Theorem 2.1 to the subsequence $S'$ of $S$ with $\lfloor \frac{m}{2} \rfloor$-set partition $P'$, and let $A = A_1, \dots, A_{\lfloor \frac{m}{2} \rfloor}$ be the resulting set partition, and $H_a$ the corresponding subgroup of index $a$.

Suppose that

$$\Big| \sum_{i=1}^{\lfloor m/2 \rfloor} A_i \Big| \geq m = \sum_{i=1}^{\lfloor m/2 \rfloor} |A_i| - \left\lfloor \frac{m}{2} \right\rfloor + 1.$$

Then applying Theorem 2.2 to $A$ and $S'$ yields a subsequence $S''$ of $S'$ of length $m$ with an $\lfloor \frac{m}{2} \rfloor$-set partition $A' = A'_1, \dots, A'_{\lfloor \frac{m}{2} \rfloor}$ satisfying $| \sum_{i=1}^{\lfloor m/2 \rfloor} A'_i| \geq m$. Then given any $\lceil \frac{m}{2} \rceil$-term subsequence $T$ of $S \setminus S''$, we can find a selection of $\lfloor \frac{m}{2} \rfloor$ terms from $A'_1, \dots, A'_{\lfloor \frac{m}{2} \rfloor}$ that sum to the additive inverse of the sum of the terms from $T$. Consequently, there will be at least $\binom{n-m}{\lceil \frac{m}{2} \rceil}$ $m$-term zero-sum subsequences. Thus, since $n \leq 3m + \lceil \frac{2m-1}{3} \rceil - 2$, the proof is complete by Lemma 3.1. So we may assume that

$$\Big| \sum_{i=1}^{\lfloor m/2 \rfloor} A_i \Big| < m = \sum_{i=1}^{\lfloor m/2 \rfloor} |A_i| - \left\lfloor \frac{m}{2} \right\rfloor + 1.$$

From Theorem 2.1 it follows that $N(A', H_a) = 1$ and $E(A', H_a) \leq a - 2$, with $H_a$ a nontrivial, proper subgroup. Hence all but at most $a - 2$ terms of $S$ are from the same $H_a$-coset, say $\alpha + H_a$. Let $H_b$ be a minimal cardinality nontrivial, proper subgroup of index $b$ such that all but at most $b - 2$ terms of $S$ are all from the same $H_b$-coset, say $\beta + H_b$, and there exists an $(n - m + 1)$-set partition $B = B_1, \dots, B_{n-m+1}$ of the terms of $S$ from $\beta + H_a$ with $|B_i| = 1$ for $i > \lfloor \frac{m}{2} \rfloor$ (such a subgroup exists in view of the previous two sentences, and taking $B_i = A'_i \cap (\alpha + H_a)$ for $i \leq \lfloor \frac{m}{2} \rfloor$, and appending on an additional $n - m + 1 - \lfloor \frac{m}{2} \rfloor$ singleton sets using the terms from $S \setminus S''$). By translation we may assume $\beta = 0$. Let $S_b$ be the subsequence of $S$ consisting of terms from $H_b$, and let $S'_b$ be the subsequence of $S_b$ partitioned by the set partition $B' = B_1, \ldots, B_{\lfloor \frac{m}{2} \rfloor}$. Apply Theorem 2.1 to the subsequence $S'_b$ of $S_b$ with $\lfloor \frac{m}{2} \rfloor$-set partition $B'$ and with $G = H_b$, and let $B'' = B'_1, \ldots, B'_{\lfloor \frac{m}{2} \rfloor}$ be the resulting set partition and $H_{kb}$ the corresponding subgroup with $[H_b : H_{kb}] = k$. If $N(B'', H_{kb}) = 1$ and $E(B'', H_{kb}) \leq k - 2$, with $H_{kb}$ a nontrivial, proper subgroup, then all but at most $k - 2 + b - 2 \leq kb - 2$ terms of $S$ will be from the same $H_{kb}$-coset, contradicting the minimality of $H_b$ (the needed $(n - m + 1)$-set partition can be induced from $B''$ as was done for showing the existence of $B$). Therefore we may assume otherwise, whence Theorem 2.1 yields

$$\Big|\sum_{i=1}^{\lfloor m/2 \rfloor} B'_i\Big| \geq \min\left\{\frac{m}{b},\ |S'_b| - \left\lfloor \frac{m}{2} \right\rfloor + 1\right\} = \frac{m}{b}.$$

Thus applying Proposition 2.2 to $B''$ shows that there exists an $\lfloor \frac{m}{2} \rfloor$-set partition $B''' = B''_1, \ldots, B''_{\lfloor \frac{m}{2} \rfloor}$ of a subsequence $S''_b$ of $S'_b$ with $|S''_b| \leq \lfloor \frac{m}{2} \rfloor + \frac{m}{b} - 1$ such that $|\sum_{i=1}^{\lfloor m/2 \rfloor} B''_i| = \frac{m}{b}$. Consequently, as in the previous paragraph, there are at least

$$\binom{n - (\lfloor \frac{m}{2} \rfloor + \frac{m}{b} - 1) - (b - 2)}{\lceil \frac{m}{2} \rceil} \geq \binom{n - m}{\lceil \frac{m}{2} \rceil}$$

$m$-term zero-sum subsequences. Thus, since $n \leq 3m + \lceil \frac{2m-1}{3} \rceil - 2$, the proof is complete by Lemma 3.1. So we may assume that $n \geq 3m + \lceil \frac{2m-1}{3} \rceil - 1$.

Step 4 ($S$ essentially trichromatic): Suppose that every term of $S$, with at most $\lfloor \frac{m-4}{3} \rfloor$ exceptions, is equal to one of three elements $x, y, z \in G$. Let $n_x, n_y, n_z$ be the respective multiplicities of $x$, $y$ and $z$ in $S$, and assume $n_x \geq n_y \geq n_z$. Let $l \leq \lfloor \frac{m-4}{3} \rfloor$ be the number of terms $t$ of $S$ with $t \notin \{x, y, z\}$. In view of Steps 2 and 3, for $w \in \{x, y, z\}$ there are at least $\lfloor \frac{2m-1}{3} \rfloor - \lfloor \frac{m-4}{3} \rfloor \geq \lfloor \frac{m-4}{3} \rfloor + 2 \geq l + 2$ terms of $S$ equal to $w$.

