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Integers free of prime divisors from an interval, II

by

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1. Introduction. Let $\Gamma(x,y,z)$ be the number of positive integers not exceeding x which are free of prime divisors from the interval $(z,y]$. By Ψ and Φ we denote the well-known functions given by $\Psi(x,z)=\Gamma(x,x,z)$ and $\Phi(x,y)=\Gamma(x,y,1)$. See Tenenbaum [12] for an overview on results on Ψ and Φ.

Throughout we will use the notation

$$u=\frac{\log x}{\log y},\qquad v=\frac{\log x}{\log z},\qquad r=\frac{u}{v}=\frac{\log z}{\log y}.$$

For $0<u\le v$, we let

$$\eta(u,v):=\varrho(v)+\int_0^u \varrho(tv/u)\omega(u-t)\,dt, \tag{1}$$

where ϱ and ω denote Dickman's function and Buchstab's function, respectively.

In [15] we have shown that, uniformly for $x\ge y\ge z\ge 3/2$,

$$\Gamma(x,y,z)=x\eta(u,v)+O\left(\frac{x}{\log y}\right). \tag{2}$$

Furthermore, we derived difference-differential equations for $\eta(u,v)$, and used these equations to study the behaviour of $\eta(u,v)$ as u, respectively v, grow unbounded.

In [13] Tenenbaum considered the more general problem of estimating the number of positive integers not exceeding x which have exactly k prime factors in the interval $[z,y)$. For $k=0$, his estimate in [13, Theorem C(v)] is equivalent to (2).

In this paper, we will use the saddle-point method to sharpen the results obtained in [15].

Let $\Theta(x,y,z)$ be the number of integers not exceeding x, all of whose prime divisors are in the interval $(z,y]$. This function has been studied by

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Friedlander [5] and by Saias [9–11]. The function

$$\sigma(u,v) := \frac{u}{v}\,\varrho'(u) + \int_0^\infty \varrho\Big(u - \frac{u}{v}\,t\Big)\,d\omega(t) \qquad (v \ge u > 0, u \ne 1)$$

arises in the study of $\Theta(x,y,z)$ (see [5], [9]). Let $\sigma_r(u) := \sigma(u,u/r)$ for $0 < r \le 1$. The results on σ and Θ obtained by Friedlander and Saias are of crucial importance for our estimates.

Let

$$L_\varepsilon(t) = \exp\{(\log t)^{3/5-\varepsilon}\}.$$

The conditions $(H_\varepsilon(x,y))$ and (V_ε) are defined by

$$(H_\varepsilon(x,y)) \qquad x \ge 3, \quad \exp\{(\log\log x)^{5/3+\varepsilon}\} \le y \le x,$$

and

$$(V_\varepsilon) \qquad z \ge 1, \quad y \ge z(1 + L_\varepsilon^{-1}(z)).$$

By (G_c) we denote the domain

$$(G_c) \qquad z \ge 3/2, \quad x \ge y \ge z^{1+c\sqrt{(\log(2u))/u}}.$$

Let

$$\mu_{y,z}(u) := \int_{-\infty}^{\infty} \eta_r(u-t)\,d\Big(\frac{[y^t]}{y^t}\Big)$$

if $x = y^u$ is not an integer and $\mu_{y,z}(u) := \mu_{y,z}(u+)$ if y^u is an integer. Throughout, $c_0, c_1, \ldots$ will denote some absolute positive constants. Define

$$W(x,y,z) := x\mu_{y,z}(u)\,\frac{\log y}{\log z} \prod_{z<p\le y} \Big(1 - \frac{1}{p}\Big),$$

$$H_r(u) := \exp(u \min{}^2(1-r, (\log(2u))^{-1})),$$

$$E(x,y,z) := (\log y)^{-1} H_r(u)^{-c_0} + e^{-u(1-\varepsilon)}.$$

The main purpose of this paper is to establish the following result.

Theorem 1.1. *There exists a positive constant c such that for all $\varepsilon > 0$, under the condition (G_c), we have*

(i) $\Gamma(x,y,z) - W(x,y,z) \ll_\varepsilon \Theta(x,y,z)E(x,y,z)L_\varepsilon^{-1}(z)$ *in* $(H_\varepsilon(x,y))$,

(ii) $$\Gamma(x,y,z) - x\prod_{z<p\le y}\Big(1 - \frac{1}{p}\Big) \ll \Theta(x,y,z) \qquad \textit{elsewhere.}$$

Note that outside the domain (G_c) the asymptotic behavior of $\Gamma(x,y,z)$ is completely described in [15, (4) and Theorem 3.2].

If $x = y$, then

$$\mu_{y,z}(u) = \mu_{x,z}(1) = \int_{-\infty}^{\infty} \eta_r(1-t)\, d\biggl(\frac{[x^t]}{x^t}\biggr) = \int_{-\infty}^{\infty} \varrho\biggl(\frac{1-t}{r}\biggr)\, d\biggl(\frac{[x^t]}{x^t}\biggr)$$
$$= \int_{-\infty}^{\infty} \varrho(v-s)\, d\biggl(\frac{[z^t]}{z^t}\biggr) =: \frac{\Lambda(x,z)}{x}.$$

Thus it follows directly from Theorem 1.1(i) that, for $v \ge v_0$,

$$\Psi(x,z) = \Lambda(x,z)\,\frac{\log x}{\log z} \prod_{z<p\le x}\biggl(1-\frac{1}{p}\biggr) + O_\varepsilon\biggl(\frac{\Phi(x,z)}{L_\varepsilon(z)}\biggr).$$

With the strong form of Mertens' formula

$$\prod_{z<p\le x}\biggl(1-\frac{1}{p}\biggr) = \frac{\log z}{\log y}\,(1+O_\varepsilon(L_\varepsilon(z)^{-1})), \tag{3}$$

we get, for $v \ge v_0$,

$$\Psi(x,z) = \Lambda(x,z) + O_\varepsilon(xL_\varepsilon(z)^{-1}). \tag{4}$$

De Bruijn [2] introduced $\Lambda(x,z)$ as an approximation to $\Psi(x,z)$. He showed that (4) holds for $x > 1$, $y \ge 2$. Saias [8] improved that result by showing that, for $\varepsilon > 0$,

$$\Psi(x,z) = \Lambda(x,z)(1+O_\varepsilon(L_\varepsilon(z)^{-1})) \quad \text{for } (x,z) \text{ in } (H_\varepsilon(x,z)).$$

Let

$$L(x,y,z) := x\eta(u,v)\,\frac{\log y}{\log z}\prod_{z<p\le y}\biggl(1-\frac{1}{p}\biggr).$$

We will derive the following corollary from Theorem 1.1.

Corollary 1.2. *There exists a positive constant c such that for all $\varepsilon > 0$, under the conditions (G_c) and $(H_\varepsilon(x,y))$, we have*

(i) $$\Gamma(x,y,z) = L(x,y,z) + O_\varepsilon\biggl(x\sigma_r(u)\biggl(\frac{H_r(u)^{-c_1}}{\log y} + \frac{e^{-u(1-\varepsilon)}}{L_\varepsilon(z)}\biggr)\biggr),$$

(ii) $$\Gamma(x,y,z) = x\eta(u,v) + O_\varepsilon\biggl(x\sigma_r(u)\biggl(\frac{H_r(u)^{-c_1}}{\log y} + \frac{e^{-u(1-\varepsilon)}}{L_\varepsilon(z)}\biggr) + \frac{x}{(\log y)L_\varepsilon(z)}\biggr),$$

(iii) $$\Gamma(x,y,z) = x\prod_{z<p\le y}\biggl(1-\frac{1}{p}\biggr) + O_\varepsilon\biggl(x\sigma_r(u)\biggl(\frac{H_r(u)^{-c_1}\log z}{\log y} + \frac{e^{-u(1-\varepsilon)}}{L_\varepsilon(z)}\biggr)\biggr).$$

The following are sample consequences of Corollary 1.2. Under the conditions (G_c), $(H_\varepsilon(x,y))$, and $(H_\varepsilon(y,z))$, we have

$$\Gamma(x,y,z) = L(x,y,z) + O_\varepsilon\left(\frac{x\sigma_r(u)H_r(u)^{-c_1}}{\log y}\right)$$

and

$$\Gamma(x,y,z) = x\eta(u,v) + O_\varepsilon\left(\frac{x}{\log y}\,(\sigma_r(u)H_r(u)^{-c_1} + L_\varepsilon(z)^{-1})\right),$$

which is an improvement of (2) if z and u grow unbounded. Under the conditions (G_c), $(H_\varepsilon(x,y))$, $(H_\varepsilon(y,z))$, and $x \ge yz$, we get

$$\Gamma(x,y,z) = L(x,y,z)\left(1 + O_\varepsilon\left(\frac{\sigma_r(u)H_r(u)^{-c_1}}{\log z}\right)\right)$$

and

$$\Gamma(x,y,z) = x\eta(u,v)\left(1 + O_\varepsilon\left(\frac{\sigma_r(u)H_r(u)^{-c_1} + L_\varepsilon(z)^{-1}}{\log z}\right)\right),$$

since $\eta(u,v) \asymp u/v$ for $x \ge yz$, according to [15, Lemma 3.4].

Saias [9, p. 351] showed that, under the condition (G_c), we have

$$\sigma_r(u) = \left(\frac{u}{1-r}\right)^{-u(1+O((\log\log(3u))/\log(2u)))}. \tag{5}$$

In Section 2 we calculate the Laplace transform of $\eta_r(u) = \eta(u, u/r)$ and estimate the inverse Laplace integral

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \widehat{\eta}_r(s)e^{us}\,ds,$$

which converges to $\eta_r(u)$ for $\mathrm{Re}(s) > 0$.

We will use two different approaches to approximate $\Gamma(x,y,z)$. First, the Möbius inversion formula gives

$$\Gamma(x,y,z) = \sum_{d\mid P}\mu(d)\left[\frac{x}{d}\right],$$

where $P = \prod_{z<p\le y} p$. The main term in Theorem 1.1(ii) is obtained by ignoring the square brackets and extending the sum to infinity.

Let

$$\zeta(s,t) = \prod_{p\le t}(1-p^{-s})^{-1}$$

and let $\zeta(s)$ denote Riemann's zeta function. With

$$a_n = \begin{cases} 1 & \text{if } p\mid n \Rightarrow p \notin (z,y], \\ 0 & \text{else,} \end{cases}$$

the series

$$\zeta(s)\,\frac{\zeta(s,z)}{\zeta(s,y)}=\sum_{n=1}^{\infty}\frac{a_n}{n^s}$$

is the Dirichlet series associated with the counting function $\Gamma(x,y,z)=\sum_{n\le x}a_n$. Perron's formula (see for example [12, Theorem II.2.1]) shows that

$$\Gamma(x,y,z)=\frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty}\zeta(s)\,\frac{\zeta(s,z)}{\zeta(s,y)}\cdot\frac{x^s}{s}\,ds\qquad(\kappa>1,x\notin\mathbb{N}).\tag{6}$$

This will be the starting point of the proof of Theorem 1.1(i). We will make use of the saddle-point method to evaluate the complex integral in (6).

2. Study of $\eta(u,v)$ by the saddle-point method. Let $\widehat{f}(s)$ denote the value of the Laplace transform of the function f at s, that is,

$$\widehat{f}(s):=\int_0^{\infty}e^{-su}f(u)\,du.$$

Let $I_r(s)$ be the entire function defined by

$$I_r(s):=\int_{rs}^{s}\frac{e^t-1}{t}\,dt=\int_r^1\frac{e^{st}-1}{t}\,dt.$$

For $0<r\le 1$, let $\eta_r(u)=\eta(u,u/r)$ and $\sigma_r(u)=\sigma(u,u/r)$. We put $\eta_0(u)=0$ and $\sigma_0(u)=e^{-\gamma}\varrho(u)$.

LEMMA 2.1. *For $0\le r<1$ we have*

(i) $\qquad\widehat{\varrho}(s)=\exp\{\gamma+I_0(-s)\}\qquad(s\in\mathbb{C}),$

(ii) $\qquad\widehat{\omega}(s)=[s\exp\{\gamma+I_0(-s)\}]^{-1}-1\qquad(\mathrm{Re}(s)>0),$

(iii) $\qquad\widehat{\sigma}_r(s)=\exp(I_r(-s))-r\qquad(s\in\mathbb{C}),$

(iv) $\qquad\widehat{\eta}_r(s)=\dfrac{r}{s}\exp(-I_r(-s))\qquad(\mathrm{Re}(s)>0).$

Proof. The Laplace transforms of ϱ, ω and σ_r have been calculated by Bovey [1, Lemma 1], Fouvry and Tenenbaum [4, (6.14)], and Saias [9, Lemma 1], respectively. We only show (iv). From the definition of $\eta(u,v)$ in (1) we have

$$\eta_r(u)=\eta(u,u/r)=\varrho(u/r)+\int_0^u\varrho(t/r)\omega(u-t)\,dt.$$

Hence, for $\mathrm{Re}(s)>0$,

$$\widehat{\eta}_r(s)=r\widehat{\varrho}(rs)+r\widehat{\varrho}(rs)\widehat{\omega}(s)=(\widehat{\omega}(s)+1)r\widehat{\varrho}(rs).$$

Together with (i) and (ii) this gives

$$\widehat{\eta_r}(s) = \frac{r}{s}\exp\{I_0(-rs) - I_0(-s)\} = \frac{r}{s}\exp(-I_r(-s)),$$

for $\mathrm{Re}(s) > 0$.

