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Horizontal sections of connections on curves and transcendence

by

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1. Introduction. Many questions in transcendence theory may be summed up in this "meta-question": Suppose that U is a variety defined over a number field K and $G(F, F^{(1)}, \ldots, F^{(n)}) = 0$ is an algebraic system of differential equations defined over U (the functions defining G are in $K(U)$). Suppose that $F := (F_1, \ldots, F_n)$ is a local solution of the system. Let $q \in U(K)$; what can we say about $\mathrm{Trdeg}_{\mathbb{Q}}(K(F(q)))$? Apart from the fact that this transcendence degree is bounded from above by $\mathrm{Trdeg}_{K(U)}(K(U)(F))$, we cannot say much about this question in general.

Siegel–Shidlovskiĭ theory gives us a very powerful and satisfactory answer when we restrict our attention to *systems of linear differential equations* over the projective line and nonsingular over the multiplicative group $\mathbb{G}_{\mathrm{m}}$. Let us recall the main result of the theory (in a simplified version; cf. for instance [La]):

Let

$$\frac{dY}{dz} = AY \quad \text{with} \quad A \in M_n(\mathbb{Q}(z)) \tag{1.1.1}$$

be a linear system of differential equations. Suppose $F = (f_1(z), \ldots, f_n(z))$ is a solution of (1.1.1) with the following properties:

(a) the functions $f_1(z), \ldots, f_n(z)$ are algebraically independent over $\mathbb{C}(z)$;
(b) each $f_i(z)$ has a Taylor expansion $f_i(z) = \sum_{j=0}^{\infty} a_{ij} z^j / j!$ with

$$a_{ij} \in \mathbb{Q}, \quad \text{and for each } i, \quad H(a_{i,0} : \cdots : a_{ij} : 1) \ll_\epsilon j^{\epsilon j}$$

($H(\cdot)$ being the exponential height).

Then, for every $q \in \mathbb{Q}^*$, we have $\mathrm{Trdeg}_{\mathbb{Q}}(\mathbb{Q}(f_1(q), \ldots, f_n(q))) = n$. Recall that functions with property (b) above are called *E-functions*.

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It is well known that the criterion above (and its extension to number fields) has many important consequences; in particular, the Hermite–Lindemann Theorem is a special case of it (take $f_i(z) = e^{\alpha_i z}$). Many nontrivial transcendence properties of special values of hypergeometric and Bessel functions can be deduced from it.

Of course, if one could generalize Siegel–Shidlovskiĭ theory to arbitrary varieties, the general "meta-question" above would have a satisfactory answer in the linear case. As a consequence of the main results of [Ga], we may deduce the following:

1.1. Theorem. *Let $X/_{\mathbb{Q}}$ be a smooth projective curve and $D \subseteq X$ be a reduced divisor. Denote by X^o the affine curve $X \setminus D$. Let (E, ∇) be a fiber bundle with connection over X having meromorphic singularities in D. Let $f_{\mathbb{C}} : X^o(\mathbb{C}) \to E(\mathbb{C})$ be a Zariski dense horizontal section of finite order of growth ρ. Then*

$$\mathrm{Card}(f(X^o(\mathbb{Q})) \cap E(\mathbb{Q})) \leq \frac{\mathrm{rk}(E) + 2}{\mathrm{rk}(E)} \rho.$$

(The theorem above is not explicitly stated in [Ga], but it can obtained as a particular case of Theorem 1.1 there.) If we apply 1.1 to the symmetric power of E with the induced connection and the induced section, we obtain:

1.2. Corollary. *Under the hypotheses of Theorem* 1.1 *we have*

$$\mathrm{Card}(f(X^o(\mathbb{Q})) \cap E(\mathbb{Q})) \leq \rho.$$

Thus, if the order of growth of f is ρ, then there are at most ρ rational points on the image of f (for a p-adic version of Theorem 1.1, see Theorem 4.6 of the recent thesis [He]). Examples show that if f is a horizontal section of a vector bundle over an affine curve which has order of growth ρ and takes less than ρ rational values at rational points, we cannot say anything about the algebraic independence of its values at other algebraic points. In this paper we show that when the number of rational values of the section is the same as the order of growth, we can say more (for a similar observation in the context of the Schneider–Lang Theorem, see [Be0, §4]).

In order to explain the main theorem of this paper, we need to explain the definition of E-sections of arithmetic type of a vector bundle over a curve. These are a generalization, over arbitrary curves, of the concept of E-functions over the affine line developed by Shidlovskiĭ (cf. for instance [La]). The precise definition of E-section of arithmetic type requires the introduction of some notation so we refer to §6 for it. Here we give just the idea of the definition. Let X^o be a smooth affine curve defined over $\mathbb{Q}$, and V be a vector bundle defined over it. We fix $p_1, \ldots, p_s \in X^o(\mathbb{Q})$ and a local trivialization of V near them (in particular this trivialization is defined over $\mathbb{Q}$).

Suppose we have an analytic section $f : X^o(\mathbb{C}) \to E(\mathbb{C})$. It is said to be an *$E$-section of arithmetic type with respect to* $\{p_1, \dots, p_s\}$ if:

- The order of growth of f is s.
- Using the trivialization fixed above, locally around each of the p_j's, we may write $f = (f_{1,p_j}(z), \dots, f_{m,p_j}(z))$ with

$$f_{i,p_j}(z) = \sum_{j=0}^{\infty} a_{ij} \frac{z^j}{j!} \quad \text{and} \quad a_{ij} \in \mathbb{Z}[1/N],\ N^j a_{ij} \in \mathbb{Z}.$$

Notice that the number s of points involved and the order of growth ρ are related. E-sections of arithmetic type are a good generalization of E-functions over arbitrary curves. Nevertheless it is important to notice that while the local behavior and the growth behavior of an E-function are summarized in its definition as a power series, the local and global properties of E-sections are defined separately via formal geometry and Nevanlinna theory. In [Be0] the author proves a generalization of the Schneider–Lang criterion just imposing a local (at the point at infinity) Gevrey condition which is very similar to our definition of E-sections of arithmetic type. With this definition in mind we can state our main theorem (here, for simplicity, we state it just over $\mathbb{Q}$; for the general statement see 6.1):