Claim 1. *If $n_x \leq \lfloor \frac{n}{2} \rfloor - l$, then for each $w \in \{x, y, z\}$ there exists an $\lceil \frac{n}{2} \rceil$-set partition $A^{(w)} = A_1, \ldots, A_{\lceil \frac{n}{2} \rceil}$ of $S$ into cardinality at most two sets such that if either $t \in A_j$ with $t \notin \{x, y, z\}$, or if $|A_j| = 1$, then $w \in A_j$.*

Since $n_w \geq l + 2$, for $i$ with $\lfloor \frac{n}{2} \rfloor - l + 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, let $A_i = \{w, t_i\}$, where the $t_i$ are the terms with $t_i \notin \{x, y, z\}$, and if $n$ is odd, then let $A_{\lceil \frac{n}{2} \rceil} = \{w\}$. Let $S'$ be the subsequence of $S$ obtained by deleting all terms contained in the $A_i$ with $i \geq \lfloor \frac{n}{2} \rfloor - l + 1$. To show the claim it suffices to show $S'$ has an $(\lfloor \frac{n}{2} \rfloor - l)$-set partition with all sets of cardinality at most two. However, in view of Proposition 2.1, this will be the case provided no term of $S'$ has multiplicity at least $\lceil \frac{n}{2} \rceil - l + 1$, which we have by assumption of Claim 1. Thus the claim is established.

Claim 2. *If $n_x \geq \lfloor \frac{n}{2} \rfloor - l + 1$, then for each $w \in \{y, z\}$ there exists an $\lceil \frac{n}{2} \rceil$-set partition $A^{(w)} = A_1, \ldots, A_n$ of $S$ into cardinality at most two sets such that: either $x \in A_j$ or $w \in A_j$ for all $j$; if $|A_j| = 1$, then $A_j = \{w\}$; and $A_j \neq \{y, z\}$ for all $j$.*

Let $w'$ be the remaining element in $\{y, z\} \backslash \{w\}$. Rearrange the sequence $S$ so that all the terms equal to $x$ precede all the terms equal $w$, which precede all the terms equal to $w'$, which precede all the terms $t$ with $t \notin \{x, y, z\}$, and let $x_1, \ldots, x_n$ be the resulting sequence. Let $A_i = \{x_i, x_{i+\lceil \frac{n}{2} \rceil}\}$ for $i \leq \frac{n}{2}$, and if $n$ is odd, then let $A_{\lceil \frac{n}{2} \rceil} = \{x_{\lceil \frac{n}{2} \rceil}\}$. In view of Step 1, $n_x \leq \lceil \frac{n}{2} \rceil - 1$. Hence, since $n_w \geq \lfloor \frac{m-4}{3} \rfloor + 2 \geq l + 2$ and $n_x \geq \lfloor \frac{n}{2} \rfloor - l + 1$, the partition $A^{(w)} = A_1, \ldots, A_{\lceil \frac{n}{2} \rceil}$ satisfies the claim.

Let $A^{(w)} = A_1, \ldots, A_{\lceil \frac{n}{2} \rceil}$ be the $\lceil \frac{n}{2} \rceil$-set partition constructed using $w$ from Claim 1 (if $n_x \leq \lfloor \frac{n}{2} \rfloor - l$) or from Claim 2 (if $n_x \geq \lfloor \frac{n}{2} \rfloor - l + 1$), and re-index $A^{(w)}$ so that if $n$ is odd, then $|A_{\lceil \frac{n}{2} \rceil}| = 1$, and $A_j \subsetneq \{x, y, z\}$ precisely for $j$ satisfying $\lfloor \frac{n}{2} \rfloor - l + 1 \leq j \leq \lfloor \frac{n}{2} \rfloor$.

If $n_x \leq \lfloor \frac{n}{2} \rfloor - l$, then suppose that for some $w \in \{x, y, z\}$ the difference of the elements in $\{x, y, z\} \setminus \{w\}$ generates a subgroup $H_b$ of index $b \leq 2$, and if $n_x \geq \lfloor \frac{n}{2} \rfloor - l + 1$, then suppose that for some $w \in \{y, z\}$ the difference of the elements in $\{x, y, z\} \setminus \{w\}$ generates a subgroup $H_b$ of index $b \leq 2$. Let $A_{i_1}, \ldots, A_{i_m}$ be a selection of $m$ sets $A_i$ from $A^{(w)}$.

First suppose that $b = 1$. As seen in Step 2, it is sufficient to show that any such selection $A_{i_1}, \ldots, A_{i_m}$ is good. We proceed to show this claim. If $|\sum_{j=1}^{m} A_{i_j}| \geq m$, then the selection is good in view of Proposition 2.2. Therefore we may assume that $|\sum_{j=1}^{m} A_{i_j}| < m$. Then Kneser's Theorem implies that $\sum_{j=1}^{m} A_{i_j}$ is maximally $H_a$-periodic for some proper, nontrivial subgroup $H_a$ of index $a$, and $|A_{i_j}| > |\phi_a(A_{i_j})|$ for at least $m - 1 - (a - 2)$ sets $A_{i_j}$. Since there are at most $\lfloor \frac{m-1}{3} \rfloor < m - a + 1$ sets $A_i$ with either $|A_i| = 1$ or $A_i \subsetneq \{x, y, z\}$, it follows that $|A_{i_{j'}}| < |\phi_a(A_{i_{j'}})|$ for some $A_{i_{j'}}$ with $A_{i_{j'}} \subseteq \{x, y, z\}$ and $|A_{i_{j'}}| = 2$. Hence $w \in A_{i_{j'}}$, since the difference of the pair from $\{x, y, z\}$ not containing $w$ generates $G$. Thus the pigeonhole principle and the definition of $A^{(w)}$ show that every set $A_{i_j}$ will contain a representative from the common $H_a$-coset $w + H_a$ (the representative being either $w$ or the other element from $A_{i_{j'}}$, which under the case of Claim 2 will be $x$). If $n$ is odd, then let $A_{i_k} = A_{\lceil \frac{n}{2} \rceil}$. Otherwise, since there are at least $m - a + 1 \geq \frac{m}{a}$ sets $A_{i_j}$ with $|A_{i_j}| > |\phi_a(A_{i_j})| = 1$, Proposition 2.2 applied to these $\frac{m}{a}$ sets yields a set $A_{i_k}$ with $|A_{i_k}| > |\phi_a(A_{i_k})|$ such that