REMARK 2.2. $\widehat{\omega}(s)$ and $\widehat{\eta_r}(s)$ extend to analytic functions on $\mathbb{C}$ except for a simple pole at $s = 0$, given explicitly by the expressions in Lemma 2.1(ii), (iv), for $s \neq 0$.

COROLLARY 2.3. *We have, for $u \geq 0$ and $0 < r \leq 1$,*

(i) $$\int_0^u \eta_r(t)\varrho(u-t)\,dt = r\int_0^{u/r} \varrho(t)\,dt,$$

(ii) $$\int_0^u \omega(t)\,dt + 1 = \eta_r(u) + \frac{1}{r}\int_0^u \eta_r(u-t)\omega(t/r)\,dt,$$

(iii) $$\int_0^u \eta_r(t)\sigma_r(u-t)\,dt = r(1-\eta_r(u)).$$

Proof. By Lemma 2.1 we have, for $\mathrm{Re}(s) > 0$,

$$\widehat{\varrho}(s)\widehat{\eta_r}(s) = \frac{r}{s}\,\widehat{\varrho}(rs), \qquad \frac{\widehat{\omega}(s)+1}{s} = \widehat{\eta_r}(s)(\widehat{\omega}(rs)+1),$$
$$\widehat{\sigma_r}(s)\widehat{\eta_r}(s) = r\left(\frac{1}{s} - \widehat{\eta_r}(s)\right).$$

The result follows.

COROLLARY 2.4. *We have*

$$\int_0^\infty \{\eta_r(t) - r\}\,dt = r(1-r).$$

Proof. We derive the expansion of $\widehat{\eta_r}(s)$ at $s = 0$ of order 1 as follows:

$$\begin{aligned}\widehat{\eta_r}(s) &= \frac{r}{s}\exp\left(-\int_{-rs}^{-s}\frac{e^t-1}{t}\,dt\right) = \frac{r}{s}\exp\left(-\int_{-rs}^{-s}\{1+O(t)\}\,dt\right)\\ &= \frac{r}{s}\exp\{(1-r)s + O(s^2)\} = \frac{r}{s}\{1+(1-r)s+O(s^2)\}\\ &= \frac{r}{s} + r(1-r) + O(s).\end{aligned}$$

Since $\widehat{r} = r/s$ it follows that

$$(\widehat{\eta_r - r})(s) = r(1-r) + O(s),$$

which implies the result.

It follows from Corollary 4.6 of [15] that $\eta_r'(u)$ is of bounded variation on any bounded interval. Together with the rapid decrease of $\eta_r'(u)$ at infinity (Proposition 5.3 in [15]), this implies that the inverse Laplace integral

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \widehat{\eta_r'}(s)e^{us}\,ds \tag{7}$$

converges to the value of $\eta_r'(u)$ for $u \neq 0, 1$, for any abscissa of integration α (see [16, II.7.3 or II.7.5]). We will evaluate (7) with the saddle-point method.

According to the saddle-point method (see [3, Chapter 5]), the integral in (7) will be dominated by the contribution from a small neighborhood of the real point α, if we choose α to be a zero of the derivative of the integrand. We have

$$\widehat{\eta_r'}(s) = s\widehat{\eta}_r(s) - 1 = r\exp(-I_r(-s)) - 1.$$

If we temporarily ignore the influence of the term -1 in the last expression, a zero of the derivative of the integrand in (7) is a solution of the equation

$$e^{-w} - e^{-rw} = uw, \tag{8}$$

which has no real solution other than $w = 0$. We consider instead the related equation

$$e^{-w} - e^{-rw} = -uw.$$

Following Saias [9], we write $w = -\xi$ and define, for $u > 1 - r$, $\xi_r(u)$ to be the unique positive solution ξ of the equation

$$e^{\xi} = e^{r\xi} + u\xi. \tag{9}$$

We let $\xi_r(1-r) = 0$. The following result, due to Saias [9, Lemma 3], shows that (8) has a solution which is close to $-\xi_r(u)$, suggesting that $\alpha = -\xi_r(u)$ is a good choice for the abscissa of integration.

Lemma 2.5 (Saias). *For $0 \le r < 1$, $s \in \mathbb{C}$ and $|s| > c_0$, the equation*

$$e^{\xi} = e^{r\xi} + s\xi$$

has a unique solution satisfying

$$|\xi - \xi_r(|s|) - i\arg(s)| < 1.$$

We are thus able to define a function $\xi = \xi_r(s)$ which is analytic for

$$\{s \in \mathbb{C} : |s| > c_0, \arg(s) \neq 3\pi/2\}.$$

The following asymptotic formula for $\sigma_r(u)$, due to Saias [9, Theorem 1], will allow us to bound $\eta_r'(u)$ in terms of $\sigma_r(u)$.

Lemma 2.6 (Saias). *Let n and k be integers ≥ 0. If $u > 2$, then there exists a constant $c = c(n,k)$ such that, under the condition (G_c), we have*

$$\sigma_r^{(n)}(u) = (-\xi_r(u))^n \left(\frac{\xi_r'(u)}{2\pi}\right)^{1/2} \times \exp\left\{-\int_{1-r}^{u} \xi_r(t)\,dt\right\}\left(1 + f_{r,n}(u) + O_{n,k}\left(\frac{1}{u^k}\right)\right)$$

where $f_{r,n}$ *denotes a* C^∞ *function satisfying*

$$f_{r,n}^{(l)} \ll_{n,l} \frac{1}{u^{l+1}} \qquad (l \geq 0).$$

REMARK 2.7. With (9) we can transform the main term in Lemma 2.6 using the identity

$$-\int_{1-r}^{u} \xi_r(t)\,dt = -u\xi_r(u) + \int_0^{\xi_r(u)} \frac{e^s - e^{rs}}{s}\,ds = -u\xi_r(u) + I_r(\xi_r(u)).$$

LEMMA 2.8 (Saias [8, Lemma 7]). *For* $s \in \mathbb{C} \setminus [0, +\infty)$, *we have*

$$I_0(s) = -\log(-s) - \gamma + O(e^s/s).$$

LEMMA 2.9 (Saias [9, Lemma 4]). *Let* $0 \leq r < 1$, $u \geq 1$ *and* $n \geq 1$. *Then*

(i) $$\xi_r(u) = \log u + \log\left(\log(2u) + \frac{1}{1-r}\right) + O(1),$$

(ii) $$\xi_r^{(n)}(u) = \frac{(-1)^{n+1}(n-1)!}{u^n}\left(1 + O_n\left(\frac{1}{\log(u/(1-r))}\right)\right) \qquad (u > c).$$

LEMMA 2.10 (Saias [9, Lemma 5]). *Let* $0 \leq r < 1$, $u \geq 1$, $s = \xi_r(u) + i\tau$ *and let* n *be an integer* ≥ 0. *Then we have, for* $\tau \ll 1$,

$$I_r^{(n)}(s) = u\left(1 + O_n\left(\frac{1}{\log(2u/(1-r))} + |\tau|\right)\right).$$

LEMMA 2.11. *Let* $r, u \in \mathbb{R}$ *satisfy* $0 \leq r < 1$ *and* $u \geq 1$. *Let* $s, \tau \in \mathbb{C}$ *satisfy* $s = \xi_r(u) + i\tau$. *Then*

(i) $$|e^{-I_r(s)}| \leq \exp\left\{-I_r(\xi_r(u)) + I_r''(\xi)\frac{\tau^2}{2}\right\} \qquad (\tau \in \mathbb{R}),$$

(ii) $$|e^{-I_r(s)}| \ll \exp\{I_r(\xi_r(u))\}H_r(u)^{-c_3} \qquad (\tau \in \mathbb{R}, 1 \leq |\tau| \leq e^\xi),$$

(iii) $$e^{-I_r(s)} = -se^\gamma\left(1 + O\left(\frac{e^\xi}{\tau} + r\tau\right)\right) \qquad (e^\xi \leq |\tau| \ll r^{-1}),$$

(iv) $$e^{-I_r(s)} = \frac{1}{r}\left(1 + O\left(\frac{e^\xi}{\tau} + \frac{e^{r\xi}}{r\tau}\right)\right) \qquad (|\tau| \geq e^\xi, r\tau \gg 1).$$

Proof. (i) Since $(1-\cos(t\tau)) \le (t\tau)^2/2$, we have

$$I_r(\xi) - \mathrm{Re}(I_r(\xi+i\tau)) = \int_r^1 \frac{e^{t\xi}(1-\cos(t\tau))}{t}\,dt \le \int_r^1 \frac{e^{t\xi}\tau^2 t^2}{2t}\,dt = \frac{\tau^2}{2}\,I_r''(\xi),$$

which implies (i).

(ii) Let

$$J := I_r(\xi) + \mathrm{Re}(I_r(\xi+i\tau)) = \int_r^1 \{e^{\xi t}(1+\cos(t\tau)) - 2\}\,\frac{dt}{t}.$$

We need to bound J from below. First consider the case $r \le 1/2$. Let J_1 be the contribution to J from the domain $r \le t \le 1/2$ and let J_2 be the contribution to J from the domain $1/2 \le t \le 1$. Since $|\tau| \le e^{\xi}$,

$$\begin{aligned} J_1 &\ge \int_r^{1/2} (\cos(t\tau)-1)\,\frac{dt}{t} = -2\int_{r\tau/2}^{\tau/4} \sin^2(t)\,\frac{dt}{t} \ge -2\int_0^{|\tau|} \sin^2(t)\,\frac{dt}{t} \\ &\ge -2\int_0^1 t\,dt - 2\int_1^{|\tau|} \frac{dt}{t} = -1 - 2\log(|\tau|) \ge -c_1\log u - c_2, \end{aligned}$$

by Lemma 2.9. Also,

$$\begin{aligned} J_2 &\ge \int_{1/2}^1 \left\{e^{\xi t}\,(1+\cos(t\tau)) - \frac{2}{t}\right\} dt = \left[\frac{e^{\xi t}}{\xi} + \mathrm{Re}\left(\frac{e^{t(\xi+i\tau)}}{\xi+i\tau}\right)\right]_{1/2}^1 - \log 4 \\ &= e^{\xi}\left\{\frac{1}{\xi} + \frac{\cos\lambda}{\sqrt{\xi^2+\tau^2}}\right\} + O\left(\frac{e^{\xi/2}}{\xi}\right), \end{aligned}$$

where $\lambda = \tau - \arctan(\tau/\xi)$. There exists some absolute constant τ_0 such that $\cos\lambda \ge 0$ for $|\tau| \le \tau_0$. For such τ the lower bound for J_2 is thus $\gg e^{\xi}/\xi$. For $|\tau| > \tau_0$, we have

$$J_2 \ge e^{\xi}\left\{\frac{1}{\xi} - \frac{1}{\sqrt{\xi^2+\tau^2}}\right\} + O\left(\frac{e^{\xi/2}}{\xi}\right) \gg \frac{e^{\xi}}{\xi^3} + O\left(\frac{e^{\xi/2}}{\xi}\right) \gg \frac{e^{\xi}}{\xi^3}.$$

From Lemma 2.9 it follows that $J_2 \gg u/(\log u)^2$, which shows the result in the case $r \le 1/2$.

If $r > 1/2$, we have

$$\begin{aligned} I_r(\xi) + \mathrm{Re}(I_r(\xi+i\tau)) &= \int_r^1 e^{\xi t}(1+\cos(t\tau))\,\frac{dt}{t} + 2\log r \\ &\ge \int_r^1 \frac{e^{\xi t}-1}{t}\,(1+\cos(t\tau))\,dt - 2\log 2. \end{aligned} \tag{10}$$

We put

$$Q(\tau) := \int_r^1 \frac{e^{\xi t}-1}{t}\,(1+\cos(t\tau))\,dt. \tag{11}$$

By the second mean value theorem we have

$$\int_r^1 \frac{e^{\xi t}-1}{t}\cos(t\tau)\,dt \geq -(e^{\xi}-1)\frac{2}{|\tau|},$$

since $(e^{\xi t}-1)/t \leq e^{\xi}-1$. Thus, for $|\tau| \geq 1$ and k a positive integer, we obtain

$$\begin{aligned} Q(k\tau) &= \int_r^1 \frac{e^{\xi t}-1}{t}\,(1+\cos(k\tau t))\,dt \geq I_r(\xi) - \frac{2(e^{\xi}-1)}{k|\tau|} \\ &\gg u\left(1-\frac{ce^{\xi}}{ku}\right), \end{aligned} \tag{12}$$

by Lemma 2.10. One easily verifies by induction that the inequality

$$|1+e^{ika}| \leq k|1+e^{ia}|$$

holds for all odd positive integers k and $a \in \mathbb{R}$. Squaring both sides implies $(1+\cos(ka)) \leq k^2(1+\cos a)$, which gives

$$Q(k\tau) \leq k^2 Q(\tau), \tag{13}$$

for any odd positive integer k. We choose $k = 1+2[c_0 e^{\xi}/u]$. Then (12) and (13) imply

$$Q(\tau) \geq \frac{Q(k\tau)}{k^2} \gg \frac{u}{k^2} \gg \frac{u^3}{e^{2\xi}}.$$

From Lemma 2.9, we have $e^{\xi} \ll u(\log(2u)+1/(1-r))$, hence

$$Q(\tau) \gg \frac{u}{\left(\log(2u)+\frac{1}{1-r}\right)^2} \gg u\min^2\left(1-r, \frac{1}{\log(2u)}\right).$$

Together with (10) and (11), this gives the desired result.