1.3. Theorem. *Let $X/_{\mathbb{Q}}$ be a smooth projective curve. Let D be a reduced effective divisor on X and (E, ∇) be a fiber bundle of rank $m > 1$ with connection with meromorphic singularities on D. Let $p_1, \dots, p_s \in X(\mathbb{Q})$ be rational points, $D' := D - \{p_1, \dots, p_s\}$ and $X^o := X \setminus D'$. Let $f : X^o(\mathbb{C}) \to E(\mathbb{C})$ be an analytic* horizontal section *with respect to the connection which is an E-section of arithmetic type with respect to the points p_j. Suppose that the image of f is Zariski dense in E. Let $q \in X^o(\mathbb{Q}) \setminus \{p_1, \dots, p_s\}$. Then*

$$\mathrm{Trdeg}_{\mathbb{Q}}(\mathbb{Q}(f(q))) = m.$$

Observe that if $X^0 = \mathbb{P}^1$, $D = 0 + \infty$, and we have only one point $p = 0$, we find the classical theorem by Siegel and Shidlovskiĭ. The requirement that the image is Zariski dense is equivalent to the requirement that the entries of f are algebraically independent over $\overline{\mathbb{Q}}(X)$.

Even in the case when $X = \mathbb{P}^1$ but D is arbitrary, Theorem 1.3 is stronger than the classical theorem by Siegel and Shidlovskiĭ. Indeed, we do not require that the solution is an E-function, so in particular an entire function, but it may have several essential singularities on D (as in [Be1], [Ga] and [He] in the Schneider–Lang context).

This paper is organized as follows. In §2 we prove a zero lemma over an arbitrary curve, which replaces the classical Shidlovskiĭ Lemma; the statement is formally similar to the Shidlovskiĭ Lemma, but the proof clarifies the classical proof and uses some tools from algebraic geometry: vector bundles, Hilbert schemes, etc. In §3 we explain the tools from Nevanlinna theory which are needed; we use a version of Nevanlinna theory (developed in [Ga]) which allows one to prove powerful lemmas of Schwarz's type over (a special kind of) Riemann surfaces. In §4 and §5 we develop the notion of E-sections of arithmetic type and we explain their main properties. Finally, in §6 we state and prove the main theorem of the paper.

1.1. Two applications. We can give two applications of the main theorem.

First application: Connections with isomorphic monodromy. The first application concerns nonisomorphic connections with the same monodromy. Let X be a curve and (E_1,∇_1) and (E_2,∇_2) two integrable connections of rank n over it. Let $\rho_i:\pi_1(X)\to GL_n$ be the monodromy representation associated to (E_i,∇_i). Suppose that ρ_1 is equivalent to ρ_2; thus the trivial representation is a subrepresentation of $\rho_1\otimes\rho_2^\vee$; consequently, we get a global horizontal section of $E_1\otimes E_2^\vee$. Provided that it has the right order of growth, this section is a typical section to which we can apply the criterion.

We may guarantee the right order of growth by the classical Gronwall Lemma. In particular it guarantees that if we have a connection on a projective curve with poles of order at most two, then a horizontal section defined on the complement of the poles of the connection will define an E-section of arithmetic type (cf. Definition 5.1) with respect to any rational point where the section takes an algebraic value.

From this we can obtain the following: Let X be a smooth projective curve over $\mathbb{Q}$. Let D be a reduced effective divisor over X. Denote by X^o the affine curve $X\setminus D$. Let (E_1,∇_1) and (E_2,∇_2) be two fiber bundles with connections having poles of order at most two on D. Suppose that the corresponding representations $\rho_i:\pi_1(X^o)\to GL_N$ are isomorphic. Let $p\in X^o(\mathbb{Q})$. We can find an analytic isomorphism $\varphi:E_1\to E_2$ over X^o which restricts to the identity over p.

1.4. THEOREM. *Let V be an analytic neighborhood of p, and let $q\in V\cap X^o(\mathbb{Q})$ be different from p. Let F be a horizontal section of (E,∇_1) defined over V. Then* $\mathrm{Trdeg}_{\mathbb{Q}}(\mathbb{Q}(\varphi(F(q))))=\mathrm{Trdeg}_{\mathbb{Q}(X)}(\mathbb{Q}(\varphi))$.

The proof is a direct application of Theorem 6.1. Observe that, since, a priori, $F(q)$ is not a rational point of E, one should apply 6.1 over the field of definition of it and use Remark 5.2(c).

A nontrivial way to construct examples where we can apply Theorem 1.4 is the following: Let B be a reduced divisor in $\mathbb{A}^1_{\mathbb{Q}}$. Let X be a smooth projective curve defined over $\mathbb{Q}$. Let D be a reduced divisor over X. Over $X\times\mathbb{A}^1$ consider the divisors $H_1=D\times\mathbb{A}^1$ and $H_2=X\times B$ and $H=H_1+H_2$.

Suppose that (E,∇) is a fiber bundle with *integrable* connection over $X\times\mathbb{A}^1$ with poles on H and which are of order at most two on H_1.

Then, for every $x\in\mathbb{A}^1(\mathbb{Q})\setminus B$, the restriction (E_x,∇_x) of (E,∇) to $X\times\{x\}$ is a vector bundle with integrable connection having poles of order at most two on D.

By construction, for every couple $x_1,x_2\in\mathbb{A}^1(\mathbb{Q})$, the vector bundles (E_{x_i},∇_{x_i}) have conjugate monodromy. Thus the theorem applies in this case.

An explicit example. Let $a,b,c\in\mathbb{Q}$, and for every $x\in\mathbb{Q}$ consider the linear system of differential equations

$$\nabla_x:\ \frac{dY}{dz}=\left(\frac{1}{z^2}\cdot\begin{pmatrix}a & (a-b)x\\ 0 & b\end{pmatrix}+\frac{1}{z}\cdot\begin{pmatrix}1-x & -x^2\\ 1 & 1\end{pmatrix}+\frac{1}{z-1}\cdot\begin{pmatrix}1 & c\\ 0 & 1\end{pmatrix}\right)\cdot Y.$$

Then, up to conjugation, for every couple $x_0,x_1\in\mathbb{Q}$ the linear systems ∇_{x_0} and ∇_{x_1} have the same monodromy.