$\sum_{j=1,\, j\neq k}^{m} A_{i_j} = \sum_{j=1}^{m} A_{i_j}$. Since $A_{i_k} \subset w + H_a$, since every $A_{i_j}$ contains a representative from the common $H_a$-coset $w + H_a$, and since $\sum_{j=1}^{m} A_{i_j}$ is $H_a$-periodic, it follows that $0 \in H_a = mw + H_a \subseteq t + \sum_{j=1,\, j\neq k}^{m} A_{i_j}$ for every $t \in A_{i_k}$, whence the selection is good. So we may assume that $b = 2$, and consequently from the definition of $A^{(w)}$ the difference of elements from every set $A_i$ with $A_i \subseteq \{x, y, z\}$ generates a proper subgroup.

If $|\sum_{j=1}^{m} A_{i_j}| \geq m$, then as seen in the previous paragraph, the selection $A_{i_1}, \ldots, A_{i_m}$ is good. If this is not the case, then Kneser's Theorem shows that $\sum_{j=1}^{m} A_{i_j}$ is maximally $H_a$-periodic with $H_a$ a nontrivial, proper subgroup of index $a$. Also, if there is a set $A_{i_j} \subseteq \{x, y, z\}$ with $w \in A_{i_j}$ and $|A_{i_j}| > |\phi_a(A_{i_j})|$, then, as in the previous paragraph, every set $A_{i_j}$ will contain a representative from the common $H_a$-coset $w + H_a$ implying that the selection $A_{i_1}, \ldots, A_{i_m}$ is again good. Hence if a selection is not good, then all sets $A_{i_j}$ with $|\phi_a(A_{i_j})| = 1$ must satisfy one of the following conditions: (a) $|A_{i_j}| = 1$, or (b) $A_{i_j} \not\subseteq \{x, y, z\}$, or (c) $A_{i_j} = \{x, y, z\} \setminus \{w\}$. Since $|\sum_{j=1}^{m} A_{i_j}| < m$, Kneser's Theorem shows that there can be at most $a - 2$ sets $A_{i_j}$ with $|\phi_a(A_{i_j})| = 2$, and consequently, in view of the previous sentence, at most $a - 2$ sets $A_{i_j}$ with $A_{i_j} \subseteq \{x, y, z\}$, $|A_{i_j}| = 2$, and $w \in A_{i_j}$.

Since there are at most $\lfloor \frac{m-1}{3} \rfloor < m-a+2$ sets $A_i$ satisfying (a) or (b), and there are at least $m-a+2$ sets $A_{i_j}$ with $|\phi_a(A_{i_j})| = 1$, at least one $A_{i_j}$ must be contained in an $H_a$-coset and satisfy (c). Hence $|\phi_a(\{x, y, z\} \setminus \{w\})| = 1$, implying that the subgroup $H_b$ generated by the difference of the elements in $\{x, y, z\} \setminus \{w\}$ is a subgroup of $H_a$. Hence $H_b = H_a$, since $H_a$ is a proper subgroup, and $H_b$ has index $b = 2$. Consequently, as noted in the previous paragraph, there can be at most $a - 2 = b - 2 = 0$ sets $A_{i_j}$ with $A_{i_j} \subseteq \{x, y, z\}$, $|A_{i_j}| = 2$, and $w \in A_{i_j}$.

Since $n_w \geq l + 2$, there exists a subset $A_k \subseteq \{x, y, z\}$ with $w \in A_k$ and $|A_k| = 2$. In view of the previous paragraph, any selection $A_{i_1}, \ldots, A_{i_m}$ that includes $A_k$ will be a good selection. Thus there are at least $2\binom{\lfloor \frac{n}{2} \rfloor - 1}{m-1} = \binom{\lfloor \frac{n}{2} \rfloor - 1}{m-1} + \binom{\lceil \frac{n}{2} \rceil - 1}{m-1}$ in case $n$ even, and $2\binom{\lfloor \frac{n}{2} \rfloor - 1}{m-1} + \binom{\lfloor \frac{n}{2} \rfloor - 1}{m-2} = \binom{\lfloor \frac{n}{2} \rfloor - 1}{m-1} + \binom{\lceil \frac{n}{2} \rceil - 1}{m-1}$ in case $n$ odd, $m$-term zero-sum subsequences that use one of the two terms contained in $A_k$. Hence by induction hypothesis there are at least

$$\binom{\lfloor \frac{n}{2} \rfloor - 1}{m - 1} + \binom{\lceil \frac{n}{2} \rceil - 1}{m - 1} + \min\left\{ \binom{\lfloor \frac{n}{2} \rfloor - 1}{m} + \binom{\lceil \frac{n}{2} \rceil - 1}{m}, \binom{n - m - 2}{\lceil \frac{2m-1}{3} \rceil} \right\} \tag{9}$$

$m$-term zero-sum subsequences. In view of the Pascal identity, it follows that $\binom{\lfloor \frac{n}{2} \rfloor - 1}{m-1} + \binom{\lceil \frac{n}{2} \rceil - 1}{m-1} + \binom{\lfloor \frac{n}{2} \rfloor - 1}{m} + \binom{\lceil \frac{n}{2} \rceil - 1}{m} = \binom{\lfloor \frac{n}{2} \rfloor}{m} + \binom{\lceil \frac{n}{2} \rceil}{m}$. Hence in view of (9), the proof will be complete unless

$$\binom{n-m-2}{\lceil\frac{2m-1}{3}\rceil} < \binom{\lceil\frac{n}{2}\rceil-1}{m} + \binom{\lfloor\frac{n}{2}\rfloor-1}{m}, \tag{10}$$

and

$$\binom{\lceil\frac{n}{2}\rceil-1}{m-1} + \binom{\lfloor\frac{n}{2}\rfloor-1}{m-1} + \binom{n-m-2}{\lceil\frac{2m-1}{3}\rceil} < \binom{n-m}{\lceil\frac{2m-1}{3}\rceil}.$$