(iii) We write $-I_r(s) = -I_0(s) + I_0(rs)$. Since $rs \ll 1$, we clearly have $I_0(rs) \ll rs$. To estimate $I_0(s)$, we apply Lemma 2.8. This gives

$$-I_r(s) = \log(-s) + \gamma + O(e^s/s + rs),$$

which yields the result.

(iv) Here we use Lemma 2.8 to approximate both $I_0(s)$, and $I_0(rs)$. We obtain

$$-I_r(s) = \log(-s) - \log(-rs) + O\left(\frac{e^s}{s} + \frac{e^{rs}}{rs}\right).$$

This shows (iv) and hence concludes the proof of Lemma 2.11.

Remark 2.12. Note that $e^{\xi} + e^{r\xi}/r = O(e^{\xi}+1/r)$. Indeed, if $e^{r\xi}/r > e^{\xi}$, then $\xi < (1-r)^{-1}\log(r^{-1})$, which implies that $e^{r\xi} \ll 1$.

LEMMA 2.13. *We have*

$$\int\limits_{\substack{s=-\xi+i\tau\\|\tau|\ge e^{\xi}}} \widehat{\eta_r'}(s)e^{us}\,ds = O\biggl(e^{-u\xi}\,\frac{e^{\xi}}{u-1}\biggr).$$

Proof. Note that $\widehat{\eta_r'}(s) = s\widehat{\eta_r}(s) - 1 = r\exp(-I_r(-s)) - 1$, by Lemma 2.1. Using integration by parts, we can write

$$\int\limits_{\substack{s=-\xi+i\tau\\|\tau|\ge e^{\xi}}} (r\exp(-I_r(-s))-1)e^{us}\,ds = \int\limits_{\substack{s=\xi+i\tau\\|\tau|\ge e^{\xi}}} (r\exp(-I_r(s))-1)e^{-us}\,ds$$
$$= \frac{1}{u}\,[(re^{-I_r(s)}-1)e^{-us}]_{\tau=-e^{\xi}}^{\tau=e^{\xi}} + \frac{r}{u}\int\limits_{\substack{s=\xi+i\tau\\|\tau|\ge e^{\xi}}} \frac{e^s-e^{rs}}{s}\,e^{-I_r(s)-us}\,ds.$$

Let A denote the first term and let B denote the integral in the last expression. To bound A we consider the two cases $1\le re^{\xi}$, and $1>re^{\xi}$, and apply Lemma 2.11(iii) respectively (iv) to obtain $A \ll e^{-u\xi}/u$. To bound B we assume first that $re^{\xi}<1$. The contribution to B from the interval $[e^{\xi},1/r]$ is, by Lemma 2.11(iii), bounded by

$$\frac{r}{u}\int\limits_{e^{\xi}<|\tau|\le 1/r} \frac{e^s-e^{rs}}{s}\,O(s)e^{-us}\,ds = O\biggl(e^{-u\xi}\,\frac{e^{\xi}}{u}\biggr).$$

Finally, the contribution to B from $\tau>\max(e^{\xi},1/r)$ is bounded by

$$\frac{r}{u}\int\limits_{|\tau|>\max(e^{\xi},1/r)} \frac{e^s-e^{rs}}{rs}\biggl(1+\frac{O(e^{\xi}+1/r)}{\tau}\biggr)e^{-us}\,ds \ll \frac{e^{-u\xi}}{u(u-1)} + \frac{1}{u}\,e^{-u\xi}e^{\xi},$$

by Lemma 2.11(iv). This concludes the proof of Lemma 2.13.

PROPOSITION 2.14. *Let $u>1$ and $0<r<1$. Then*

$$\eta_r'(u) \ll e^{I_r(\xi)-u\xi+\xi}H_r(u)^{-c_1}. \tag{14}$$

Proof. If $1<u\le 3/2$, then Lemmas 2.9 and 2.10 show that the right hand side of (14) is $\gg 1-r$. On the other hand, $\eta_r'(u)=0$ if $r>3/4$ and $1<u\le 3/2$. This follows from Proposition 5.3(ii) of [15] and the fact that $\sigma(u,v)=0$ if $u>1$ and $1<v\le 2$ (see [5, Theorem 2(B)]). If $r\le 3/4$ we clearly have $\eta_r'(u)\ll 1-r$. This shows the result for $1<u\le 3/2$.

Now let $u>3/2$. By (7) we may write

$$2\pi i\eta_r'(u) = \int_{-\xi-i\infty}^{-\xi+i\infty} \widehat{\eta_r'}(s)e^{us}\,ds = \int_{\xi-i\infty}^{\xi+i\infty} (re^{-I_r(s)}-1)e^{-us}\,ds$$
$$= \int_{\xi-ie^{\xi}}^{\xi+ie^{\xi}} (re^{-I_r(s)}-1)e^{-us}\,ds + O\biggl(\frac{e^{-u\xi+\xi}}{u-1}\biggr),$$

by Lemma 2.13. We clearly have

$$\int_{\xi-ie^{\xi}}^{\xi+ie^{\xi}} e^{-us}\,ds = O(e^{-u\xi}).$$

Lemmas 2.10 and 2.11(i) show that $\exp\{-I_r(s)\} \ll 1$ for $|\tau| \le 1$. This yields

$$\int\limits_{\substack{|\tau|\le 1\\ s=\xi+i\tau}} re^{-I_r(s)-us}\,ds = O(e^{-u\xi}).$$

Lemma 2.11(ii) finally allows us to write

$$\int\limits_{\substack{1\le|\tau|\le e^{\xi}\\ s=\xi+i\tau}} re^{-I_r(s)-us}\,ds \ll \exp\{I_r(\xi)-u\xi+\xi\}H_r(u)^{-c_1},$$

which completes the proof of Proposition 2.14.

Corollary 2.15. *There exists a positive constant c such that, under the condition (G_c), we have*

$$\eta_r'(u) \ll \sigma_r(u)H_r(u)^{-c_1}.$$

Proof. If $u \le u_0$, the result follows from [15, Prop. 5.3(ii)]. Let $u > u_0$ for some sufficiently large constant u_0. By Lemma 2.9 we have $\xi_r'(u) \gg 1/u$ and $e^{\xi_r(u)} \ll u(\log(2u)+(1-r)^{-1})$. Thus, in the domain (G_c), we have

$$(\xi_r'(u))^{-1/2}e^{\xi_r(u)}H_r(u)^{-c_0} \ll H_r(u)^{-c_1}.$$

The result now follows from Lemma 2.6 and Proposition 2.14.

We now turn our attention to $\eta_r(u)$. We start with the following lemma.

Lemma 2.16. *Let $u \ge 2$. Then*

$$J := \int\limits_{\substack{\sigma=-\xi\\ |\tau|>e^{\xi}}} \widehat{\eta}_r(s)e^{us}\,ds \ll \frac{e^{-u\xi}}{u}\min(1, re^{\xi}).$$

Proof. Integration by parts shows that

$$J = \frac{e^{us}}{u}\cdot\frac{r}{s}\,e^{-I_r(-s)}\Big|_{|\tau|=e^{\xi}}^{|\tau|=\infty} + \int\limits_{\substack{\sigma=-\xi\\ |\tau|\ge e^{\xi}}} \frac{e^{us}}{u}\cdot\frac{r}{s}\,e^{-I_r(-s)}\left(\frac{1}{s}-\frac{e^{-s}-e^{-rs}}{s}\right)ds. \tag{15}$$

Lemma 2.11(iii) and (iv) shows that the first term in (15) is $\ll re^{-u\xi}/u$. Let J_1 denote the contribution to the integral in (15) from the domain $e^{\xi} \le |\tau| \le \max(e^{\xi}, 1/r)$ and J_2 the contribution from $|\tau| > \max(e^{\xi}, 1/r)$. To bound J_1, we may assume that $e^\xi < 1/r$, otherwise $J_1 = 0$. Lemma 2.11(iii) shows that

$$J_1 = \int\limits_{\substack{\sigma=-\xi \\ e^\xi \le |\tau| \le 1/r}} \frac{e^{us}}{u} \cdot \frac{r}{s} s e^\gamma \left\{1 + O\left(\frac{e^\xi}{\tau} + r\tau\right)\right\}\left(\frac{1}{s} - \frac{e^{-s} - e^{-rs}}{s}\right) ds$$
$$\ll r\,\frac{e^{-u\xi}}{u} \cdot \frac{1}{u-1} + r\,\frac{e^{-u\xi}}{u}\, e^\xi \ll \frac{e^{-u\xi}}{u} \min(1, re^\xi),$$

since $e^\xi < 1/r$. Lemma 2.11(iv) allows us to write

$$J_2 = \int\limits_{\substack{\sigma=-\xi \\ |\tau| \ge \max(e^\xi, 1/r)}} \frac{e^{us}}{u} \cdot \frac{r}{s} O\left(\frac{1}{r}\right)\left(\frac{1}{s} - \frac{e^{-s} - e^{-rs}}{s}\right) ds$$
$$\ll \frac{e^{-u\xi}}{u} \cdot \frac{e^\xi}{\max(e^\xi, 1/r)} = \frac{e^{-u\xi}}{u} \min(1, re^\xi).$$

This completes the proof of Lemma 2.16.

PROPOSITION 2.17. *Let $u \ge 2$. Then*

$$|\eta_r(u) - r| \ll r\xi e^{I_r(\xi) - u\xi} H_r(u)^{-c_1}.$$

Proof. Let $s = \sigma + i\tau$. Then the inverse Laplace transform

$$\frac{1}{2\pi i} \int\limits_{\sigma - i\infty}^{\sigma + i\infty} \widehat{\eta}_r(s) e^{us}\, ds$$

converges to $\eta_r(u)$ for $\sigma > 0$, according to Widder [16, II.7.3]. Now $\widehat{\eta}_r(s) = rs^{-1}\exp\{-I_r(-s)\}$ has a simple pole at $s = 0$ with residue r. Also, Lemma 2.11(iv) shows that $\widehat{\eta}_r(s) \ll 1/s$ for $|\tau| > \max(e^\xi, 1/r)$. This allows us to move the abscissa of integration to $\sigma = -\xi$ to obtain

$$\eta_r(s) - r = \frac{1}{2\pi i} \int\limits_{-\xi - i\infty}^{-\xi + i\infty} \widehat{\eta}_r(s) e^{us}\, ds.$$

We have

$$\int\limits_{\substack{\sigma=-\xi \\ |\tau| \le 1}} \frac{r}{s} e^{-I_r(-s) + us}\, ds \ll re^{-u\xi},$$

since $\exp\{-I_r(-s)\} \ll 1$ for $|\tau| \le 1$, by Lemmas 2.10 and 2.11(i). Also

$$\int\limits_{\substack{\sigma=-\xi \\ 1 < |\tau| \le e^\xi}} \frac{r}{s} e^{-I_r(-s) + us}\, ds \ll r\xi \exp\{I_r(\xi) - u\xi\} H_r(u)^{-c_1}.$$

By Lemma 2.16, we have

$$\int_{\substack{\sigma=-\xi\\|\tau|>e^{\xi}}} \widehat{\eta}_r(s)e^{us}\,ds \ll \frac{e^{-u\xi}}{u}\min(1, re^{\xi}).$$

Thus it suffices to show that $u^{-1}\min(1, re^{\xi}) \ll r\xi$. If $r \le 1/2$ note that $\xi_r(u) = \log u + \log_2 u + O(1)$. Thus $re^{\xi}/u \ll r\xi$. If $r > 1/2$ we have $1/u \ll \xi/2 < r\xi$. This completes the proof of Proposition 2.17.

Corollary 2.18. *There exists a positive constant c such that, under the conditions (G_c) and $u \ge 2$, we have*

$$\eta_r(u) - r \ll r\sigma_r(u)H_r(u)^{-c_1}.$$

The proof is analogous to the proof of Corollary 2.15.

3. Proof of Theorem 1.1 and its corollary. To prove Theorem 1.1 we will establish the following propositions.

Proposition 3.1. *Let $\varepsilon > 0$. Under the conditions $(H_\varepsilon(x,y))$ and (V_ε), we have*

$$\Gamma(x,y,z) - W(x,y,z) \ll_\varepsilon \frac{xe^{-u\xi}}{L_\varepsilon(z)}\biggl(\frac{e^{I_r(\xi)}H_r(u)^{-c_2}}{\log y} + 1\biggr).$$

Proposition 3.2. *There exists a constant c such that under the conditions (G_c) and $u \ge \log y$ we have*

$$\Gamma(x,y,z) - x\prod_{z<p\le y}\biggl(1 - \frac{1}{p}\biggr) \ll \Theta(x,y,z).$$

Proof of Theorem 1.1. We first assume that $(x,y,z) \in (H_\varepsilon(x,y))$. If $y \le y_0(\varepsilon)$, then $(H_\varepsilon(x,y))$ implies that $x \le x_0(\varepsilon)$ and the result is trivial. Thus we may assume that $y > y_0(\varepsilon)$ for some sufficiently large constant $y_0(\varepsilon)$. Theorems 1 and 4 of [10] show that, under conditions $(H_\varepsilon(x,y))$ and (G_c), we have

$$\Theta(x,y,z) \asymp x\frac{e^{-u\xi+I_r(\xi)}}{\sqrt{u}\log z}.$$

For a suitable constant c, condition (G_c) implies that $c_2u(1-r)^2 \ge \log u$. Also note that conditions $(H_\varepsilon(x,y))$ and (G_c) imply $(V_{\varepsilon/5})$ for c large enough. Applying Proposition 3.1 with $\varepsilon/5$ in place of ε yields the result.

If $(x,y,z) \notin (H_\varepsilon(x,y))$, the result follows directly from Proposition 3.2.