To see this, fix local coordinates (z,y) over $\mathbb{P}^1\times\mathbb{A}^1$. Denote by ω the matrix of differential forms

$$\omega:=\left(\frac{1}{z^2}\cdot A(y)+\frac{1}{z}\cdot B(y)+\frac{1}{z-1}\cdot\begin{pmatrix}1 & c\\ 0 & 1\end{pmatrix}\right)dz+\frac{1}{y}\cdot\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}dy$$

with $A(y)$ and $B(y)$ unknown matrices to be determined. The system of differential equations

$$\mathcal{E}:\ \nabla(Y)=\omega\cdot Y$$

defines a fiber bundle with integrable connection if and only if

$$d\omega=\omega\wedge\omega.$$

Thus $\mathcal{E}$ is integrable if and only if $A(y)$ and $B(y)$ are solutions of the linear differential system

$$\frac{dW(y)}{dy}=\frac{\left[W(y);\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\right]}{y}. \tag{1.5.1}$$

A basis of solutions of the system (1.5.1) is

$$\left\{\begin{pmatrix}1 & \log(y)\\ 0 & 0\end{pmatrix};\ \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix};\ \begin{pmatrix}-\log(y) & -\log^2(y)\\ 1 & 0\end{pmatrix};\ \begin{pmatrix}0 & -\log(y)\\ 0 & 1\end{pmatrix}\right\}.$$

Choose ∇_0 to be $(y=1)$

$$\nabla_0:\ \frac{dY}{dz}=\left(\frac{1}{z^2}\cdot\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}+\frac{1}{z}\cdot\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}+\frac{1}{z-1}\cdot\begin{pmatrix}1 & c\\ 0 & 1\end{pmatrix}\right)\cdot Y.$$

Thus if we put $x=\log(y)$, the conclusion follows.

Second application: Connections and coverings of curves. Let X be a smooth projective curve defined over $\mathbb{Q}$, and (E_1, ∇_1) and (E_2, ∇_2) two vector bundles with meromorphic connections over X. Suppose that the poles of ∇_1 and ∇_2 are both contained in a fixed divisor D. Denote by X^o the affine curve $X \setminus D$. Suppose that we can find a point $p \in X(\mathbb{Q})$ and analytic horizontal sections f_1 and f_2 of (E_1, ∇_1) and (E_2, ∇_2) respectively over X^o which are E-sections of arithmetic type with respect to p.

We can apply the main theorem to both f_i's and of course, if $f_1 \otimes f_2$ is Zariski dense in $E_1 \otimes E_2$, we can apply the main theorem to it too.

A small generalization of this argument may obtained by applying the full force of Theorem 1.3: Suppose that $g_1 : Y \to X$ and $g_2 : Y \to X$ are finite coverings of degree d with $g_1^{-1}(p) = g_2^{-1}(p) = \{p_1, \ldots, p_d\}$ (and p not contained in the branch loci of g_i's). Then $g_1^*(f_1) \otimes g_2^*(f_2)$ is an E-section of arithmetic type with respect to $p_1, \ldots, p_d$ which is a horizontal section of $g_1^*(E_1) \otimes g_2^*(E_2)$ (with the induced connection). Thus Theorem 1.3 applies to it: Let $q \in Y(\mathbb{Q})$ be such that $g_i(q) \in X^o \setminus \{p\}$. Then, if the image of $g_1^*(f_1) \otimes g_2^*(f_2)$ is Zariski dense, $\mathrm{Trdeg}_{\mathbb{Q}}(g_1^*(f_1) \otimes g_2^*(f_2)(q)) = \mathrm{rk}(E_1) \cdot \mathrm{rk}\, E_2)$.

For instance: let $F(x)$, $G_1(x)$ and $G_2(x)$ be polynomials of degree d with coefficients in $\mathbb{Z}$, with no common zeros, and $F(x)$ with d distinct roots defined over $\mathbb{Q}$. Let $J_0(z)$ be the Bessel function ([La, p. 76]). Then for every rational number t such that $F(t)G_1(t)G_2(t) \neq 0$ the numbers

$$J_0\biggl(\frac{F(t)}{G_1(t)}\biggr) \exp\biggl(\frac{F(t)}{G_2(t)}\biggr) \quad \text{and} \quad J_0'\biggl(\frac{F(t)}{G_1(t)}\biggr) \exp\biggl(\frac{F(t)}{G_2(t)}\biggr)$$

are algebraically independent.

1.5. Remark. The algebraic independence of the values of the functions above can be deduced from the classical Siegel–Shidlovskiĭ theorem. Nevertheless we proposed it as an explicit example of the principle above.

2. Connections and the Zero Lemma. In this section we will prove a theorem which is a generalization to every curve of the classical Shidlovskiĭ Zero Lemma. The statement of the classical lemma may be found for instance in [La, VII.3], and a generalization of it in [Be2].

We will start by fixing some notations and recalling some standard facts:

• X will be a smooth projective curve defined over the field of complex numbers. We will denote by R the field $\mathbb{C}(X)$.

• If $D = \sum_i n_i P_i$ is a divisor on X, we denote by $|D|$ the divisor $\sum_i \min\{1, |n_i|\} P_i$. If $D_1 = \sum_h n_{1,h} P_h$ and $D_2 = \sum_h n_{2,h} P_h$ are two effective divisors on X, we denote by $\mathrm{l.c.m.}(D_1, D_2)$ the divisor $\sum_i \max_h \{n_{1,h}, n_{2,h}\} P_j$.

• We fix an effective divisor D such that if we denote by T_X the tangent bundle of X, then $T_X(D)$ is generated by its global section. Denote by H the line bundle $\mathcal{O}_X(D)$ and by $s \in H^0(X, D)$ a section such that $\mathrm{div}(s) = D$. If F is a coherent sheaf on X and x an integer, we will denote by $F(x)$ the sheaf $F \otimes H^{\otimes x}$; in particular $\mathcal{O}_X(x) = \mathcal{O}_X(xD)$.

• The standard derivation $d : \mathcal{O}_X \to \Omega^1_X$ induces, for every point $P \in X(\mathbb{C})$, a singular connection $\nabla^P : \mathcal{O}_X(P) \to \mathcal{O}_X(P) \otimes \Omega^1_X(P)$. Thus, for every divisor S, the line bundle $\mathcal{O}(S)$ is equipped with a canonical connection $\nabla^S : \mathcal{O}(S) \to \mathcal{O}(S) \otimes \Omega^1_X(|S|)$. In particular, the line bundle $H := \mathcal{O}_X(D)$ is canonically equipped with a singular connection $\nabla^H : H \to H \otimes \Omega^1_X(D)$ because $H = \mathcal{O}_X(D)$.