From the above inequality and the Pascal identity, it follows that

$$\binom{\lceil\frac{n}{2}\rceil-1}{m-1} + \binom{\lfloor\frac{n}{2}\rfloor-1}{m-1} < \binom{n-m-2}{\lceil\frac{2m-1}{3}\rceil-1} + \binom{n-m-1}{\lceil\frac{2m-1}{3}\rceil-1}. \tag{11}$$

From (10) it follows that $n \geq 2m+2$. Hence applying to (11) the binomial identity given in (3), as well as the binomial identity $\binom{n}{m} = \frac{n}{n-m}\binom{n-1}{m}$, implies

$$\binom{\lceil\frac{n}{2}\rceil-1}{m-1} + \binom{\lfloor\frac{n}{2}\rfloor-1}{m-1}$$
$$< \frac{\lceil\frac{2m-1}{3}\rceil}{n-m-\lceil\frac{2m-1}{3}\rceil-1}\left(1+\frac{n-m-1}{n-m-\lceil\frac{2m-1}{3}\rceil}\right)\binom{n-m-2}{\lceil\frac{2m-1}{3}\rceil}.$$

Applying (3) to the above inequality yields

$$\binom{\lceil\frac{n}{2}\rceil-1}{m} + \binom{\lfloor\frac{n}{2}\rfloor-1}{m}$$
$$< \frac{\lceil\frac{n}{2}\rceil-m}{m}\cdot\frac{\lceil\frac{2m-1}{3}\rceil}{n-m-\lceil\frac{2m-1}{3}\rceil-1}\left(1+\frac{n-m-1}{n-m-\lceil\frac{2m-1}{3}\rceil}\right)\binom{n-m-2}{\lceil\frac{2m-1}{3}\rceil}.$$

Hence from (10) it follows that

$$1 < \frac{\frac{n+1}{2}-m}{m}\cdot\frac{\frac{2m+1}{3}}{n-m-\frac{2m+1}{3}-1}\left(1+\frac{n-m-1}{n-m-\frac{2m+1}{3}}\right),$$

implying that $3(m-1)n^2-(10m^2+7m+1)n+9m^3+17m^2+8m+2<0$. Hence the quadratic formula yields $8m^4-44m^3-177m^2-86m-25 \leq 0$, else the square root of the discriminant will be imaginary. However, Theorem 2.3 shows the roots of $8m^4-44m^3-177m^2-86m-25$ are bounded from above by 9, whence $8m^4-44m^3-177m^2-86m-25>0$ for $m>10$, a contradiction. So we may assume that if $n_x \leq \lfloor\frac{n}{2}\rfloor - l$, then none of $x-z$, $x-y$, and $y-z$ generates a subgroup $H_b$ of index $b \leq 2$, and if $n_x \geq \lfloor\frac{n}{2}\rfloor - l + 1$, then none of $x-y$ and $x-z$ generates a subgroup $H_b$ of index $b \leq 2$ in $G$.

For $t \in \{x,y,z\}$ if $n_x \leq \lfloor\frac{n}{2}\rfloor - l$, and for $t \in \{y,z\}$ if $n_x \geq \lfloor\frac{n}{2}\rfloor - l+1$, let $H_{b_t}$ be the subgroup of index $b_t$ generated by the difference of the elements in

$\{x,y,z\}\setminus\{t\}$. From the conclusion of the previous paragraph, it follows that $b_t>2$ for each $t$. Thus given any selection $A_{i_1},\ldots,A_{i_m}$ with all $A_{i_j}$ satisfying $A_{i_j}\subseteq\{x,y,z\}$ and $|A_{i_j}|=2$, it follows from the pigeonhole principle that there are at least $\frac{m}{b_t}-1$ sets $A_{i_j}$ equal to $\{x,y,z\}\setminus\{t\}$ for some $t$. Note that $\sum_{i=1}^{m/b_t-1}\{x,y,z\}\setminus\{t\}$ is an $H_{b_t}$-coset, implying that $\sum_{j=1}^m A_{i_j}$ is maximally $H_a$-periodic with $H_{b_t}\le H_a$. Thus Proposition 2.2 applied with elements considered modulo $H_{b_t}$ shows that there exists a re-indexing such that

$$\Big|\sum_{j=1}^{m/b_t-1+b_t-1}A_{i_j}\Big|=\Big|\sum_{j=1}^{m}A_{i_j}\Big|; \tag{12}$$

furthermore, Kneser's Theorem yields $|\phi_a(A_{i_j})|=1$ for $i>\frac{m}{b_t}+b_t-2$, since otherwise

$$\Big|\phi_a\Big(\sum_{j=1}^{m/b_t+b_t-2}A_{i_j}\Big)\Big|<\Big|\phi_a\Big(\sum_{j=1}^{m/b_t+b_t-2}A_{i_j}\Big)+\phi_a\Big(\sum_{j=m/b_t+b_t-1}^{m}A_{i_j}\Big)\Big|,$$

implying

$$\Big|\sum_{j=1}^{m/b_t+b_t-2}A_{i_j}\Big|<\Big|\Big(\sum_{j=1}^{m/b_t+b_t-2}A_{i_j}\Big)+\sum_{j=m/b_t+b_t-1}^{m}A_{i_j}\Big|=\Big|\sum_{j=1}^{m}A_{i_j}\Big|,$$