Lemma 3.3. *There exists a constant c such that for all $\varepsilon > 0$, under the conditions (G_c) and $(H_\varepsilon(x,y))$, we have*

$$\mu_{y,z}(u) = \eta_r(u) + O\biggl(\frac{\sigma_r(u)H_r(u)^{-c_2}}{\log y}\biggr).$$

Proof. Using integration by parts, we can write

$$\mu_{y,z}(u) = \eta_r(u) - \frac{\{x\}}{x} - \int_0^u \eta_r'(u-t)\,\frac{\{y^t\}}{y^t}\,dt.$$

We need to estimate

$$\int_0^u \eta_r'(u-t)\,\frac{dt}{y^t} = \frac{1}{x}\int_0^u \eta_r'(t)y^t\,dt.$$

Since $\eta_r'(u) \ll 1$, we have

$$\frac{1}{x}\int_0^{u/2} \eta_r'(t)y^t\,dt \ll \frac{u\sqrt{x}}{x} \ll x^{-1/3}.$$

This is clearly acceptable since the conditions (G_c) and $(H_\varepsilon(x,y))$ imply that

$$\frac{\sigma_r(u)H_r(u)^{-c_2}}{\log y} \gg \frac{u^{-2u}}{\log y} \gg x^{-1/3},$$

according to (5).

Note that if (u,v) is in (G_c) then $(u/2, v/2)$ is in $(G_{c/2})$. By Corollary 2.15 we have

$$\int_{u/2}^u \eta_r'(t)y^t\,dt \ll \int_{u/2}^u H_r(t)^{-c_1}\sigma_r(u)y^t\,dt \ll H_r(u)^{-c_2}\int_0^u \sigma_r(u)y^t\,dt.$$

Saias [10, p. 297] showed that, under the conditions (G_c) and $y \ge (\log x)^2$, we have

$$\int_0^u \sigma_r(u)y^t\,dt \asymp \frac{x\sigma_r(u)}{\log y}.$$

This completes the proof of the lemma.

Proof of Corollary 1.2. (i) By Lemma 3.3 and Mertens' formula we have

$$W(x,y,z) = L(x,y,z) + O\left(\frac{x\sigma_r(u)H_r(u)^{-c_1}}{\log y}\right).$$

The result now follows from Theorem 1.1 since, under the conditions (G_c) and $(H_\varepsilon(x,y))$, we have

$$\varTheta(x,y,z) \asymp \frac{x\sigma_r(u)}{\log z},$$

according to Saias [10, Theorem 2].

Part (ii) follows directly from (i) since

$$L(x,y,z) = x\eta(u,v)(1 + O_\varepsilon(L_\varepsilon(z)^{-1})),$$

due to the strong form of Mertens' formula (3).

(iii) If $1 \le u < 2$, the result follows since $\Gamma(x,y,z) \ll xr$ according to [15, Theorem 3.1]. If $u \ge 2$ we combine Lemma 3.3 and Corollary 2.18 to get

$$W(x,y,z) = x \prod_{z<p\le y} \left(1 - \frac{1}{p}\right) + O(xr\sigma_r(u)H_r(u)^{-c_1}).$$

The result now follows from Theorem 1.1.

4. Proof of Proposition 3.1. Before we are able to evaluate the integral in (6) we need several auxiliary results. In the following, we write (H_ε) to mean $(H_\varepsilon(x,y))$.

LEMMA 4.1. *Let $\varepsilon > 0$ and write $s = \sigma + i\tau$ with $\sigma, \tau \in \mathbb{R}$. There exists a constant $t_0 = t_0(\varepsilon)$ such that, under the conditions*

$$t \ge t_0(\varepsilon), \qquad 2 \ge \sigma \ge 1 - (\log t)^{-2/5-\varepsilon}, \qquad |\tau| \le L_\varepsilon(t),$$

we have

$$\text{(i)} \qquad \frac{\zeta'(s,t)}{\zeta(s,t)} = \frac{\zeta'(s)}{\zeta(s)} + \frac{t^{1-s}}{s-1} + O_\varepsilon\left(\frac{1}{L_\varepsilon(t)}\right),$$

$$\text{(ii)} \qquad \zeta(s,t) = \zeta(1,t)(s-1)\zeta(s)\exp\{I_0((1-s)\log t)\}\left(1 + O_\varepsilon\left(\frac{s-1}{L_\varepsilon(t)}\right)\right).$$

Proof. The estimate in (i) is equivalent to the estimate in [12, (67) in Chapter III.5]. We obtain (ii) by integrating both sides of the equation in (i) on the straight line between 1 and s and subsequent exponentiation.

LEMMA 4.2. *Let $\varepsilon > 0$. There exists a $y_0 = y_0(\varepsilon)$ such that under the conditions*

$$y \ge y_0(\varepsilon), \qquad 2 \ge \sigma \ge 1 - (\log y)^{-2/5-\varepsilon}, \qquad |\tau| \le L_\varepsilon(z),$$

we have

$$\frac{\zeta(s,z)}{\zeta(s,y)} = \frac{\zeta(1,z)}{\zeta(1,y)} \exp\{-I_r((1-s)\log y)\}\left(1 + O_\varepsilon\left(\frac{1}{L_\varepsilon(z)}\right)\right).$$

Proof. We have

$$\left(\frac{\zeta(s,z)}{\zeta(s,y)}\right)' \left(\frac{\zeta(s,z)}{\zeta(s,y)}\right)^{-1} = \frac{\zeta'(s,z)}{\zeta(s,z)} - \frac{\zeta'(s,y)}{\zeta(s,y)}.$$

We apply Lemma 4.1 with $t = y$ and $\varepsilon/3$ in place of ε to obtain, under the given conditions,

$$\left(\frac{\zeta(s,z)}{\zeta(s,y)}\right)' \frac{\zeta(s,y)}{\zeta(s,z)} = \frac{z^{1-s} - y^{1-s}}{s-1} - \frac{\zeta'(s)}{s} + \frac{\zeta'(s,z)}{\zeta(s,z)} - \frac{z^{1-s}}{s-1} + O_\varepsilon\left(\frac{1}{L_{\varepsilon/2}(y)}\right).$$

We integrate both sides of the last equation along the straight line between 1 and s and subsequently exponentiate both sides of the equation. This gives

$$\frac{\zeta(s,z)}{\zeta(s,y)} = \frac{\zeta(1,z)}{\zeta(1,y)} \exp\{-I_r((1-s)\log y)\}(1+G(s,z))\left(1+O_\varepsilon\left(\frac{1}{L_\varepsilon(y)}\right)\right),$$

with

$$G(s,z) := \exp\left\{\int_1^s \left(\frac{\zeta'(w,z)}{\zeta(w,z)} - \frac{\zeta'(w)}{w} - \frac{z^{1-w}}{w-1}\right)dw\right\} - 1.$$

If $z \geq z_0(\varepsilon)$, we integrate the formula in Lemma 4.1(i) between 1 and s to obtain

$$G(s,z) \ll L_\varepsilon(z)/L_{\varepsilon/2}(z) \ll 1/L_\varepsilon(z).$$

If $z \leq z_0(\varepsilon)$, we have, under the given conditions,

$$\frac{\zeta'(w,z)}{\zeta(w,z)} - \frac{\zeta'(w)}{w} - \frac{z^{1-w}}{w-1} \ll 1,$$

uniformly for w on the straight line between 1 and s. Thus $G(s,z) \ll (s-1) \ll_\varepsilon 1$ in this case. This concludes the proof of Lemma 4.2.

LEMMA 4.3. *Let $\varepsilon > 0$. There exists a $y_0 = y_0(\varepsilon)$ such that under the conditions*

$$y \geq y_0(\varepsilon), \qquad \sigma \geq 1-(\log y)^{-2/5-\varepsilon}, \qquad \frac{y^{1-\sigma}}{\log y} \leq |\tau| \leq L_\varepsilon(y),$$

we have

$$\zeta(s)\,\frac{\zeta(s,z)}{\zeta(s,y)} \ll \log z \cdot e^{I_0((1-\sigma)\log z)}.$$

Proof. We have

$$|\zeta(s,z)| \leq \zeta(\sigma,z) \ll \log z \cdot e^{I_0((1-\sigma)\log z)},$$

where the last estimate is from Lemma 4.1 and Mertens' formula. Lemma 4.1 allows us to write, under the given conditions,

$$\frac{\zeta(s)}{\zeta(s,y)} \ll \frac{e^{-I_0((1-s)\log y)}}{(s-1)\log y}. \tag{16}$$

We have

$$\frac{e^{(1-\sigma)\log y}}{|\tau|\log y} \leq 1.$$

Thus, Lemma 2.8 shows that the right hand side of (16) is $\ll 1$, which concludes the proof of the lemma.

LEMMA 4.4. *Let $\varepsilon > 0$, $r \leq 1/2$ and let the conditions (H_ε) and (V_ε) be satisfied. Then, for $s = \beta + i\tau$ and $\xi/\log y \leq |\tau| \leq \exp\{(\log y)^{3/2-\varepsilon}\}$, we have*

$$\frac{\zeta(s,z)}{\zeta(s,y)} \ll \frac{\zeta(\beta,y)}{\zeta(\beta,z)} \cdot e^{-cu}.$$

Proof. An easy calculation shows that

$$|(1-p^{-s})(1-p^{-\beta})| = (1-p^{-2\beta})\left(1-\frac{2(1+\cos(\tau\log p))}{p^\beta(1+p^{-\beta})^2}\right)^{1/2} \le \exp\left\{-\frac{1+\cos(\tau\log p)}{4p^\beta}\right\}.$$

This implies that

$$\left|\frac{\zeta(s,z)}{\zeta(s,y)}\cdot\frac{\zeta(\beta,z)}{\zeta(\beta,y)}\right| \le e^{-V}$$

with

$$V := \sum_{z<p\le y}\frac{1+\cos(\tau\log p)}{p^\beta}.$$

We need to show that $V \gg u$. The result holds trivially when $u < u_0$, since $V \ge 0$. Thus we may assume that $u \ge u_0$. We have

$$V + 2S \ge \frac{1}{\log y}\sum_{z<n\le y}\Lambda(n)(1+\cos(\tau\log n))n^{-\beta}, \tag{17}$$

where Λ denotes von Mangoldt's function and

$$S := \frac{1}{\log y}\sum_{\nu\ge 2}\sum_{p^\nu\le y} p^{-\nu\beta}\log p \ll 1,$$

since $\beta \ge 1-(\log y)^{-2/5-\varepsilon}$. We have

$$\begin{aligned}\frac{1}{\log y}\sum_{n\le z}&\Lambda(n)(1+\cos(\tau\log n))n^{-\beta} \\ &\le \frac{2}{\log y}\sum_{n\le z}\Lambda(n)n^{-\beta} \le \frac{3z^{1-\beta}}{(1-\beta)\log y}+O(1) = \frac{3e^{r\xi}}{\xi}+O(1).\end{aligned} \tag{18}$$

Lemma 6 of [6] shows that, for $\varepsilon < 1$ and $|\tau| \le \exp\{(\log y)^{3/2-\varepsilon}\}$,

$$\sum_{n\le y}\Lambda(n)n^{-s} = \frac{y^{1-\beta-i\tau}}{1-\beta-i\tau} + O_\varepsilon\left(\frac{1}{1-\beta}\left(1+\frac{y^{1-\beta}}{e^{(\log y)^{\varepsilon/2}}}\right)\right).$$

Since $(1+\cos(\tau\log n))n^{-\beta} = \operatorname{Re}(n^{-\beta}+n^{-\beta-i\tau})$, this gives

$$\begin{aligned}\frac{1}{\log y}&\sum_{n\le y}\Lambda(n)(1+\cos(\tau\log n))n^{-\beta} \\ &= \frac{1}{\log y}\operatorname{Re}\left\{\frac{y^{1-\beta}}{1-\beta}+\frac{y^{1-\beta-i\tau}}{1-\beta-i\tau}\right\} \\ &\quad + O_\varepsilon\left(\frac{1}{(1-\beta)\log y}\left(1+\frac{y^{1-\beta}}{\exp\{(\log y)^{\varepsilon/2}\}}\right)\right) \\ &= \frac{e^\xi}{\xi}\left(1+\frac{\cos\delta}{\sqrt{1+(\tau(\log y)/\xi)^2}}\right) + O_\varepsilon\left(\frac{1+\exp\{\xi-(\log y)^{\varepsilon/2}\}}{\xi}\right),\end{aligned}$$

with $\delta = -\tau \log y - \arctan(-\tau/(1-\beta))$. Together with (17) and (18) this shows that

$$\begin{aligned} V &\geq \frac{e^\xi}{\xi}\biggl(1 - \frac{1}{\sqrt{1+(\tau(\log y)/\xi)^2}}\biggr) - \frac{3e^{r\xi}}{\xi} + O_\varepsilon\biggl(1 + \frac{1}{\xi}\exp\{\xi - (\log y)^{\varepsilon/2}\}\biggr) \\ &\gg e^\xi/\xi \gg u, \end{aligned}$$

for $|\tau| \geq \xi/\log y$. This completes the proof of the lemma.