• If $\nabla_i : F_i \to F_i \otimes \Omega^1_X(D_i)$ $(i = 1, 2)$ are fiber bundles on X with connections having singularities on the divisors D_i respectively, then the tensor product $F_1 \otimes F_2$ is naturally equipped with a singular connection $\nabla_{1,2} : F_1 \otimes F_2 \to F_1 \otimes F_2 \otimes \Omega^1_X(\mathrm{l.c.m.}(D_1, D_2))$.

• Fix a point $Q \in X(\mathbb{C})$, which *may be in the support of* D. Let ∂ be a global section of $T_X(D)$ which does not vanish at Q. We fix a section $s' \in H^0(X, H)$ such that $s'(Q) \neq 0$.

• We will denote by $\mathcal{D}$ the (noncommutative) ring $R[\partial]$.

• Denote by $\hat{X}_Q$ the completion of X around Q and if E, $\mathbb{L}$, f, Z etc. is a vector bundle, a coherent sheaf, a section, a scheme, etc. defined over X, we denote by E_Q, $\mathbb{L}_Q$, f_Q, Z_Q etc. its restriction to $\hat{X}_Q$.

• Similarly, if E, $\mathbb{L}$, f, Z etc. is a vector bundle, a coherent sheaf, a section, a scheme, etc. defined over X, we denote by E_R, $\mathbb{L}_R$, f_R, Z_R etc. its restriction to the generic point of X.

We fix a vector bundle (E, ∇^E) on X of rank m with a singular connection

$$\nabla^E : E \to E \otimes \Omega^1_{X_K}(D).$$

Then:

• For every integer x, the vector bundle $E(x)$ is equipped with a singular connection $\nabla^x : E(x) \to E(x) \otimes \Omega^1_X(D)$.

• The derivation ∂ and the connection ∇^x induce a derivation

$$\nabla^x_\partial : E(x) \to E(x+2).$$

• The restriction of $(E(x), \nabla^x)$ to the generic point of X is a $\mathcal{D}$-module which we will denote (E_R, ∇^x).

• If F is a vector bundle on X and $G \hookrightarrow F$ is a subsheaf, we will say that G is a *subbundle* if the quotient F/G is without torsion. In this case G, F and F/G are locally free.

2.1. Definition. Suppose that (F, ∇) is a vector bundle equipped with a (possibly singular) connection, and $G \hookrightarrow F$ is a subbundle. We will say that G is a *subbundle with connection* of F if the image of G via ∇ is contained in $G \otimes \Omega^1_X(D)$ (where D is the divisor involved in the definition of the connection on F).

2.2. Lemma. *Let $G \hookrightarrow E(x)$ be a subbundle. The following properties are equivalent:*

(a) *G is a subbundle with connection.*
(b) *The R-vector space G_R is a $\mathcal{D}$-submodule of $E(x)_R$, namely $\nabla(G_R)$ is contained in $(G \otimes \Omega^1_X(D))_R \subseteq (E(x) \otimes \Omega^1_X(D))_R$.*
(c) *The image of the R-vector space G_R under the map ∇^x_∂ is contained in $G(2)_R \subseteq E(x+2)_R$.*

The proof is left to the reader.

Let $P \in H^0(X, E(x))$; denote $P = P_0$ and $P_{i+1} := \nabla^{x+2i}_\partial(P_i)$. By construction, P_i is an element of $H^0(X, E(x+2i))$. Fix a positive integer $r \le m$. The sections $\tilde{P}_i := P_i \otimes (s')^{\otimes 2(r-i)}$ are elements of $H^0(X, E(x+2r))$. Let $G \subseteq E(x+2r)$ be the vector subbundle generated by the $\tilde{P}_i$. From the lemma above we deduce

2.3. Lemma. *Suppose that the $\tilde{P}_i$ are linearly dependent as elements of $E(x+2r)_R$. Then G is a subbundle with connection of $E(x+2r)$.*

Fix a global section $P \in H^0(X, E(x))$, suppose that $\tilde{P}_0, \ldots, \tilde{P}_\ell$, with some $\ell \le r-1$, are linearly independent over R and suppose that $\tilde{P}_0, \ldots, \tilde{P}_{\ell+1}$ are linearly dependent. In this case, a simple local computation implies that $\tilde{P}_0, \ldots, \tilde{P}_r$ are also linearly dependent. Denote by G the vector subbundle of $E(x+2r)$ generated by the $\tilde{P}_i$'s. It is a subbundle with connection, and every subbundle with connection containing $P \otimes (s')^{\otimes 2r}$ contains G. This motivates the following definition:

2.4. Definition. Given a global section $P \in E(x)$, we will call the subbundle $G \hookrightarrow E(x+2r)$ constructed above the *minimal subbundle with connection generated by P*.

In particular we remark that if $\tilde{P}_0, \ldots, \tilde{P}_{m-1}$ are linearly independent over R, then $G = E(x+2m)$.

Let $E^\vee$ be the dual of E and let f be a horizontal section of $E^\vee_Q$ (the dual of E_Q), that is, $\nabla^{E^\vee_Q}(f) = 0$. The natural evaluation map $\langle \cdot, \cdot \rangle : E(x) \otimes E^\vee \to \mathcal{O}_X(x)$ induces a linear map

$$\mathrm{ev} : H^0(X, E(x)) \to H^0(X_Q, \mathcal{O}_{X_Q}(x)), \qquad P \mapsto \langle P, f \rangle.$$

We will denote by F_i the sections $\mathrm{ev}(\tilde{P}_i) \in H^0(X_Q, \mathcal{O}_{X_Q}(x+2r))$.

The main theorem of this section is the following Zero Lemma:

2.5. Theorem. *Suppose that the above hypotheses hold, and that $f \notin H^0(X_Q, K_Q)$ for every proper algebraic subbundle $K \hookrightarrow E^\vee$. Then there* *exists a constant C, depending only on E, f and the fixed connections, but independent of P, such that*

$$\operatorname{ord}_Q(F_0) \leq x \operatorname{rk}(G) + C.$$

Observe that $\operatorname{ord}_Q(F_0) = \operatorname{ord}_Q(\langle P, f\rangle)$.

2.6. Remark. (a) The condition on f means that f is not algebraically degenerate: once we fix an algebraic trivialization of E_F, the coordinates of f are linearly independent over F.