which contradicts (12). Since $A_{i_j}\subseteq\{x,y,z\}$ with $|A_{i_j}|=2$ for all $j$, and $|\phi_{b_t}(\{x,y,z\}\setminus\{t\})|=1$ implies $|\phi_a(\{x,y,z\}\setminus\{t\})|=1$ (since $H_{b_t}\le H_a$), the pigeonhole principle shows that every $A_{i_j}$ contains a representative from $\{x,y,z\}\setminus\{t\}+H_a$, whence from (12) and the previous sentence there are at least $2^{m-\frac{m}{b_t}-b_t+2}>0$ ways to select a term from each $A_{i_j}$ and have the resulting $m$-term sequence be zero-sum. Thus we conclude that there are at least $2^{m-\frac{m}{b_t}-b_t+2}\binom{\lfloor\frac{n}{2}\rfloor-\lfloor\frac{m-4}{3}\rfloor}{m}$ $m$-term zero-sum subsequences. If $b_t\neq\frac{m}{2}$ for every such selection $A_{i_1},\ldots,A_{i_m}$, then in view of $b_t>2$,

$$2^{m-\frac{m}{b_t}-b_t+2}\ge 2^{\frac{2}{3}m-1}=2\cdot 4^{\frac{m}{3}-1}\ge 2\cdot 3^{\lfloor\frac{m-1}{3}\rfloor}\quad\text{for } m\ge 30,$$

whence the proof is complete in view of Lemma 3.3 and Step 3. So we may assume $b_t=\frac{m}{2}$ for some such selection $A_{i_1},\ldots,A_{i_m}$; and it suffices to further show that each selection $A_{i_1},\ldots,A_{i_m}$, with all $A_{i_j}$ satisfying $A_{i_j}\subseteq\{x,y,z\}$ and $|A_{i_j}|=2$, and with $b_t=\frac{m}{2}$, also has at least $2\cdot 3^{\lfloor\frac{m-1}{3}\rfloor}$ ways to select an $m$-term zero-sum subsequence. We proceed to show this, which will complete the proof of Step 4.

Since $b_t=\frac{m}{2}$, by translation we may assume $\{x,y,z\}\setminus\{t\}=\{0,s\}$, where $s$ has order 2. Since $t-0=t$ does not generate a subgroup with index $b\le 2$, implying the order of $t$ is strictly less than $\frac{m}{2}$, and since $|G/H_{b_t}|=\frac{m}{2}$, it follows that $\phi_{b_t}(t)$ generates a proper subgroup $H_{b'}$ of $G/H_{b_t}$ with index $b' \geq 2$ in $G/H_{b_t}$.

Suppose there are at least $2\lfloor\frac{m-1}{3}\rfloor + 2$ sets $A_{i_j}$ with $|\phi_{b_t}(A_{i_j})| = 1$. Then, since $|H_{b_t}| = 2$ implies that $|A_{i_{j_1}} + A_{i_{j_2}}| = |A_{i_{j_1}}|$ when $|\phi_{b_t}(A_{i_{j_1}})| = |\phi_{b_t}(A_{i_{j_2}})| = 1$, it follows that we can re-index so that

$$\Big|\sum_{j=1}^{m-(2\lfloor(m-1)/3\rfloor+1)} A_{i_j}\Big| = \Big|\sum_{j=1}^{m} A_{i_j}\Big|,$$

with $|\phi_{b_t}(A_{i_j})| = 1$ for $j > m - (2\lfloor\frac{m-1}{3}\rfloor + 1)$. Since there are at least $2^{m-\frac{m}{b_t}-b_t+2} > 0$ ways to select an $m$-term zero-sum from the selection $A_{i_1}, \ldots, A_{i_m}$, it follows that $0 \in \sum_{j=1}^{m} A_{i_j}$. Thus, by the above display, there will be at least $2^{2\lfloor\frac{m-1}{3}\rfloor+1} \geq 2 \cdot 3^{\lfloor\frac{m-1}{3}\rfloor}$ ways to select an $m$-term zero-sum subsequence from the selection $A_{i_1}, \ldots, A_{i_m}$, completing the proof as noted earlier. So, we may assume there are at least $m - (2\lfloor\frac{m-1}{3}\rfloor + 1) \geq \lceil\frac{m-1}{3}\rceil \geq \frac{m}{2b'} - 1 = \frac{b_t}{b'} - 1$ sets $A_{i_j}$ with $|\phi_{b_t}(A_{i_j})| = 2$.

Hence, since $|\phi_{b_t}(\{0, s\})| = 1$, and $\phi_{b_t}(\{0, t\}) = \phi_{b_t}(\{s, t\})$ (as $|\phi_{b_t}(\{0, s\})| = 1$ implies $\phi_{b_t}(0) = \phi_{b_t}(s)$), it follows that there are at least $\frac{b_t}{b'} - 1$ sets $A_{i_j}$ that modulo $H_{b_t}$ have the difference of their elements generating the subgroup $H_{b'} = \langle\phi_{b_t}(t)\rangle$. Note that $\sum_{i=1}^{b_t/b'-1} \phi_{b_t}(\{0, t\}) = H_{b'}$. Hence, since there are at least $\frac{b_t}{b'} - 1$ sets $A_{i_j}$ with $|\phi_{b_t}(A_{i_j})| = 2$, and since there are at least $\frac{m}{b_t} - 1$ sets $A_{i_j}$ equal to $\{x, y, z\} \setminus \{t\} = \{0, s\}$, Proposition 2.2 applied with elements considered in $(G/H_{b_t})/H_{b'}$ shows that there exists a re-indexing such that

$$\Big|\sum_{j=1}^{m/b_t-1+b_t/b'-1+b'-1} A_{i_j}\Big| = \Big|\sum_{j=1}^{m} A_{i_j}\Big|;$$

furthermore, Kneser's Theorem yields $|\phi_a(A_{i_j})| = 1$ for $i > \frac{m}{b_t} - 1 + \frac{b_t}{b'} - 1 + b' - 1 = \frac{b_t}{b'} + b' - 1$, since otherwise

$$\Big|\sum_{j=1}^{b_t/b'+b'-1} A_{i_j}\Big| < \Big|\Big(\sum_{j=1}^{b_t/b'+b'-1} A_{i_j}\Big) + \sum_{j=b_t/b'+b'}^{m} A_{i_j}\Big|,$$

a contradiction. Thus, since $\frac{b_t}{b'} + b' - 1 \leq \frac{m}{4} + 1$, and since $0 \in \sum_{j=1}^{m} A_{i_j}$, it follows that there will be at least $2^{\frac{3}{4}m-1} \geq 2 \cdot 3^{\lfloor\frac{m-1}{3}\rfloor}$ ways to select an $m$-term zero-sum subsequence from $A_{i_1}, \ldots, A_{i_m}$, completing the proof of Step 4 as noted earlier. So we may assume that given any $x, y, z \in G$, there are at least $\lfloor\frac{m-1}{3}\rfloor$ terms of $S$ not equal to $x$ or $y$ or $z$.