Lemma 4.5. *Let $\varepsilon > 0$. Under the conditions (H_ε) and (V_ε) we have, for $1 \leq |\tau| \leq L_\varepsilon(y)$,*

$$\zeta(\beta + i\tau) \ll (\log(|\tau|+1))^{2/3}.$$

Proof. Korobov [7] and Vinogradov [14] established the upper bound

$$\zeta(\sigma + i\tau) \ll (1 + \tau^{c(1-\sigma)^{3/2}})(\log \tau)^{2/3}$$

for $\sigma \geq 0$, $\tau \geq 2$ and for some positive constant c. According to Lemma 2.9, the conditions (H_ε) and (V_ε) imply that

$$\beta = 1 - \frac{\xi_r}{\log y} \geq 1 - (\log y)^{-2/5-\varepsilon}.$$

Hence

$$\zeta(\beta + i\tau) \ll \biggl\{1 + \exp\biggl(c\,\frac{\log|\tau|}{(\log y)^{3/5+3\varepsilon/2}}\biggr)\biggr\}(\log|\tau|)^{2/3}$$

for $|\tau| \geq 2$. This yields the desired result since

$$\log|\tau| \leq \log(L_\varepsilon(y)) = (\log y)^{3/5-\varepsilon}$$

and $\zeta(\beta + i\tau) \ll 1$ for $1 \leq |\tau| \leq 2$.

Lemma 4.6. *Let $\varepsilon > 0$. Under the conditions (H_ε) and (V_ε), we have*

$$\begin{aligned} E_3 &:= x\,\frac{\zeta(1,z)}{\zeta(1,y)} \int\limits_{\substack{\sigma=-\xi \\ |\tau|\geq L_{\varepsilon/4}(z)\log y}} \zeta\biggl(1 + \frac{s}{\log y}\biggr)\frac{e^{-I_r(-s)+us}}{s+\log y}\,ds \\ &\ll x\,\frac{e^{-u\xi}}{L_{\varepsilon/2}(z)}\biggl(\frac{e^{I_r(\xi)-c_2u/\log^2 u}}{\log y} + 1\biggr). \end{aligned}$$

Proof. Define $M := \max(L_{\varepsilon/4}(z)\log y, e^\xi)$. Let $E_{3,1}$ denote the contribution to E_3 from the domain $L_{\varepsilon/4}(z)\log y \leq |\tau| \leq M$ and let $E_{3,2}$ be the contribution from $|\tau| > M$. We first study $E_{3,2}$. Since $|\tau| > e^\xi$ and $\tau \gg 1/r$, Lemma 2.11(iv) yields

$$e^{-I_r(-s)} = \frac{1}{r}\biggl(1 + O\biggl(\frac{e^\xi}{\tau} + \frac{e^{r\xi}}{r\tau}\biggr)\biggr).$$

Thus, by Mertens' formula,

$$\begin{aligned}
E_{3,2} &\ll x \int\limits_{\substack{|\tau|>M\\ \sigma=-\xi}} \zeta\left(1+\frac{s}{\log y}\right)\frac{e^{us}}{s+\log y}\,ds \\
&\qquad + O\left(xe^{-u\xi}(e^{\xi}+1/r)\int\limits_{|\tau|>M}\frac{1}{\tau}\,\zeta\left(1+\frac{s}{\log y}\right)\frac{ds}{s+\log y}\right) \\
&= \int\limits_{\substack{\sigma=1-\xi/\log y\\ |\tau|>M/\log y}} \zeta(s)\,\frac{x^s}{s}\,ds + O\left(xe^{-u\xi}(e^{\xi}+1/r)\,\frac{1}{\sqrt{M\log y}}\right),
\end{aligned}$$

since $\zeta(1+s/\log y) \ll \sqrt{\frac{\tau}{\log y}}$. Now let $\beta = 1-\xi/\log y$ and let $T := M/\log y = \max(L_{\varepsilon/4}(z), e^{\xi}/\log y)$. In order to estimate the last integral, we approximate $\zeta(s)$ using Corollary II.3.5.1 of [12]. This gives

$$\begin{aligned}
\int\limits_{\substack{|\tau|>T\\ \sigma=\beta}} \zeta(s)\,\frac{x^s}{s}\,ds &= \int\limits_{\substack{|\tau|>T\\ \sigma=\beta}} \Big(\sum_{n\le|\tau|} n^{-s} + O(|\tau|^{-\sigma})\Big)\frac{x^s}{s}\,ds \\
&= \sum_{n=1}^{\infty} \int\limits_{\substack{|\tau|\ge\max(n,T)\\ \sigma=\beta}} \left(\frac{x}{n}\right)^{s}\frac{ds}{s} + O\left(\frac{x^{\beta}}{T^{\beta}}\right).
\end{aligned}$$

The last term is clearly acceptable. The estimate (7) in Chapter II.2 of [12] allows us to estimate the sum over n by

$$\ll \sum_{n=1}^{\infty}\left(\frac{x}{n}\right)^{\beta}\frac{1}{1+(n+T)|\log(x/n)|}.$$

The contribution from positive integers n with $|x-n| \le x^{3/4}$ is clearly $\ll x^{3/4}$. If $|x-n| > x^{3/4}$, one easily verifies that $n^{\beta}|\log(x/n)| \gg \sqrt{n}$, since $\beta > 3/4$ for $y > y_0$. Hence the last sum is

$$\ll \sum_{n=1}^{\infty}\frac{x^{\beta}}{n^{3/2}+T} + x^{3/4} \ll \frac{x^{\beta}}{T^{1/3}} + x^{3/4} \ll \frac{x^{\beta}}{T^{1/3}} \ll x\,\frac{e^{-u\xi}}{L_{\varepsilon/2}(z)},$$

since $T \ll x^{\varepsilon}$. Thus $E_{3,2}$ is acceptable.

We now turn to $E_{3,1}$. We may assume that $e^{\xi} > L_{\varepsilon/4}(z)\log y$, otherwise we have $E_{3,1} = 0$. This implies that $r \le 1/2$ and $\xi \ll \log u$. It follows from usual estimates for the zeta function that

$$\zeta\left(1+\frac{s}{\log y}\right)\frac{1}{s+\log y} \ll |s|^{-1/2}.$$

Thus, Lemma 2.11(ii) allows us to write

$$
\begin{aligned}
E_{3,1} &\ll xr \int\limits_{\substack{\sigma=-\xi \\ L_{\varepsilon/4}(z)\log y\le|\tau|\le e^{\xi}}} \zeta\Big(1+\frac{s}{\log y}\Big)\frac{e^{-I_r(-s)+us}}{s+\log y}\,ds \\
&\ll xr \int\limits_{\substack{\sigma=-\xi \\ L_{\varepsilon/4}(z)\log y\le|\tau|\le e^{\xi}}} \frac{\exp\{I_r(\xi)-u\xi\}H_r(u)^{-c_1}}{|s|^{1/2}}\,ds \\
&\ll xr\exp\{\xi-u\xi+I_r(\xi)-c_1u/\log^2(2u)\} \ll x\,\frac{e^{-u\xi+I_r(\xi)-c_2u/\log^2(2u)}}{L_{\varepsilon/2}(z)\log y},
\end{aligned}
$$

which is clearly acceptable. This completes the proof of Lemma 4.6.

LEMMA 4.7. *Let $\varepsilon>0$. Under the conditions (H_ε) and (V_ε) we have*

$$
\begin{aligned}
E_2 &:= \frac{x}{L_{\varepsilon/4}(z)}\cdot\frac{\zeta(1,z)}{\zeta(1,y)}\int_{-\xi-iL_{\varepsilon/4}(z)\log y}^{-\xi+iL_{\varepsilon/4}(z)\log y}\bigg|\zeta\Big(1+\frac{s}{\log y}\Big)\frac{e^{-I_r(-s)+us}}{s+\log y}\bigg|\,d|s| \\
&\ll x\,\frac{e^{-u\xi}}{L_{\varepsilon/2}(z)}\bigg(\frac{\exp\{I_r(\xi)\}H_r(u)^{-c_2}}{\log y}+1\bigg).
\end{aligned}
$$

Proof. Let $L:=L_{\varepsilon/4}(z)$ and let

$$
J:=E_2\,\frac{Le^{u\xi}}{rx}\ll\int_{-\xi-iL\log y}^{-\xi+iL\log y}\bigg|\zeta\Big(1+\frac{s}{\log y}\Big)\frac{e^{-I_r(-s)}}{s+\log y}\bigg|\,d|s|,
$$

by Mertens' formula. We divide the interval $[0,L\log y)$ into four subintervals determined by the endpoints

$$
0,\quad 1,\quad \min(L\log y,e^{\xi}),\quad \max(\min(L\log y,e^{\xi}),1/r),\quad L\log y.
$$

Let J_1, J_2, J_3 and J_4 denote the corresponding contributions to J.

To bound J_1, note that it follows from Lemmas 2.10 and 2.11(i) that $\exp\{-I_r(s)\}\ll1$ for $|\tau|\le1$. Also, $\zeta(t)$ has a simple pole at $t=1$. Finally, $|s+\log y|\gg\log y$ under the conditions (H_ε) and (V_ε). Thus

$$
J_1\ll1.
$$

For J_2 we make use of Lemma 2.11(ii) to obtain

$$
J_2\ll e^{I_r(\xi)}H_r(u)^{-c_1}\int\limits_{1\le|\tau|\le\min(L\log y,e^{\xi})}\bigg|\frac{\zeta(1+s/\log y)}{s+\log y}\bigg|\,d|s|. \tag{19}
$$

Due to the pole of $\zeta(t)$ at $t=1$ we subdivide the domain of the last integral again. Let $J_{2,1}$ and $J_{2,2}$ be the contributions to the integral in (19) from the domains $1\le|\tau|\le\min(\log y,e^{\xi})$ and $\min(\log y,e^{\xi})\le|\tau|\le\min(L\log y,e^{\xi})$, respectively. Then

$$J_{2,1} \ll \int_{1 \le |\tau| \le \min(\log y, e^{\xi})} \frac{\log y}{\tau} \frac{d\tau}{\log y} = \min(\log\log y, \xi).$$

If $e^{\xi} \le \log y$, then $J_{2,2} = 0$. If $e^{\xi} > \log y$, Lemma 4.5 implies that

$$\begin{aligned} J_{2,2} &\ll \int_{\log y \le |\tau| \le \min(L\log y, e^{\xi})} \frac{(\log(|\tau|/\log y + 1))^{2/3}}{|\tau|} \, d\tau \\ &= \int_{1 \le |\tau| \le \min(L, e^{\xi}/\log y)} \frac{(\log(|\tau| + 1))^{2/3}}{|\tau|} \, d\tau \ll (\log L)^{5/3} < \log z. \end{aligned}$$

If $\log\log y < \log z$, we have $J_{2,1} \ll \log z$. Otherwise $J_{2,1} \ll \xi \ll \log u$. Hence

$$J_2 \ll (\log z) e^{I_r(\xi)} H_r(u)^{-c_2}.$$

To bound J_3, we may assume that $e^{\xi} \le L \log y$ and $e^{\xi} < 1/r$, otherwise $J_3 = 0$. This means that $|\tau|/\log y \le 1/\log z$ and hence $\zeta(1 + s/\log y) \ll (\log y)/s$. By Lemma 2.11(iii), we have

$$J_3 \ll \int_{e^{\xi} \le |\tau| \le 1/r} \left| \frac{\zeta(1 + s/\log y)}{s + \log y} \right| |s| \, d|s| \ll \int_{e^{\xi} \le |\tau| \le 1/r} 1 \, d\tau \ll 1/r.$$

To bound J_4, we may assume that $e^{\xi} < L \log y$, otherwise $J_4 = 0$. By Lemma 2.11(iv), we have

$$J_4 \ll \int_{\max(e^{\xi}, 1/r) \le |\tau| \le L \log y} \left| \frac{\zeta(1 + s/\log y)}{s + \log y} \right| \frac{1}{r} \, d\tau. \tag{20}$$

Due to the pole of $\zeta(t)$ at $t = 1$, we subdivide the domain of the last integral. Let $J_{4,1}$ and $J_{4,2}$ denote the contributions to the integral in (20) from the domains $\max(e^{\xi}, 1/r) \le |\tau| \le \max(\log y, e^{\xi}, 1/r)$ and $\max(\log y, e^{\xi}, 1/r) \le |\tau| \le L \log y$, respectively. If $\log y \le e^{\xi}$, then $J_{4,1} = 0$. Otherwise, we have

$$J_{4,1} \ll \frac{1}{r} \int_{\max(e^{\xi}, 1/r) \le |\tau| \le \log y} \frac{1}{|\tau|} \, d\tau \ll \frac{1}{r} (\log\log y - \log(1/r)) = \frac{1}{r} \log\log z.$$

Lemma 4.5 shows that

$$\begin{aligned} J_{4,2} &\ll \frac{1}{r} \int_{\max(\log y, e^{\xi}, 1/r) \le |\tau| \le L \log y} \frac{(\log(1 + |\tau|/\log y))^{2/3}}{|\tau|} \, d\tau \\ &= \frac{1}{r} \int_{\max(1, e^{\xi}/\log y, 1/\log z) \le |\tau| \le L} \frac{(\log(1 + |\tau|))^{2/3}}{|\tau|} \, d\tau \\ &\ll \frac{1}{r} (\log L)^{5/3} < \frac{\log z}{r}. \end{aligned}$$

Therefore, we have

$$J_4 \ll \frac{\log z}{r}.$$

Since $J_1 + J_3 \ll J_4$, we have $J \ll J_2 + J_4$ and hence

$$E_2 \ll xre^{-u\xi}(J_2 + J_4)/L_{\varepsilon/4}(z).$$

Thus,

$$E_2 \ll x\,\frac{e^{-u\xi}}{L_{\varepsilon/2}(z)}\biggl(\frac{e^{I_r(\xi)}H_r(u)^{-c_2}}{\log y} + 1\biggr),$$

which concludes the proof of Lemma 4.7.