(b) One should compare this theorem (and its proof) with the statement (and proof) of the classical Shidlovskiĭ Lemma. More precisely, the crucial point of the proof is the existence of a *lower bound* for the degrees of all subbundles of E which are generically stable under the connection ∇^E; see Proposition 2.10 below which should be compared with [La, Lemma 2.4, p. 85]. For a different approach to the Shidlovskiĭ Lemma based on Fuchs relations see [Be2, §2].

In order to prove Theorem 2.5, we need to generalize to higher rank the notion of the order of vanishing of a section:

Let V be a vector bundle on X_K and $f \in H^0(X_Q, V_Q)$ be a nonzero section. If we fix a trivialization of V_Q, we may write f as $(f_1, \dots, f_r)$ where r is the rank of V and f_i are power series in one variable.

2.7. Definition. The *order of vanishing* of f at Q is the integer $\min_i \operatorname{ord}_Q(f_i)$.

One easily sees that the order of vanishing of f is independent of the choice of the trivialization.

The theorem will be a consequence of the following lemma:

2.8. Lemma. *There is a constant C depending only on the vector bundle E with connection and f with the following property: Let $\mathcal{F}$ be a vector bundle with connection and $\alpha : E^\vee \twoheadrightarrow \mathcal{F}$ be a surjective morphism of vector bundles with connections. Let $[f] := \alpha(f) \in H^0(X_Q, \mathcal{F}_Q)$. Then*

$$\operatorname{ord}_Q([f]) \leq C.$$

Recall the following standard properties of vector bundles (cf. for instance [Se]):

(a) (Cramer rule) If G is a vector bundle of rank r then there is a canonical isomorphism

$$\det(G) \otimes G^\vee \simeq \bigwedge^{r-1} G.$$

(b) There is a constant C depending only on E such that if $G \hookrightarrow E(x)$ is a subbundle of rank r, then $\deg(G) \leq rx + C$.

Let us show how the lemma implies the theorem.

Proof of Theorem 2.5. First of all we claim that $\mathrm{ord}_Q(F_i) \geq \mathrm{ord}_Q(F_0) - i$. Indeed, by definition

$$F_i = \langle P_i \otimes (s')^{2(r-i)}, f\rangle = \langle P_i, f\rangle \otimes (s')^{2(r-i)};$$

thus, since s' does not vanish at Q, $\mathrm{ord}_Q(F_i) = \mathrm{ord}_Q(\langle P_i, f\rangle)$. Suppose that e is a local generator of $H^{\otimes x+2i}$ and z is a local coordinate around Q. Then we may suppose that $\langle P_i, f\rangle = z^a \cdot e$ for some positive integer a. The evaluation map

$$\mathrm{ev} : E(x+2i) \otimes E^\vee \to \mathcal{O}_X(x+2i)$$

is a morphism of vector bundles with connection; thus, we may find an analytic function h in a neighborhood of Q such that

$$az^{a-1}he + z^a \nabla_\partial(e) = \nabla_\partial \langle P_i, f\rangle = \langle \nabla_\partial^{x+2i} P_i, f\rangle + \langle P_i, \nabla_\partial f\rangle = \langle P_{i+1}, f\rangle.$$

The claim follows by induction on i.

Denote by r the rank of G. The inclusion $G \hookrightarrow E(x+2r)$ gives rise to a surjection $\alpha : E^\vee \twoheadrightarrow G^\vee(x+2r)$. Denote by $[f]$ the image of f in $H^0(X_Q, G_Q^\vee(x+2r))$.

We may suppose that $\tilde{P}_0, \ldots, \tilde{P}_{r-1}$ are linearly independent elements of G_R, so $\tilde{P}_0 \wedge \tilde{P}_1 \wedge \cdots \wedge \tilde{P}_{r-1}$ is a nonzero global section of $\bigwedge^r G$. Since, by property (b) above, there is a constant C_1 depending only on E such that $\deg(\bigwedge^r G) \leq xr + C$, we have $\mathrm{ord}_Q(\tilde{P}_0 \wedge \tilde{P}_1 \wedge \cdots \wedge \tilde{P}_{r-1}) \leq xr + C_1$. By Lemma 2.8 above, there is a constant C_2 such that $\mathrm{ord}_Q([f]) \leq C_2$. The isomorphism given by the Cramer rule (a) gives rise to the equality

$$(\tilde{P}_0 \wedge \tilde{P}_1 \wedge \cdots \wedge \tilde{P}_{r-1}) \otimes [f] = \sum_i (-1)^i (\tilde{P}_0 \wedge \cdots \wedge \widehat{\tilde{P}_i} \wedge \cdots \wedge \tilde{P}_{r-1}) \otimes F_i;$$

thus

$$C_1 + C_2 + rx \geq \mathrm{ord}_Q((\tilde{P}_0 \wedge \tilde{P}_1 \wedge \cdots \wedge \tilde{P}_{r-1}) \otimes [f]) \geq \inf_i \mathrm{ord}_Q(F_i) \geq \mathrm{ord}_Q(F_0) - r.$$

The conclusion of the theorem follows.

2.9. Remark. Observe that the constant C of the theorem is the sum of two terms: the first is purely geometrical, it is essentially related to the measure of the stability of E; the second term is analytical and it is related to the structure of the specific solution of the differential equation.

Proof of Lemma 2.8. We start with a proposition:

2.10. Proposition. *Let V be a vector bundle with singular connection on X. Then there exists a constant C with the following property: Let L be a line bundle with singular connection on X with a surjection $\alpha : V \twoheadrightarrow L$ (of vector bundles with singular connections). Then*

$$\deg(L) \leq C.$$

Let us show how Proposition 2.10 implies Lemma 2.8. We apply Proposition 2.10 to $V = \bigwedge^r E^\vee$ to find a constant, depending only on E, such that, for every subbundle G of E with connection, we have

$$\deg(G^\vee) \leq C.$$

This implies that the degrees of subbundles of E with connection are uniformly upper and lower bounded. Consequently, by the theory of the Hilbert scheme, we can find a projective variety $\underline{\mathrm{Hilb}_E}$, a vector bundle T on $X \times \underline{\mathrm{Hilb}_E}$ and a surjection $v : \mathrm{pr}_1^*(E) \twoheadrightarrow T$ (where $\mathrm{pr}_1 : X \times \underline{\mathrm{Hilb}_E} \to X$ is the first projection) such that, for every vector bundle V with connection which is a quotient of E, there is a point $q \in \underline{\mathrm{Hilb}_E}$ such that the surjection $E \twoheadrightarrow V$ is the restriction of v to $X \times \{q\}$. For every $q \in \underline{\mathrm{Hilb}_E}$ denote by T_q the vector bundle $T|_{X\times\{q\}}$ on X.