Step 5 (The general case): In view of Steps 1–4, and Proposition 2.1, it follows that there exists an $(n - m + 1)$-set partition $P = P_1, \ldots, P_{n-m+1}$

of $S$ with $|P_i| = 1$ for $i > \lceil \frac{m-1}{3} \rceil$. Let $P' = P_1, \ldots, P_{\lceil \frac{m-1}{3} \rceil}$, and let $S'$ be the corresponding subsequence partitioned by $P'$. Apply Theorem 2.1 to the subsequence $S'$ of $S$ with $\lceil \frac{m-1}{3} \rceil$-set partition $P'$, and let $S''$ be the resulting subsequence, $H_a$ the resulting subgroup of index $a$, and $A = A_1, \ldots, A_{\lceil \frac{m-1}{3} \rceil}$ the resulting set partition of $S''$.

Suppose that

$$\Big|\sum_{i=1}^{\lceil (m-1)/3 \rceil} A_i\Big| \geq m = \sum_{i=1}^{\lceil (m-1)/3 \rceil} |A_i| - \left\lceil \frac{m-1}{3} \right\rceil + 1.$$

Then applying Theorem 2.2 to $A$ and $S''$ yields a subsequence $T$ of $S''$ of length at most $m$ with a set partition $B = B_1, \ldots, B_{\lceil \frac{m-1}{3} \rceil}$ such that $|\sum_{i=1}^{\lceil (m-1)/3 \rceil} B_i| \geq m$. Then given any subsequence $T'$ of $S \setminus T$ of length $m - \lceil \frac{m-1}{3} \rceil = \lceil \frac{2m-1}{3} \rceil$, we can find a selection of $\lceil \frac{m-1}{3} \rceil$ terms from $T$, one from each of the $B_1, \ldots, B_{\lceil \frac{m-1}{3} \rceil}$, that sum to the additive inverse of the sum of the terms from the $\lceil \frac{2m-1}{3} \rceil$-term subsequence $T'$. Consequently, there will be at least $\binom{n-m}{\lceil \frac{2m-1}{3} \rceil}$ $m$-term zero-sum subsequences, completing the proof. So we can assume that

$$\Big|\sum_{i=1}^{\lceil (m-1)/3 \rceil} A_i\Big| < m = \sum_{i=1}^{\lceil (m-1)/3 \rceil} |A_i| - \left\lceil \frac{m-1}{3} \right\rceil + 1. \tag{13}$$

Theorem 2.1 then shows that $H_a$ is a proper, nontrivial subgroup, and that either $N(A, H_a) = 1$ and $E(A, H_a) \leq a - 2$, or else $N(A, H_a) = 2$, $|H_a| = 2$, and $E(A, H_a) \leq \frac{m}{2} - \lceil \frac{m-1}{3} \rceil - 2 \leq \lfloor \frac{m-10}{6} \rfloor$. The case $N(A, H_a) = 1$ and $E(A, H_a) \leq a - 2$ can be handled by a minor modification of the arguments from the third paragraph of Step 3 (simply replace $\lfloor \frac{m}{2} \rfloor$ by $\lceil \frac{m-1}{3} \rceil$ where appropriate). Therefore we may assume the latter case holds.

Since $N(A, H_a) = 2$, choose $x, y \in G$ so that $\phi_a(x), \phi_a(y) \in G/H_a$ are the two elements from $\phi_a(\bigcap_{i=1}^{\lceil (m-1)/3 \rceil} (A_i + H_a))$. Suppose first that $\phi_a(x - y)$ generates a proper subgroup $H_{a'}/H_a$ of $G/H_a$. If there does not exist a set $A_{j'}$ such that $\{x, y\} + H_a \subseteq A_{j'}$, then there will be at least $\lceil \frac{m-1}{3} \rceil = \lceil \frac{m-1}{3} \rceil (|H_a| - 1)$ holes contained among the sets $A_{i_j}$, which in view of the comments after Theorem 2.1 implies that (13) cannot hold, a contradiction. Therefore we may assume that there exists a set $A_{j'}$ with $\{x, y\} + H_a \subseteq A_{j'}$.

For $i = j'$, let $B_{j'} = (\{x, y\} + H_a) \cap A_{j'} = \{x, y\} + H_a$, and for $i \neq j'$, let $B_i$ be a cardinality two subset of $A_i \cap (\{x, y\} + H_a)$ with $|\phi_a(B_i)| = 2$. Since $\phi_a(x - y)$ generates a proper subgroup $H_{a'}/H_a$, and $\lceil \frac{m-1}{3} \rceil \geq \frac{m}{4} \geq |G/H_{a'}|$, it follows that $\sum_{i=1}^{\lceil (m-1)/3 \rceil} B_i$ is an $H_{a'}$-coset. Observe that all but at most

$E(A, H_a) \leq \frac{m}{2} - \lceil \frac{m-1}{3} \rceil - 2$ terms of $S$ are from the same $H_{a'}$-coset $x + H_{a'}$. Let $T$ be the subsequence of $S$ partitioned by $B = B_1, \ldots, B_{\lceil \frac{m-1}{3} \rceil}$. Since $B_i \subseteq x + H_{a'}$ for all $i$, and $\sum_{i=1}^{\lceil (m-1)/3 \rceil} B_i$ is an $H_{a'}$-coset, it follows that given any $\lceil \frac{2m-1}{3} \rceil$-term subsequence $T'$ of $S \setminus T$ with all terms from $x + H_{a'}$, we can find a selection of $\lceil \frac{m-1}{3} \rceil$ terms from $T$, one from each $B_1, \ldots, B_{\lceil \frac{m-1}{3} \rceil}$, that sums to the additive inverse of the sum of terms from $T'$. Hence, as there are at least

$$\begin{aligned} n - \left(2\left\lceil \frac{m-1}{3} \right\rceil + 2 + E(A', H_a)\right) &\geq n - \left(2\left\lceil \frac{m-1}{3} \right\rceil + 2 + \frac{m}{2} - \left\lceil \frac{m-1}{3} \right\rceil - 2\right) \\ &\geq n - m \end{aligned}$$

terms of $S \setminus T$ from $x + H_{a'}$, it follows that there are at least $\binom{n-m}{\lceil \frac{2m-1}{3} \rceil}$ $m$-term zero-sum subsequences, completing the proof. So we may assume that $\phi_a(x - y)$ generates $G/H_a$.