LEMMA 4.8. *Let $\varepsilon > 0$. Under the conditions (H_ε) and (V_ε) we have*

$$E_1 := \int\limits_{\substack{\sigma=1-\xi_r/\log y\\ |\tau|\ge L_{\varepsilon/4}(z)}} \zeta(s)\,\frac{\zeta(s,z)}{\zeta(s,y)}\cdot\frac{x^s}{s}\,ds$$

$$\ll \frac{x\exp\{-u\xi + I_r(\xi) - \min(cu, (\log y)^{3/5-\varepsilon})\}}{(\log y)L_\varepsilon(z)}.$$

Proof. We write $\beta = 1 - \xi/\log y$. Define

$$T = \max(L_{\varepsilon/4}(z), L_{\varepsilon/3}(z)\min(L_{\varepsilon/2}(y), \exp\{u + (\log\log y)^{1+\varepsilon/3}\})).$$

Let $E_{1,1}$ and $E_{1,2}$ denote the contributions to E_1 from the domains $|\tau| > T$ and $L_{\varepsilon/4}(z) \le |\tau| \le T$, respectively.

We begin by studying $E_{1,1}$. Corollary II.3.5.1 of [12] enables us to write, for $s = \beta + i\tau$,

$$\zeta(s) = \sum_{n\le|\tau|} n^{-s} + O(|\tau|^{-\beta}).$$

Since

$$\frac{\zeta(s,z)}{\zeta(s,y)} = \prod_{z<p\le y}(1 - p^{-s}) = \sum_{\substack{P^-(m)>z\\ P^+(m)\le y}} \frac{\mu(m)}{m^s},$$

we have

$$\zeta(s)\,\frac{\zeta(s,z)}{\zeta(s,y)} = \sum_{n\le|\tau|}\sum_{\substack{P^-(m)>z\\ P^+(m)\le y}} \frac{\mu(m)}{(mn)^s} + O\biggl(\frac{\zeta(s,z)}{\zeta(s,y)}\cdot\frac{1}{|\tau|^\beta}\biggr). \tag{21}$$

We clearly have

$$\biggl|\frac{\zeta(s,z)}{\zeta(s,y)}\biggr| = \biggl|\sum_{\substack{P^-(m)>z\\ P^+(m)\le y}} \frac{\mu(m)}{m^s}\biggr| \le \sum_{\substack{P^-(m)>z\\ P^+(m)\le y}} \frac{1}{m^\beta} = \frac{\zeta(\beta,y)}{\zeta(\beta,z)}.$$

Furthermore, Lemma 4 of [10] shows that, under the conditions (H_ε) and (V_ε),

$$\frac{\zeta(\beta,y)}{\zeta(\beta,z)} \ll \frac{e^{I_r(\xi)}}{r}.$$

Thus, the contribution from the error term in (21) to $E_{1,1}$ is

$$\ll x^\beta \frac{\zeta(\beta,y)}{\zeta(\beta,z)} \int\limits_{|\tau|>T} \frac{d\tau}{|\tau|^{\beta+1}} \ll \frac{x^\beta}{T^\beta}\cdot\frac{e^{I_r(\xi)}}{r} = \frac{x\exp\{-u\xi+I_r(\xi)\}}{rT^\beta}. \tag{22}$$

The estimate (7) in Chapter II.2 of [12] enables us to write, uniformly for $z>0$,

$$\int\limits_{\substack{\sigma=\beta\\|\tau|>T}} \frac{z^s}{s}\,ds \ll \frac{z^\beta}{1+T|\log z|}.$$

The contribution to $E_{1,1}$ from the main term in (21) can therefore be estimated as follows:

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{\substack{P^-(m)>z\\P^+(m)\le y}} \mu(m) \int\limits_{\substack{\sigma=\beta\\|\tau|\ge\max(T,n)}} \left(\frac{x}{mn}\right)^s \frac{ds}{s} \\ \ll \sum_{n=1}^{\infty}\sum_{\substack{P^-(m)>z\\P^+(m)\le y}} \frac{(x/(mn))^\beta}{1+(T+n)|\log(x/(mn))|}.\end{aligned} \tag{23}$$

Note that the contribution to the right hand side of (23) from pairs (m,n) such that $|\log(x/(mn))|>1$ is

$$\ll x^\beta \frac{\zeta(\beta,y)}{\zeta(\beta,z)} \sum_{n=1}^{\infty} \frac{1}{n^\beta(T+n)} \ll x^\beta \frac{\zeta(\beta,y)}{\zeta(\beta,z)}\cdot\frac{1}{\sqrt{T}}, \tag{24}$$

since $\beta\ge 1/2$. For the remaining pairs (m,n) with $|\log(x/(mn))|\le 1$, let S_1, S_2 denote the respective contributions to (23) corresponding to the cases $m\le x/T$, $m>x/T$. We have

$$(T+n)\left|\log\left(\frac{x}{mn}\right)\right| \gg n\left|1-\frac{x}{mn}\right|,$$

which gives

$$S_1 \ll \sum_{\substack{m\le x/T\\P^-(m)>z\\P^+(m)\le y}} \sum_{x/(em)\le n\le xe/m} \frac{1}{1+|n-x/m|} \ll \sum_{\substack{m\le x/T\\P^-(m)>z\\P^+(m)\le y}} \log\left(\frac{ex}{m}\right). \tag{25}$$

Using partial summation, we have

$$(26)\qquad \sum_{\substack{m\le x/T\\ P^-(m)>z\\ P^+(m)\le y}} \log\left(\frac{ex}{m}\right) = \sum_{m\le x/T} \Theta(m,y,z)\left(\log\left(\frac{ex}{m}\right) - \log\left(\frac{ex}{m+1}\right)\right) + \Theta(x/T,y,z)\log(eT)$$

$$\ll \sum_{m\le x/T} \frac{\Theta(m,y,z)}{m} + \Theta(x/T,y,z)\log T.$$

To bound Θ in the last expression, we make use of Rankin's inequality for Θ:

$$(27)\qquad \Theta(x,y,z) = \sum_{\substack{1\le m\le x\\ P^-(m)>z\\ P^+(m)\le y}} 1 \le \sum_{\substack{m\ge 1\\ P^-(m)>z\\ P^+(m)\le y}} \left(\frac{x}{m}\right)^{\beta} = x^{\beta}\,\frac{\zeta(\beta,y)}{\zeta(\beta,z)}.$$

Thus, by (25)–(27), we have

$$(28)\qquad S_1 \ll \frac{\zeta(\beta,y)}{\zeta(\beta,z)}\left(\sum_{m\le x/T} m^{\beta-1} + \frac{x^{\beta}}{T^{\beta}}\log T\right) \ll \frac{x^{\beta}}{\sqrt{T}}\cdot\frac{\zeta(\beta,y)}{\zeta(\beta,z)},$$

since $\beta\ge 3/4$.

When (m,n) is counted in S_2, we have $m>x/T$ and $1/e\le x/(mn)\le e$, hence $n\le eT$. From this we obtain

$$S_2 \le \sum_{n\le eT}\ \sum_{\substack{x/(en)\le m\le ex/n\\ P^-(m)>z\\ P^+(m)\le y}} \frac{1}{1+T|x/(mn)-1|}$$

$$\ll \sum_{n\le eT}\left\{\frac{\Theta(ex/n,y,z)}{\sqrt{T}} + \sum_{\substack{|m-x/n|<x/(n\sqrt{T})\\ P^-(m)>z\\ P^+(m)\le y}} 1\right\}.$$

We use (27) (Rankin's inequality for Θ) to get

$$(29)\qquad S_{2,1} := \sum_{n\le eT}\frac{\Theta(ex/n,y,z)}{\sqrt{T}} \ll \sum_{n\le eT}\left(\frac{ex}{n}\right)^{\beta}\frac{\zeta(\beta,y)}{\zeta(\beta,z)}\cdot\frac{1}{\sqrt{T}}$$

$$\ll \frac{x^{\beta}T^{1-\beta}}{(1-\beta)\sqrt{T}}\cdot\frac{\zeta(\beta,y)}{\zeta(\beta,z)} \ll \frac{x^{\beta}}{T^{1/4}}\cdot\frac{\zeta(\beta,y)}{\zeta(\beta,z)}.$$

It remains to bound

$$S_{2,2} := \sum_{n\le eT} R_n,$$

with

$$R_n := \Theta\left(\frac{x}{n}+\frac{x}{n\sqrt{T}},y,z\right) - \Theta\left(\frac{x}{n}-\frac{x}{n\sqrt{T}},y,z\right).$$

If $r \le 1/2$ and

$$u < (\log\log y)^{1+\varepsilon/4} + (\log z)^{3/5-2\varepsilon/5},$$

we use the trivial bound $R_n \ll x/(n\sqrt{T})$ to obtain

$$S_{2,2} \ll \frac{x\log T}{\sqrt{T}} \ll x\,\frac{\exp\{-u\xi + I_r(\xi) - u/2\}}{(\log y)L_{\varepsilon/2}(z)} \tag{30}$$

according to Lemma 2.9. Thus, we may assume in the following that

$$u \ge (\log\log y)^{1+\varepsilon/4} + (\log z)^{3/5-2\varepsilon/5} \tag{31}$$

if $r \le 1/2$. From the proof of Lemma 7 in [10] we have

$$R_n \ll \frac{1}{\sqrt{T}}\Bigg|\int\limits_{\substack{\sigma=\beta\\ T^{1/4}\le|\tau|\le\sqrt{T}}} \frac{\zeta(s,y)}{\zeta(s,z)}\left(\frac{x}{n}\right)^s\left(1-\frac{|\tau|}{\sqrt{T}}\right)ds\Bigg| + \left(\frac{x}{n}\right)^\beta \frac{\zeta(\beta,y)}{\zeta(\beta,z)}\cdot\frac{1}{T^{1/4}}. \tag{32}$$

We have

$$\sum_{n\le eT}\left(\frac{x}{n}\right)^\beta \frac{\zeta(\beta,y)}{\zeta(\beta,z)}\cdot\frac{1}{T^{1/4}} \ll \frac{x^\beta T^{1-\beta}}{(1-\beta)T^{1/4}}\cdot\frac{\zeta(\beta,y)}{\zeta(\beta,z)} \ll \frac{x^\beta}{T^{1/5}}\cdot\frac{\zeta(\beta,y)}{\zeta(\beta,z)}. \tag{33}$$

To bound the first term in (32), we consider two cases, $r \ge 1/2$ and $r < 1/2$.

If $r \ge 1/2$, then $T = L_{\varepsilon/4}(z)$. Lemma 6(i) of [10] states that, under conditions (H_ε), (V_ε), $L_\varepsilon(z) \le |\tau| \le L_{\varepsilon/3}(y)$, $r \ge 1/2$ and $y \ge y_0(\varepsilon)$,

$$\frac{\zeta(s,y)}{\zeta(s,z)} = 1 + O_\varepsilon\left(\frac{e^\xi}{|\tau\log y|} + \frac{1}{L_\varepsilon(z)}\right).$$

Applying this result with $\varepsilon/3$ instead of ε shows that, for $r \ge 1/2$, the first term on the right hand side of (32) is

$$\ll_\varepsilon \left(\frac{x}{n}\right)^\beta\left(\frac{1}{\sqrt{T}\log(x/n)} + \frac{\log T}{\sqrt{T}\log y} + \frac{1}{L_{\varepsilon/3}(z)}\right) \ll \left(\frac{x}{n}\right)^\beta \frac{1}{L_{\varepsilon/3}(z)}.$$

Thus,

$$\begin{aligned}\sum_{n\le eT}\frac{1}{\sqrt{T}}\Bigg|\int\limits_{\substack{\sigma=\beta\\ T^{1/4}\le|\tau|\le\sqrt{T}}} \frac{\zeta(s,y)}{\zeta(s,z)}\left(\frac{x}{n}\right)^s\left(1-\frac{|\tau|}{\sqrt{T}}\right)ds\Bigg| &\ll \sum_{n\le eT}\left(\frac{x}{n}\right)^\beta\frac{1}{L_{\varepsilon/3}(z)} \ll \frac{x^\beta}{L_{\varepsilon/3}(z)}\cdot\frac{T^{1-\beta}}{1-\beta}\\ &\ll \frac{x^\beta}{L_{\varepsilon/2}(z)} \ll \frac{x\exp\{-u\xi + I_r(\xi) - (\log y)^{3/5-\varepsilon}\}}{(\log y)L_\varepsilon(z)}.\end{aligned} \tag{34}$$

In the last estimate we made use of the fact that $L_{\varepsilon/2}(z) \gg L_\varepsilon(z)L_{3\varepsilon/4}(y)$, since $r \ge 1/2$.

If $r < 1/2$ we make use of estimate (5.3) of [10]: under the conditions (H_ε), (V_ε), $L_\varepsilon(z) \le |\tau| \le L_{\varepsilon/3}(y)$, $r \le 1/2$ and $y \ge y_0(\varepsilon)$, we have

$$\frac{\zeta(s,y)}{\zeta(s,z)} \ll_\varepsilon \frac{\zeta(\beta,y)}{\zeta(\beta,z)}\, e^{-cu},$$

for some positive constant c. This shows that, for $r < 1/2$, the first term on the right hand side of (32) is

$$\ll \left(\frac{x}{n}\right)^\beta \frac{\zeta(\beta,y)}{\zeta(\beta,z)}\, e^{-cu}.$$

Since $\xi \ll \log u$ for $r < 1/2$, this implies that

$$\begin{aligned}(35)\qquad \sum_{n\le eT} \frac{1}{\sqrt{T}}\Bigg| &\int\limits_{\substack{\sigma=\beta\\ T^{1/4}\le|\tau|\le\sqrt{T}}} \frac{\zeta(s,y)}{\zeta(s,z)}\left(\frac{x}{n}\right)^s\left(1-\frac{|\tau|}{\sqrt{T}}\right)ds\Bigg| \\ &\ll x^\beta\,\frac{T^{1-\beta}}{1-\beta}\cdot\frac{\zeta(\beta,y)}{\zeta(\beta,z)}\,e^{-cu} \\ &\ll x^\beta(\log y)e^\xi\,\frac{\zeta(\beta,y)}{\zeta(\beta,z)}\,e^{-cu} \ll \frac{x^\beta e^{-cu/2}}{(\log y)^2 L_{\varepsilon/2}(z)}\cdot\frac{\zeta(\beta,y)}{\zeta(\beta,z)}.\end{aligned}$$

The last estimate follows from (31).