Let $\underline{\mathrm{Hilb}_{E_Q}}$ be the completion of $X \times \underline{\mathrm{Hilb}_E}$ around the Cartier divisor $\{Q\} \times \underline{\mathrm{Hilb}_E}$. The section f defines an element of $H^0(\underline{\mathrm{Hilb}_{E_Q}}, T_Q)$; thus, for every $q \in \underline{\mathrm{Hilb}_E}$, a global section $[f_q]$ of the localization $(T_q)_Q$ of T_q at Q. Consequently, we find a function

$$\mathrm{ord}_Q : \underline{\mathrm{Hilb}_E}(\mathbb{C}) \to \mathbb{Z}, \qquad q \mapsto \mathrm{ord}_Q([f_q]).$$

The local expression of the function $\mathrm{ord}_Q([f_q])$ shows that it is upper semicontinuous for the Zariski topology, and since $\underline{\mathrm{Hilb}_E}$ is compact, the conclusion follows.

Proof of Proposition 2.10. We begin by fixing some notation. Denote by m the rank of V. We fix a point p on X which is not a singular point of the connection. Denote by k_p the completion of R with respect to the valuation induced by p. We also fix an algebraic trivialization of V near p. Since the connection is nonsingular around p, the space of horizontal sections of the module V_p with connection has dimension m. Thus the space of *algebraic* horizontal sections of $V_p^\vee$ is finite-dimensional over $\mathbb{C}$ of dimension less than or equal to m.

Every line bundle with singular connection and which is a quotient of V defines a section g, up to a scalar, of $V_p^\vee$ which is horizontal. Thus, g belongs to a finite-dimensional $\mathbb{C}$-vector space, say W. The line bundles L which are quotients of V are in bijection with points of $\mathbb{P}^{m-1}(R)$ and thus with algebraic maps $\varphi_L : X \to \mathbb{P}^{m-1}$ (modulo the action of $PGL(m)$).

Fix a basis $g_1, \ldots, g_r$ of W over $\mathbb{C}$. Each g_i corresponds to a line bundle L_i which is a quotient of V. To every line bundle with connection and which is a quotient of V, we can associate an element g of W, thus a linear combination of the g_i's. The lemma below shows that the degree of every such bundle is bounded by the maximum of the degrees of the L_i's; thus the conclusion follows.

2.11. Lemma. *Let $L_i \hookrightarrow \mathcal{O}_X^m$ ($i = 1, 2$) be subbundles of rank one. Consider the map*

$$+ : \mathcal{O}_X^m \oplus \mathcal{O}_X^m \to \mathcal{O}_X^m, \qquad (x, y) \mapsto x + y.$$

Let M be the image of $L_1 \oplus L_2$ via $+$. Then $\deg(M) \geq \min \deg(L_i)$.

The proof of the lemma is elementary once one observes that there is a surjection $L_1 \oplus L_2 \twoheadrightarrow M$.

3. Nevanlinna theory and order of growth of sections. In this section we will recall the main definitions and theorems relating to the order of growth of analytic maps. Most of these things are classical (cf. for instance [GK]), but the approach we take here is a little different. One can find details and possible generalizations in [Ga].

Let X be a smooth projective curve over $\mathbb{C}$, and D a reduced effective divisor on it. Let d be the degree of D. We denote by U the affine curve $X \setminus |D|$. If $p \in U$ then $z - p$ will be a local coordinate near it. We define the operator d^c to be $\frac{1}{4\pi i}(\partial - \overline{\partial})$; consequently, $dd^c = \frac{1}{2\pi i}\partial\overline{\partial}$.

3.1. Theorem. *Let $p \in U$. Then, up to an additive scalar, there exists a unique function $g_p : X \to [-\infty, +\infty]$ with the following properties:*

(a) *g_p satisfies the differential equation*

$$dd^c g_p = \delta_p - \frac{1}{d}\delta_D,$$

δ_p (resp. δ_D) being the Dirac operator at p (resp. on D).

(b) *g_p is a C^∞ function on $U \setminus \{p\}$.*

(c) *There is an open neighborhood V of p and a harmonic function v_p on V such that*

$$g_p|_V = \log|z - p|^2 + v_p.$$

This theorem has already been proved in [Ga] in a more general situation. We give here a sketch of proof in this case for the reader's convenience.

Proof. Fix a (Kähler) metric ω on X. Let $\Delta_{\overline{\partial}}$ be the associated Laplace operator. The operator $T := \delta_p - (1/d)\delta_D$ is orthogonal to the constants. Thus there is a $(1, 1)$ current α on X such that $\Delta_{\overline{\partial}}(\alpha) = T$. Since T is smooth on $X \setminus \{p, |D|\}$, the form α is also smooth there. The operator $L := \cdot \wedge \omega$ induces an isomorphism between $\mathcal{D}^{(0,0)}(X)$ and $\mathcal{D}^{(1,1)}(X)$ ($\mathcal{D}^{(i,i)}(X)$ being the space of (i, i) currents). Thus there is a function $\tilde{g}_p$ such that $L(\tilde{g}_p) = \alpha$. Since, for a suitable constant c, we have $dd^c(g) = cL(\Delta_{\overline{\partial}}(g))$, points (a) and (b) are easily deduced. Point (c) is similar.

The functions g_p are exhaustion functions in the sense of [GK]:

3.2. Lemma. *For every constant C, $g_p^{-1}((C,\infty])$ is a nonempty neighborhood of D in $\overline{X}$.*

Proof. Fix a metric $\|\cdot\|$ on $\mathcal{O}_X(D)$. Let $\mathbb{I}$ be the canonical section of $\mathcal{O}_X(D)$. By the Poincaré–Lelong equation, the function $g_p + (1/d)\log\|\mathbb{I}\|^2$ is smooth near D. The conclusion follows.