Let $x'$ be the other element from $x + H_a$, and $y'$ the other element from $y + H_a$. Let $n_x$, $n_{x'}$, $n_y$, and $n_{y'}$ be the respective multiplicities of $x$, $x'$, $y$, and $y'$ in $S$. Since, as noted previously, there is a set $A_{j'}$ such that $\{x, y\} + H_a \subseteq A_{j'}$, it follows that $n_x, n_{x'}, n_y, n_{y'} \geq 1$. We may assume that $n_x + n_{x'} \geq n_y + n_{y'}$, $n_x \geq n_{x'}$, and $n_y \geq n_{y'}$. Remove two terms from $S$, one equal to $x$ and one equal to $x'$, and let the resulting sequence be $T$. Let $B_0$ be the set consisting of the two removed terms. Rearrange the terms of $T$ so that all terms equal to $x$ precede all terms equal to $x'$, which precede all terms equal to $y$, which precede all terms equal to $y'$, which precede all terms $t$ with $t \notin \{x, x', y, y'\}$, and let $x_1, \ldots, x_{n-2}$ be the resulting sequence. Let $B_i = \{x_i, x_i + \lceil \frac{n}{2} \rceil - 1\}$ for $i = 1, \ldots, \lfloor \frac{n}{2} \rfloor - 1$, and, in case $n$ odd, let $B_{\lceil \frac{n}{2} \rceil - 1} = \{x_{\lceil \frac{n}{2} \rceil - 1}\}$. In view of Step 1, $B = B_1, \ldots, B_{\lceil \frac{n}{2} \rceil - 1}$ is an $(\lceil \frac{n}{2} \rceil - 1)$-set partition of $T$. As seen in the ninth paragraph of Step 4, it suffices by induction hypothesis to show that any selection $B_0, B_{i_1}, \ldots, B_{i_{m-1}}$ containing $B_0$ is good. We proceed to show this.

If $|\sum_{j=1}^{m-1} \phi_a(B_{i_j})| \geq \frac{m}{2}$, then, since $B_0$ is an $H_a$-coset, it follows that $|B_0 + \sum_{j=1}^{m-1} B_{i_j}| \geq m$, whence the selection $B_0, B_{i_1}, \ldots, B_{i_{m-1}}$ is good by Proposition 2.2. Hence we may assume that

$$\Big| \sum_{j=1}^{m-1} \phi_a(B_{i_j}) \Big| < \frac{m}{2}. \tag{14}$$

Suppose that $n_x + n_{x'} > \lceil \frac{n}{2} \rceil$. Then every set $B_{i_j}$ will contain a representative from the common $H_a$-coset $x + H_a$. Since $B_0$ is $H_a$-periodic, it follows that $0 \in B_0 + \sum_{j=1}^{m-1} B_{i_j}$.

Suppose further that $|\phi_a(B_{i_k})| = 1$ for some $B_{i_k}$ with $i_k \geq 1$. Then

$$B_0 + \sum_{\substack{j=1\\ j\neq k}}^{m-1} B_{i_j} = B_0 + \sum_{j=1}^{m-1} B_{i_j},$$

and it follows that either $|B_{i_k}| = 1$, or else there will be at least two ways to represent every $x \in B_0 + \sum_{j=1}^{m-1} B_j$. Since $0 \in B_0 + \sum_{j=1}^{m-1} B_j$, it follows that the selection $B_0, B_{i_1}, \ldots, B_{i_{m-1}}$ is good, completing the proof as noted earlier. So we may assume $|\phi_a(B_{i_k})| = 2$ for all $i_k \geq 1$.

Since $|\phi_a(B_{i_k})| = 2$ for all $i_k \geq 1$, and $|\phi_a(\{x, x'\})| = 1$, it follows that there does not exist a set $B_{i_j}$ with $i_j \geq 1$ and $B_{i_j} = \{x, x'\}$. Since there are at most $E(A, H_a) \leq \frac{m-10}{6}$ terms $t$ with $t \notin \{x, x', y, y'\}$, and every $B_{i_j}$ contains either $x$ or $x'$, it follows that there are at least $m - 2 - \frac{m-10}{6} \geq \frac{m}{2}$ sets $B_{i_j}$ with the difference of terms in $B_{i_j}$ equal modulo $H_a$ to $\phi_a(x - y)$. Thus, since $\phi_a(x - y)$ generates $G/H_a$, it follows that (14) cannot hold, a contradiction. So we may assume that $n_x + n_{x'} \leq \lceil \frac{n}{2} \rceil$.

Since $n_x + n_{x'} \leq \lceil \frac{n}{2} \rceil$, since $n_x + n_{x'} \geq n_y + n_{y'}$, and since all but at most $E(A', H_a) \leq \frac{m}{2} - \lceil \frac{m-1}{3} \rceil - 2 \leq \lfloor \frac{m-10}{6} \rfloor$ terms of $S$ are equal to one of $x$, $x'$, $y$, or $y'$, it follows that at least $(m-3) - \frac{m-10}{6} \geq \frac{m}{2}$ sets $B_{i_j}$ have $\phi_a(B_{i_j}) = \{\phi_a(x), \phi_a(y)\}$. As $\phi_a(x) - \phi_a(y)$ generates $G/H_a$, it follows that $|\sum_{j=1}^{m-1} \phi_a(B_{i_j})| \geq \frac{m}{2}$, contradicting (14) again, and completing the proof.