Now Lemma 4 of [10] shows that, under the conditions (H_ε) and (V_ε),

$$(36)\qquad \frac{\zeta(\beta,y)}{\zeta(\beta,z)} \ll \frac{e^{I_r(\xi)}}{r}.$$

Thus, (30) and (32)–(35) show that

$$S_{2,2} = \sum_{n\le eT} R_n \ll \frac{x\exp\{-u\xi + I_r(\xi) - \min(cu, (\log y)^{3/5-\varepsilon})\}}{(\log y)L_\varepsilon(z)}.$$

Finally, (29), (28), (24) and (22) allow us to conclude that

$$E_{1,1} \ll \frac{x\exp\{-u\xi + I_r(\xi) - \min(cu, (\log y)^{3/5-\varepsilon})\}}{(\log y)L_\varepsilon(z)},$$

for some positive constant c.

To conclude the proof of Lemma 4.8, we need to bound

$$E_{1,2} = \int\limits_{\substack{\sigma=\beta\\ L_{\varepsilon/4}(z)\le|\tau|\le T}} \zeta(s)\,\frac{\zeta(s,z)}{\zeta(s,y)}\cdot\frac{x^s}{s}\,ds$$

from above.

If $r = \log z/\log y > 1/2$, or if both conditions, $u < (\log\log y)^{1+\varepsilon/3} + (\log z)^{3/5-\varepsilon/2}$ and $(\log z)^{3/5-\varepsilon/2} > \log\log y$, are satisfied, choosing ε sufficiently small and $y_0(\varepsilon)$ sufficiently large gives $T = L_{\varepsilon/4}(z)$ and hence $E_{1,2} = 0$.

If $u \geq (\log\log y)^{1+\varepsilon/3} + (\log z)^{3/5-\varepsilon/2}$ and $r \leq 1/2$, Lemma 4.5, Lemma 4.4 and (36) show that

$$E_{1,2} \ll x^\beta \frac{\zeta(\beta,y)}{\zeta(\beta,z)}\, e^{-cu} \int\limits_{L_{\varepsilon/4}(z)\leq\tau\leq T} \frac{(\log\tau)^{2/3}}{\tau}\, d\tau \ll x^\beta (\log y)^2 e^{I_r(\xi)-cu},$$

which is clearly acceptable.

Finally, we consider the case where $u < (\log\log y)^{1+\varepsilon/3} + (\log z)^{3/5-\varepsilon/2}$ and $(\log z)^{3/5-\varepsilon/2} \leq \log\log y$. We define

$$\lambda(s,y,z) := \zeta(s)\,\frac{\zeta(s,z)}{\zeta(s,y)}.$$

Integration by parts applied twice yields

$$E_{1,2} = \left[\frac{x^s\lambda(s,y,z)}{is\log x}\right]_{|\tau|=L_{\varepsilon/4}(z)}^{|\tau|=T} + \frac{1}{\log x} \int\limits_{\substack{\sigma=\beta\\ L_{\varepsilon/4}(z)\leq|\tau|\leq T}} \lambda(s,y,z)x^s\,\frac{ds}{s^2}$$
$$- \frac{1}{\log x} \int\limits_{\substack{\sigma=\beta\\ L_{\varepsilon/4}(z)\leq|\tau|\leq T}} \lambda'(s,y,z)x^s\,\frac{d\tau}{s}$$

and

$$E_{1,2} = \left[\frac{x^s\lambda(s,y,z)}{is\log x} + \frac{ix^s\lambda'(s,y,z)}{s(\log x)^2}\right]_{|\tau|=L_{\varepsilon/4}(z)}^{|\tau|=T} \tag{37}$$
$$+ \int\limits_{\substack{\sigma=\beta\\ L_{\varepsilon/4}(z)\leq|\tau|\leq T}} \left(\frac{\lambda(s,y,z)}{\log x} - \frac{\lambda'(s,y,z)}{(\log x)^2}\right)\frac{x^s}{s^2}\,ds$$
$$+ \int\limits_{\substack{\sigma=\beta\\ L_{\varepsilon/4}(z)\leq|\tau|\leq T}} \frac{\lambda''(s,y,z)}{(\log x)^2}\,x^s\,\frac{d\tau}{s}.$$

We apply Lemma 4.3, with $\varepsilon/4$ replaced by ε, to obtain

$$\lambda(s,y,z) \ll (\log z)e^{I_0((1-\beta)\log z)}.$$

Now (H_ε) and (V_ε) imply that $1-\beta \leq (\log y)^{-2/5-\varepsilon}$. Since $(\log z)^{3/5-\varepsilon/2} \leq \log\log y$ this shows that $(1-\beta)\log z \ll 1$ and hence $\exp\{I_0((1-\beta)\log z)\} \ll 1$. Thus,

$$\lambda(s,y,z) \ll \log z$$

in this case. Cauchy's formula allows us to write

$$\lambda^{(k)}(s,y,z) = \frac{k!}{2\pi i}\int_{|w-s|=(\log y)^{-2/5-\varepsilon}} \frac{\lambda(w,y,z)}{(w-s)^{k+1}}\,dw \ll (\log z)(\log y)^{k(2/5+\varepsilon)}, \tag{38}$$

for $L_{\varepsilon/4}(z) \le |\tau| \le T$. Applying (38) to the expression (37) leads to

$$E_{1,2} \ll x^{\beta}\bigg(\frac{\log z}{u(\log y)L_{\varepsilon/4}(z)} + \frac{(\log T)(\log z)}{u^2(\log y)^{6/5-\varepsilon}}\bigg),$$

which is acceptable. This concludes the proof of Lemma 4.8.

Proof of Proposition 3.1. By Perron's formula (see for example [12, Theorem II.2.1]), we have, for all real $\kappa > 1$ and $x \notin \mathbb{N}$,

$$\varGamma(x,y,z) = \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} \zeta(s)\,\frac{\zeta(s,z)}{\zeta(s,y)}\cdot\frac{x^s}{s}\,ds.$$

The residue at $s = 1$ has value $x\zeta(1,z)/\zeta(1,y)$. The integrand is an analytic function of s for $s \neq 0$ or 1, and tends to 0 as $|\tau| \to \infty$ in every vertical strip $0 < \sigma_0 \le \sigma \le 1$. We can therefore move the abscissa of integration to the left as far as

$$\beta := 1 - \frac{\xi_r(u)}{\log y}.$$

Set $L := L_{\varepsilon/4}(z)$. We obtain

$$\begin{aligned}\varGamma(x,y,z) &= x\,\frac{\zeta(1,z)}{\zeta(1,y)} + \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} \zeta(s)\,\frac{\zeta(s,z)}{\zeta(s,y)}\cdot\frac{x^s}{s}\,ds \\ &= x\,\frac{\zeta(1,z)}{\zeta(1,y)} + \frac{1}{2\pi i}\int_{\beta-iL}^{\beta+iL} \zeta(s)\,\frac{\zeta(s,z)}{\zeta(s,y)}\cdot\frac{x^s}{s}\,ds + E_1,\end{aligned} \tag{39}$$

with

$$E_1 := \int_{\substack{\sigma=\beta\\|\tau|\ge L}} \zeta(s)\,\frac{\zeta(s,z)}{\zeta(s,y)}\cdot\frac{x^s}{s}\,ds.$$

The conditions $(H_\varepsilon(x,y))$ and (V_ε) imply that β satisfies

$$\beta \ge 1 - (\log y)^{-2/5-\varepsilon}, \tag{40}$$

according to Lemma 2.9. Using Lemma 4.2, we approximate the integrand in (39) by

$$\zeta(s)\,\frac{\zeta(1,z)}{\zeta(1,y)}\cdot\frac{x^s}{s}\,e^{-I_r((1-s)\log y)} = x\,\frac{\zeta(1,z)}{\zeta(1,y)}\,\zeta\bigg(1+\frac{s'}{\log y}\bigg)\frac{e^{-I_r(-s')+us'}}{1+s'/\log y},$$

where $s' = (s-1)\log y$. We obtain

$$(41)\qquad \Gamma(x,y,z) = x\,\frac{\zeta(1,z)}{\zeta(1,y)} + \frac{x}{2\pi i}\cdot\frac{\zeta(1,z)}{\zeta(1,y)} \int_{-\xi-iL\log y}^{-\xi+iL\log y} \zeta\Big(1+\frac{s}{\log y}\Big)\frac{e^{-I_r(-s)+us}}{s+\log y}\,ds + O(E_1+E_2),$$

with

$$E_2 := \frac{x}{L}\cdot\frac{\zeta(1,z)}{\zeta(1,y)} \int_{-\xi-iL\log y}^{-\xi+iL\log y} \Big|\zeta\Big(1+\frac{s}{\log y}\Big)\frac{e^{-I_r(-s)+us}}{s+\log y}\Big|\,d|s|.$$

We extend the integral in (41) to infinity by writing

$$(42)\qquad \Gamma(x,y,z) = x\,\frac{\zeta(1,z)}{\zeta(1,y)} + \frac{x}{2\pi i}\cdot\frac{\zeta(1,z)}{\zeta(1,y)} \int_{-\xi-i\infty}^{-\xi+i\infty} \zeta\Big(1+\frac{s}{\log y}\Big)\frac{e^{-I_r(-s)+us}}{s+\log y}\,ds + O(E_1+E_2+E_3),$$

with

$$E_3 := x\,\frac{\zeta(1,z)}{\zeta(1,y)} \int_{\substack{\sigma=-\xi\\ |\tau|\ge L\log y}} \zeta\Big(1+\frac{s}{\log y}\Big)\frac{e^{-I_r(-s)+us}}{s+\log y}\,ds.$$

Set $E := E_1+E_2+E_3$. The integrand in (42) has a simple pole at $s=0$ with residue $x\zeta(1,z)/\zeta(1,y)$. By moving the line of integration of (42) to the right as far as $\sigma=\kappa>0$, it follows that

$$\begin{aligned}\Gamma(x,y,z) &= \frac{x}{2\pi i}\cdot\frac{\zeta(1,z)}{\zeta(1,y)} \int_{-\kappa-i\infty}^{-\kappa+i\infty} \zeta\Big(1+\frac{s}{\log y}\Big)\frac{e^{-I_r(-s)+us}}{s+\log y}\,ds+O(E)\\ &= \frac{x}{2\pi i}\cdot\frac{\zeta(1,z)}{r\zeta(1,y)} \int_{-\kappa-i\infty}^{-\kappa+i\infty} \zeta\Big(1+\frac{s}{\log y}\Big)\frac{s}{s+\log y}\,\widehat{\eta}_r(s)e^{us}ds+O(E),\end{aligned}$$

since $\widehat{\eta}_r(s) = rs^{-1}e^{-I_r(-s)}$ by Lemma 2.1. Now, it is easily verified that, for all complex s, we have

$$\zeta\Big(1+\frac{s}{\log y}\Big)\frac{s}{s+\log y} = \int_{-\infty}^{\infty} e^{-st}\,d\Big(\frac{[y^t]}{y^t}\Big).$$

Thus

$$\Gamma(x,y,z) = \frac{x}{2\pi i}\cdot\frac{\zeta(1,z)}{\zeta(1,y)} \int_{-\kappa-i\infty}^{-\kappa+i\infty} \widehat{\mu}_{y,z}(s)e^{us}\,ds + O(E).$$

Since $\eta_r(u) = r + O(\varrho(u))$ it follows that $\mu_{y,z}(u) \ll 1$. Therefore the inverse

Laplace integral

$$\mu_{y,z}(u) = \int_{-\kappa-i\infty}^{-\kappa+i\infty} \widehat{\mu}_{y,z}(s) e^{us}\, ds \tag{43}$$

converges whenever $\kappa > 0$ and $y^u \notin \mathbb{N}$. When $x = y^u \in \mathbb{N}$, the integral (43) converges to

$$\frac{1}{2}(\mu_{y,z}(u) + \mu_{y,z}(u-)) = \mu_{y,z}(u) + \frac{1}{2x}.$$

Lemmas 4.6–4.8 complete the proof of Proposition 3.1 by showing that, under the given conditions, $E = E_1 + E_2 + E_3$ is acceptable.

5. Proof of Proposition 3.2. Let $\alpha = \alpha(x, y, z)$ be defined to be the solution of the equation

$$\sum_{z<p\le y} \frac{\log p}{p^\alpha - 1} = \log x. \tag{44}$$

We will first establish three auxiliary results.

Lemma 5.1. *Let $a \ge 1$, $u \ge u_0$, $y \ge y_0$ and $(x, y, z) \in (G_c)$ for some suitable c. Then*

$$\Theta(ax, y, z) \ll a^\alpha \Theta(x, y, z),$$

where $\alpha = \alpha(x, y, z)$ is defined in (44).