We will call such a g_p an *exhausting function for U and p*. Observe that an argument similar to the one above gives

3.3. Proposition. *Let p and q points on U. Let g_p and g_q be exhausting functions for U and p and q respectively. Then there is a constant $C_{p,q}$ and an open neighborhood V of D such that for every $z \in V$,*

$$|g_p(z) - g_q(z)| \leq C_{p,q}.$$

If $p \in U$, we fix a function g_p as in the theorem above. For every positive real number r, we consider the following two closed subsets of U:

$$B(r)_p := \{z \in U : g_p(z) \leq \log(r)\}, \qquad S(r)_p := \{z \in U : g_p(z) = \log(r)\}.$$

The function g_p is strictly related to the Green function on $B(r)$. We first recall the definition:

3.4. Definition. Let V be a regular region on a Riemann surface M and let $p \in V$. A *Green function for V and p* is a function $g_{V;p}(z)$ on V such that:

(a) $g_{V;p}(z)|_{\partial V} \equiv 0$ continuously;
(b) $dd^c g_{V;p} = 0$ on $U \setminus \{p\}$;
(c) near p, we have $g_{V;p} = -\log|z-p|^2 + \varphi$, with φ continuous around p.

One extends $g_{V;p}$ to all of V by defining $g_{V;p} \equiv 0$ outside the closure of V. We easily deduce from the definitions that $dd^c g_{V;p} + \delta_p = \mu_{\partial V;p}$ where $\mu_{\partial V;p}$ is a positive measure of total mass one and supported on ∂V. Moreover the following is true:

3.5. Proposition. *The Green function, if it exists, is unique.*

The following gives the relation between the function g_p and the Green functions on $B(r)_p$:

3.6. Proposition. *Let r be a positive real number. The function*

$$g_p^r := \log(r) - g_p|_{B(r)}$$

is the Green function for $B(r)_p$ and p. Consequently, for every p and q in U there is a constant C, depending on p and q, such that, for every r sufficiently large,

$$|g_p^r(q) - \log(r)| \leq C.$$

The proof follows from the definitions.

By the Stokes theorem, one can easily verify that, in this case, $\mu_{S(r);p}$ is the positive measure $d^c g_p|_{S(r)}$.

Let Z be a projective variety and L be an ample line bundle on it equipped with a positive metric. Denote by $c_1(L)$ its first Chern form.

Let $\gamma : U \to Z$ be an analytic map. We define the associated height function by

$$T_\gamma(r) := \int_0^r \frac{dt}{t} \int_{B(t)_p} \gamma^*(c_1(L)) = \int_U g_p^r \cdot \gamma^*(c_1(L)).$$

The *order of growth* of γ is defined to be

$$\limsup_{r\to\infty} \frac{\log T_\gamma(r)}{\log(r)}.$$

More generally, if M is an hermitian line bundle on U, we define

$$(M, U)(r) := \int_0^r \frac{dt}{t} \int_{B(t)_p} c_1(M) = \int_U g_p^r \cdot c_1(M).$$

Some remarks are in order (for the proofs, see for instance [Ga]):

• The order of growth is independent of the choice of the ample line bundle L and the metric on it, and of the choice of the point p.

• If γ is the inclusion in X, or more generally if γ is an algebraic map (cf. [GK]), then there is a constant C such that

$$\left|\frac{T_\gamma(r)}{\log(r)}\right| \le C. \tag{3.7.1}$$

• The Stokes and Poincaré–Lelong formulas give rise to the first main theorem: Let $Y \in H^0(U, M)$ be a global section. We define the *counting function* of Y: if $\mathrm{div}(Y) = \sum n_z z$ (the sum may be infinite), and for simplicity $p \notin \mathrm{div}(Y)$, then

$$N_Y(r) := \int_X g_p^r \cdot \delta_{\mathrm{div}(Y)} = \sum_{g_p^r(z)<\log(r)} n_z g_p^r(z).$$

The *First Main Theorem* (FMT) holds:

$$N_Y(r) - \int_{S(r)_p} \log \|Y\|^2 \mu_{S(r)_p} = (M, U)(r) + \log \|Y\|^2(p).$$

The term $-\int_{S(r)_p} \log\|Y\|^2 \mu_{S(r)_p}$ is often denoted by $m_Y(r)$ and called the *proximity function* of Y.

Let $E \to X$ be an hermitian vector bundle and $p : \mathbb{P} := \underline{\mathrm{Proj}}(\mathcal{O} \oplus E^\vee) \to X$ be the associated compactification. Let $\mathbb{M}$ be the tautological line bundle of $\mathbb{P}$; since E is hermitian, $\mathbb{M}$ is naturally equipped with the relative Fubini–Study metric. The surjection $\mathcal{O} \oplus E^\vee \to E^\vee$ defines an inclusion $\mathbb{P}(E) \hookrightarrow \mathbb{P}$ (the divisor at infinity) and the image is a global section of $\mathbb{M}$. It is well known that if M is a sufficiently ample line bundle on $\overline{X}$ then $\mathbb{M} \otimes p^*(M)$ is a very ample line bundle on $\mathbb{P}$.

Let $f : U \to E$ be an analytic section of E. It canonically defines an analytic map $f_{\mathbb{P}} : U \to \mathbb{P}$. By definition, the *order of growth* of $f_{\mathbb{P}}$ is

$$\limsup_{r\to\infty} \frac{\log\left(f_{\mathbb{P}}^*(\mathbb{M} \otimes p^*(M)), X)(r)\right)}{\log(r)}.$$

Observe that by (3.7.1) the order of growth of $f_{\mathbb{P}}$ is independent of M.

3.7. Definition. We define the *order of growth* of the section f to be the number

$$\rho := \limsup_{r\to\infty} \frac{\log\left(f_{\mathbb{P}}^*(\mathbb{M}), U)(r)\right)}{\log(r)}.$$

3.8. Lemma. *Suppose that f is a section of order strictly less than ρ. Then there is a constant C such that*

$$\int\limits_{S(r)_p} \log \|f\| \, \mu_{S(r)_p} \le C r^\rho.$$

Proof. Observe that $f_{\mathbb{P}}$ does not intersect $\mathbb{P}(E)$, and if $q \notin \mathbb{P}(E)$, then $\|\mathbb{P}(E)\|^2(q) = \frac{1}{1+\|q\|^2}$. Thus, by FMT, there is a constant C such that

$$(f_{\mathbb{P}}^*(\mathbb{M}), U)(r) = \frac{1}{2} \int\limits_{S(r)_p} \log(1 + \|f\|^2) \, \mu_{S(r)_p} + C.$$

The conclusion easily follows.