## References


[1] N. Alon and M. Dubiner, *Zero-sum sets of prescribed size*, in: Combinatorics, Paul Erdős is Eighty, Vol. 1, Bolyai Soc. Math. Stud., János Bolyai Math. Soc., Budapest, 1993, 33–50.

[2] A. Bialostocki, *Some combinatorial number theory aspects of Ramsey theory*, research proposal, 1989.

[3] A. Bialostocki and P. Dierker, *On the Erdős–Ginzburg–Ziv theorem and the Ramsey numbers for stars and matchings*, Discrete Math. 110 (1992), 1–8.

[4] A. Bialostocki, P. Dierker, D. Grynkiewicz, and M. Lotspeich, *On some developments of the Erdős–Ginzburg–Ziv Theorem II*, Acta Arith. 110 (2003), 173–184.

[5] A. Bialostocki and M. Lotspeich, *Some developments of the Erdős–Ginzburg–Ziv theorem I*, in: Sets, Graphs and Numbers (Budapest, 1991), Colloq. Math. Soc. János Bolyai 60, North-Holland, 1992, 97–117.

[6] B. Bollobás and I. Leader, *The number of $k$-sums modulo $k$*, J. Number Theory 78 (1999), 27–35.

[7] W. Brakemeier, *Eine Anzahlformel von Zahlen modulo $n$*, Monatsh. Math. 85 (1978), 277–282.

[8] Y. Caro, *Remarks on a zero-sum theorem*, J. Combin. Theory Ser. A 76 (1996), 315–322.

[9] H. Davenport, *On the addition of residue classes*, J. London Math. Soc. 10 (1935), 30–32.

[10] P. Erdős, A. Ginzburg, and A. Ziv, *Theorem in additive number theory*, Bull. Res. Council Israel 10F (1961), 41–43.

[11] C. Flores and O. Ordaz, *On the Erdős–Ginzburg–Ziv theorem*, Discrete Math. 152 (1996), 321–324.

[12] Z. Füredi and D. J. Kleitman, *The minimal number of zero sums*, in: Combinatorics, Paul Erdős is Eighty, Vol. 1, Bolyai Soc. Math. Stud., János Bolyai Math. Soc., Budapest, 1993, 159–172.

[13] L. Gallardo and G. Grekos, *On Brakemeier's variant of the Erdős–Ginzburg–Ziv problem*, Number Theory (Liptovský Ján, 1999), Tatra Mt. Math. Publ. 20 (2000), 91–98.

[14] L. Gallardo, G. Grekos, L. Habsieger, F. Hennecart, B. Landreau, and A. Plagne, *Restricted addition in $\mathbb{Z}/n\mathbb{Z}$ and an application to the Erdős–Ginzburg–Ziv problem*, J. London Math. Soc. (2) 65 (2002), 513–523.

[15] L. Gallardo, G. Grekos, and J. Pihko, *On a variant of the Erdős–Ginzburg–Ziv problem*, Acta Arith. 89 (1999), 331–336.

[16] W. D. Gao and Y. O. Hamidoune, *Zero sums in abelian groups*, Combin. Probab. Comput. 7 (1998), 261–263.

[17] W. Gao and X. Jin, *Weighted sums in finite cyclic groups*, Discrete Math. 283 (2004), 243–247.

[18] D. Grynkiewicz, *A weighted Erdős–Ginzburg–Ziv theorem*, Combinatorica, to appear.

[19] —, *An extension of the Erdős–Ginzburg–Ziv theorem to hypergraphs*, European J. Combin., to appear.

[20] —, *On a conjecture of Hamidoune for subsequence sums*, Integers, to appear.

[21] —, *On four colored sets with nondecreasing diameter and the Erdős–Ginzburg–Ziv Theorem*, J. Combin. Theory Ser. A 100 (2002), 44–60.

[22] —, *On a partition analog of the Cauchy–Davenport theorem*, Acta Math. Hungar. 107 (2005), 161–174.

[23] H. Halberstam and K. F. Roth, *Sequences*, Springer, New York, 1983.

[24] Y. O. Hamidoune, *Subsequence sums*, Combin. Probab. Comput. 12 (2003), 413–425.

[25] —, *On weighted sums in abelian groups*, Discrete Math. 162 (1996), 127–132.

[26] —, *On weighted sequence sums*, Combin. Probab. Comput. 4 (1995), 363–367.

[27] Y. O. Hamidoune, O. Ordaz, and A. Ortuño, *On a combinatorial theorem of Erdős, Ginzburg and Ziv*, ibid. 7 (1998), 403–412.

[28] X. Hou, K. Leung, and Q. Xiang, *A generalization of an addition theorem of Kneser*, J. Number Theory 97 (2002), 1–9.

[29] J. H. B. Kemperman, *On small sumsets in an abelian group*, Acta Math. 103 (1960), 63–88.

[30] M. Kisin, *The number of zero sums modulo $m$ in a sequence of length $n$*, Mathematika 41 (1994), 149–163.

[31] M. Kneser, *Ein Satz über abelsche Gruppen mit Anwendungen auf die Geometrie der Zahlen*, Math. Z. 64 (1955), 429–434.

[32] —, *Abschätzung der asymptotischen Dichte von Summenmengen*, ibid. 58 (1953), 459–484.

[33] H. B. Mann, *Two addition theorems*, J. Combin. Theory 3 (1967), 233–235.

[34] M. B. Nathanson, *Additive Number Theory: Inverse Problems and the Geometry of Sumsets*, Grad. Texts in Math. 165, Springer, New York, 1996.

[35] J. E. Olson, *An addition theorem for finite abelian groups*, J. Number Theory 9 (1977), 63–70.
[36] —, *On a combinatorial problem of Erdős, Ginzburg and Ziv*, ibid. 8 (1976), 52–57.

Mathematics 253–37
Caltech
Pasadena, CA 91125, U.S.A.
E-mail: diambri@hotmail.com

*Received on 18.7.2005* (5039)