Proof. Let $\alpha_1 := \alpha(ax, y, z)$. Then $\alpha_1 \le \alpha$. Following Saias [11, Theorem 1], we can write

$$\Theta(ax, y, z) = \frac{(ax)^{\alpha_1}\zeta(\alpha_1, y, z)}{\alpha_1\sqrt{2\pi\phi_2(\alpha_1, y, z)}}\biggl\{1 + O\biggl(\frac{1}{u} + (\log y)^2\,\frac{\sqrt{y-z}}{y}\biggr)\biggr\}, \tag{45}$$

where

$$\zeta(s, y, z) := \prod_{z<p\le y} (1 - p^{-s})^{-1}, \qquad \phi(s, y, z) := \log\zeta(s, y, z)$$

and, for $k \ge 1$,

$$\phi_k(s, y, z) = \frac{\partial}{\partial s^k}\,\phi(s, y, z).$$

By definition of α_1 we have

$$(ax)^{\alpha_1}\zeta(\alpha_1, y, z) \le (ax)^\alpha\zeta(\alpha, y, z).$$

A routine calculation shows that $\alpha \mapsto \alpha\sqrt{\phi_2(\alpha, y, z)}$ is a decreasing function of α. After replacing α_1 by α on the right hand side of (45) we apply Theorem 1 of [11] a second time to obtain the desired inequality.

Lemma 5.2. *Let $\varepsilon > 0$. Let $y \ge y_0$, $z \ge 1$ and $x \ge y \ge z + z^{7/12}$ with y_0 sufficiently large. Let $s = \alpha + i\tau$ with $\tau \in \mathbb{R}$ and $\overline{u} = \min(u, (y-z)/\log y)$. Then*:

(i) *if* $|\tau| \le 1/\log y$,

$$\left|\frac{\zeta(s,z)}{\zeta(s,y)}\right| \ll \zeta(\alpha,y,z)\exp\{-c_0\overline{u}\},$$

(ii) *if* $1/\log y < |\tau| \le \exp\{(\log y)^{3/2-\varepsilon}\}$ *and* $y \ge \max(2z, y_0)$,

$$\left|\frac{\zeta(s,z)}{\zeta(s,y)}\right| \ll \zeta(\alpha,y,z)\exp\left\{-c_1(\varepsilon)\,\frac{\overline{u}\tau^2}{(1-\alpha)^2+\tau^2+(\log(y/z))^{-2}}\right\},$$

(iii) *if* $1/\log y < |\tau| \le c_2 y/\log y$,

$$\left|\frac{\zeta(s,z)}{\zeta(s,y)}\right| \ll \zeta(\alpha,y,z)\exp\left\{-c_3\,\frac{\log(y/(|\tau|\log y))}{\log y}\cdot\frac{\overline{u}\tau^2}{(1-\alpha)^2+\tau^2+(\log(y/z))^{-2}}\right\}.$$

Proof. An easy calculation shows that

$$\begin{aligned}|(1-p^{-s})(1-p^{-\alpha})| &= (1-p^{-2\alpha})\left(1-\frac{2(1+\cos(\tau\log p))}{p^\alpha(1+p^{-\alpha})^2}\right)^{1/2}\\ &\le \exp\left\{-\frac{1+\cos(\tau\log p)}{4p^\alpha}\right\}.\end{aligned}$$

This implies that

$$\left|\frac{\zeta(s,z)}{\zeta(s,y)}\cdot\frac{\zeta(\alpha,z)}{\zeta(\alpha,y)}\right| \le e^{-V},$$

with

$$V := \sum_{z<p\le y}\frac{1+\cos(\tau\log p)}{p^\alpha}.$$

If $|\tau|\log y \le 1$, we have

$$V \ge \frac{1}{\log y}\sum_{z<p\le y}\frac{\log p}{p^\alpha} \gg \frac{y^{1-\alpha}-z^{1-\alpha}}{(1-\alpha)\log y} \gg \overline{u},$$

by Lemma 1(ii) and Lemma 8 of [11].

To show (ii) and (iii) we argue as in Lemma 10 of [11], where

$$\sum_{z<p\le y}\frac{1-\cos(\tau\log p)}{p^\alpha}$$

is bounded from below. The calculations are identical and we obtain the desired bounds.

Lemma 5.3. *Let* $y-z > \log x$, $u \ge (\log\log y)^2$ *and* $(x,y,z)\in(G_c)$ *for some suitable* c. *Then*

$$M(x,y,z) := \sum_{\substack{n\le x\\ n\mid P}}\mu(n) \ll \Theta(x,y,z)\left(\frac{1}{u}+(\log y)^2\,\frac{\sqrt{y-z}}{y}\right),$$

where $P := \prod_{z<p\le y} p$.

Proof. We only sketch the proof since it is almost identical to the proof of Theorem 1 in [11] by Saias. The result being trivial for $u < u_0$ or $y < y_0$, we may assume that u and y are sufficiently large. The first effective Perron formula (see for example [12, Theorem II.2.2]) enables us to write

$$M(x,y,z) = \int_{\alpha-iT}^{\alpha+iT} \frac{\zeta(s,z)}{\zeta(s,y)} \cdot \frac{x^s}{s}\,ds + O\biggl(x^\alpha \sum_{n\mid P} \frac{|\mu(n)|}{n^\alpha(1+T|\log(x/n)|)}\biggr), \tag{46}$$

where $P := \prod_{z<p\le y} p$. Following Saias, we choose $\alpha = \alpha(x,y,z)$ as defined in (44) and

$$T := \begin{cases} [\exp\{-(\log y)^{3/2-\varepsilon}\} + \exp\{-c_1 u(\log(2u))^{-2}\}]^{-1} & (y \ge 2z), \\ \displaystyle\biggl[c_2 \frac{(\log y)^{3/2}}{y} + (\log(2u))\exp\biggl\{-c_3 \frac{u}{(\log(2u) + \log y/\log(y/z))^2}\biggr\}\biggr]^{-1} & \\ & (y < 2z). \end{cases}$$

From the proof of Lemma 12 in [11], we see that the error term in (46) is

$$\ll \frac{1}{T}\,(\Theta(x,y,z) + (\log T)x^\alpha\zeta(\alpha,y,z)).$$

Let $s = \alpha + i\tau$. The contribution to the integral in (46) from the domain $1/\log y \le |\tau| \le T$ is $\ll x^\alpha\zeta(\alpha,y,z)/T$ by Lemma 5.2. This is shown in the same manner as in [11] or [6]. It remains to estimate the contribution to the integral in (46) from the domain $|\tau| < 1/\log y$. Lemma 5.2 allows us to write

$$\begin{aligned} \int_{|\tau|<1/\log y} \frac{\zeta(s,z)}{\zeta(s,y)} \cdot \frac{x^s}{s}\,ds &\ll x^\alpha\zeta(\alpha,y,z)e^{-cu} \int_{|\tau|<1/\log y} \frac{d\tau}{\alpha+\tau} \\ &= x^\alpha\zeta(\alpha,y,z)e^{-cu}\log\biggl(1 + \frac{1}{\alpha\log y}\biggr) \\ &\ll x^\alpha\zeta(\alpha,y,z)e^{-cu} \\ &\ll \frac{x^\alpha}{T}\,\zeta(\alpha,y,z), \end{aligned}$$

since $\alpha\log y \gg 1$ for $y - z > \log x$, according to Lemma 4 in [11]. Following the proof of Theorem 1 in [11], we conclude that $M(x,y,z)$ is

$$\ll \Theta(x,y,z)\biggl(\frac{1}{u} + (\log y)^2\,\frac{\sqrt{y-z}}{y}\biggr),$$

which is the desired inequality.

Proof of Proposition 3.2. Let $P = \prod_{z<p\le y} p$. Then

$$\Gamma(x,y,z) = \sum_{n\mid P} \mu(n)\biggl[\frac{x}{n}\biggr] = x\sum_{n\mid P}\frac{\mu(n)}{n} - \sum_{n\mid P}\mu(n)\biggl\{\frac{x}{n}\biggr\}$$

$$= x \prod_{z<p\le y}\left(1-\frac{1}{p}\right) - x\sum_{\substack{n>x\\ n|P}} \frac{\mu(n)}{n} - \sum_{\substack{n\le x\\ n|P}} \mu(n)\left\{\frac{x}{n}\right\}$$
$$= x \prod_{z<p\le y}\left(1-\frac{1}{p}\right) - x\sum_{\substack{n>x\\ n|P}} \frac{\mu(n)}{n} + O(\Theta(x,y,z)).$$

If $y \le y_0$ for some fixed y_0 then the last sum vanishes for $x > \prod_{p\le y_0} p$. Since the result is trivial for bounded x we can assume that $y > y_0$ for some sufficiently large y_0. By partial summation, we have

$$x\sum_{\substack{n>x\\ n|P}} \frac{\mu(n)}{n} \ll x\int_x^\infty |M(t,y,z)|\,\frac{dt}{t^2},$$

with

$$M(x,y,z) := \sum_{\substack{n\le x\\ n|P}} \mu(n).$$

Let $\alpha = \alpha(x,y,z)$ be as in (44). If $y - z \le \log x$ then $\alpha \ll 1/\log y$ according to Saias [11, Lemma 4]. Thus Lemma 5.1 allows us to write

$$x\int_x^\infty |M(t,y,z)|\,\frac{dt}{t^2} \ll x\int_x^\infty \Theta(t,y,z)\,\frac{dt}{t^2} \ll x^{1-\alpha}\Theta(x,y,z)\int_x^\infty t^{\alpha-2}\,dt$$
$$= \frac{1}{1-\alpha}\,\Theta(x,y,z) \ll \Theta(x,y,z).$$

If $y - z > \log x$ and $u \ge u_0 = \log y_0$ then $(1-\alpha)\log y \gg 1$ according to Saias [11, Theorem 2]. Thus Lemma 5.3 and Lemma 5.1 yield

$$x\int_x^{e^{y-z}} |M(t,y,z)|\,\frac{dt}{t^2} \ll \frac{x}{\log y}\int_x^{e^{y-z}} \Theta(t,y,z)\,\frac{dt}{t^2} \ll \frac{x^{1-\alpha}}{\log y}\,\Theta(x,y,z)\int_x^{e^{y-z}} t^{\alpha-2}\,dt$$
$$\ll \frac{\Theta(x,y,z)}{(1-\alpha)\log y} \ll \Theta(x,y,z).$$

With $\alpha = \alpha(x,y,z)$ and $\widetilde{\alpha} = \alpha(e^{y-z},y,z)$ we have, by Lemma 5.1,

$$\begin{aligned} x\int_{e^{y-z}}^\infty |M(t,y,z)|\,\frac{dt}{t^2} &\ll x\int_{e^{y-z}}^\infty \Theta(x,y,z)\,\frac{dt}{t^2} \\ &\ll xe^{-(y-z)\widetilde{\alpha}}\Theta(e^{y-z},y,z)\int_{e^{y-z}}^\infty t^{\widetilde{\alpha}-2}\,dt \\ &\ll \frac{x}{e^{y-z}}\,\Theta(e^{y-z},y,z) \\ &\ll \left(\frac{x}{e^{y-z}}\right)^{1-\alpha}\Theta(x,y,z) \ll \Theta(x,y,z), \end{aligned}$$

since $\widetilde{\alpha} = \alpha(e^{y-z}, y, z) \ll 1/\log y$, by Saias [11, Lemma 4]. This completes the proof of Proposition 3.2.

References

[1] J. D. Bovey, *On the size of prime factors of integers*, Acta Arith. 33 (1977), 65–80.

[2] N. G. de Bruijn, *On the number of positive integers $\le x$ and free of prime factors $> y$*, Proc. Konink. Nederl. Akad. Wetensch. 54 (1951), 50–60.

[3] —, *Asymptotic Methods in Analysis*, 3rd ed., North-Holland, Amsterdam, 1970; new printing: Dover, New York, 1981.

[4] E. Fouvry and G. Tenenbaum, *Entiers sans grand facteur premier en progressions arithmétiques*, Proc. London Math. Soc. 63 (1991), 449–494.

[5] J. B. Friedlander, *Integers free from small and large primes*, ibid. 33 (1976), 565–576.

[6] A. Hildebrand and G. Tenenbaum, *On integers free of large prime factors*, Trans. Amer. Math. Soc. 296 (1986), 265–290.

[7] N. M. Korobov, *Estimates of Weyl's sums and the distribution of primes*, Dokl. Akad. Nauk SSSR 123 (1958), 28–31 (in Russian).

[8] E. Saias, *Sur le nombre des entiers sans grand facteur premier*, J. Number Theory 32 (1989), 78–99.

[9] —, *Entiers sans grand ni petit facteur premier I*, Acta Arith. 61 (1992), 347–374.

[10] —, *Entiers sans grand ni petit facteur premier II*, ibid. 63 (1993), 287–312.

[11] —, *Entiers sans grand ni petit facteur premier III*, ibid. 71 (1995), 351–379.

[12] G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, Cambridge Univ. Press, Cambridge, 1995.

[13] —, *Lois de repartition des diviseurs, 2*, Acta Arith. 38 (1980), 1–36.

[14] I. M. Vinogradov, *A new estimate for $\zeta(1 + it)$*, Izv. Akad. Nauk SSSR Ser. Mat. 22 (1958), 161–164 (in Russian).

[15] A. Weingartner, *Integers free of prime divisors from an interval, I*, Acta Arith. 98 (2001), 117–131.

[16] D. V. Widder, *The Laplace Transform*, Princeton Univ. Press, Princeton, N.J., 1946.

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