We will show that, given a section with finite order of growth, and two points, we can estimate the size of a related section at one point if we know that the section vanishes to a high order at the other point.

Fix two points p and q in U.

Suppose that E is an algebraic vector bundle over X. Fix an ample line bundle H on X. We suppose that E and H are equipped with smooth metrics. For every positive integer x, denote by $E(x)$ the vector bundle $E \otimes H^{\otimes x}$. Denote by $E^\vee$ the dual of E.

Fix an *analytic section* $f \in H^0(U, E)$ having order of growth ρ. For every $P \in H^0(X, E^\vee(x))$ denote by F the analytic section $\langle f, P\rangle \in H^0(U, H^{\otimes x})$.

We will show that one can bound the size of F at q in terms of the sup norm of P, the order of vanishing of F at p and the order of growth of f.

3.9. Theorem. *There exists a constant c_1 depending only on H, a constant c_2 depending only on f, and a constant c_3 depending only on p and q, for which the following holds: for every $x \gg 0$ and every section*

$P \in H^0(X, E^\vee(x))$, if A, B and b are positive constants such that

$$\log \sup\{\|P\|\} \le B, \qquad \mathrm{ord}_p(F) \ge Ax - b,$$

then

$$\log \|F\|(q) \le B - \frac{Ax}{\rho}\log(x) + (c_2 + c_3)x + \frac{c_1}{\rho} x \log(x).$$

Observe that the constant c_1 depends only on H.

Proof. By the Stokes formula, for every real number r,

$$\int_U \log \|F\| \cdot dd^c g_q^r = \int_U dd^c \log \|F\| \cdot g_q^r. \tag{3.10.1}$$

By the definition of Green function, the left hand side of (3.10.1) is

$$\int_{S(r)_q} \log \|F\| \, \mu_{S(r)_q} - \log \|F\|(q).$$

The Cauchy–Schwarz inequality implies that $\|F\| \le \|P\| \cdot \|f\|$; thus

$$\int_{S(r)_q} \log \|F\| \, \mu_{S(r)_q} \le \int_{S(r)_q} \log \|P\| \, \mu_{S(r)_q} + \int_{S(r)_q} \log \|f\| \, \mu_{S(r)_q}.$$

The hypotheses and Lemma 3.8 imply that $\int_{S(r)_q} \log \|F\| \, \mu_{S(r)_q}$ is bounded above by

$$B + c_2 r^\rho$$

where c_2 depends only on f. The fact that $\mathrm{ord}_{p_1}(F) \ge Ax - b$, Proposition 3.6, and the Poincaré–Lelong formula imply that the right hand side of (3.10.1) is surely greater than

$$(Ax - b)(\log(r) + c_3) - x(H, U)(r),$$

where c_3 depends only on p_1 and p_2. Since H is algebraic, with a metric smooth at infinity, the last term of this sum is surely lower bounded by $-x \cdot c_1 \log(r)$, for a suitable c_1 depending only on H. The conclusion follows by taking $r = x^{1/\rho}$.

4. Order of growth at finite places. In this section we will recall the definitions and principal properties of *LG*-germs. This notion is defined in [Ga] and developed there in a greater generality, and here we just recall it (and explain in the special situation we need) for the reader's convenience. The notion of *LG* the germ is similar to the notion of *E*-function developed by Siegel, Shidlovskiĭ and others. When we are dealing with *LG*-germs, we can estimate the order of growth of sections at all finite places at the same time. It is our opinion that the notions of *LG*-germs and of the order of growth of sections (or more generally of analytic maps) are two concepts which may be in contrast; and from this contrast we may deduce nontrivial results.

We fix some notations:

- K is a number field and O_K its ring of integers.
- We will denote by M_{fin} the set of finite places of K.
- If $v \in M_{\text{fin}}$, we denote by K_v the completion of K with respect to v and by O_v its ring of integers; we normalize the norm of K_v by the condition $\|p\|_v = p^{-1}$ (where p is the associated prime number). If M is an O_K-module, we denote by M_v the K_v-vector space $M \otimes K_v$ and by M_{O_v} the O_v-module $M \otimes_{O_K} O_v$.
- We fix a smooth projective curve X_K over K and an ample line bundle H_K over it.
- We fix a vector bundle E_K of rank m over X_K. Denote by $E_K^\vee$ the dual of E_K.
- Let $\mathcal{X} \to \operatorname{Spec}(O_K)$ be a regular projective model of X_K. We may suppose that H_K (resp. E_K) extends to a line bundle H (resp. to a vector bundle E) over $\mathcal{X}$. We will denote by $E^\vee$ the dual of E.
- For every integer x, denote by G_x the O_K-module $H^0(\mathcal{X}, E \otimes H^{\otimes x})$.
- If $p_K \in X_K(K)$ is a rational point, we may extend it to a section $p : \operatorname{Spec}(O_K) \to \mathcal{X}$. Denote by $\hat{X}_p$ the completion of X_K near p_K and by $\hat{\mathcal{X}}_p$ the completion of $\mathcal{X}$ near p.
- Denote by H_p, E_p etc. (resp. $H_{p,K}$, $E_{p,K}$ etc.) the restriction of H, E etc. to $\hat{\mathcal{X}}_p$ (resp. of H_K, E_K etc. to $\hat{X}_p$).
- Extending the base K if necessary, we may suppose that $\hat{\mathcal{X}}_p$ is isomorphic to $\operatorname{Spf}(O_K[\![Z]\!])$; we fix such an isomorphism.
- Since $\hat{\mathcal{X}}_p$ is an affine formal scheme, we may identify every coherent sheaf on it with the corresponding module of global sections. The $O_K[\![Z]\!]$-modules H_p and E_p are isomorphic to the trivial modules of the corresponding rank; we fix such isomorphisms.
- For every positive integer i, denote by $\hat{\mathcal{X}}_p^i$ (resp. $\hat{X}_p^i$) the ith infinitesimal neighborhood of p (resp. p_K) in $\mathcal{X}$ (resp. X_K). Similarly we denote by $H_{p,i}$, $E_{p,i}$ etc. the restriction of H, E etc. to $\hat{\mathcal{X}}_{p_j}^i$.
- Let $p_K \in X_K(K)$ and $p : \operatorname{Spec}(O_K) \to \mathcal{X}$ the corresponding section. The sheaf of Kähler differentials $\Omega^1_{\mathcal{X}/O_K}$ is locally free in a neighborhood of p, indeed the morphism $\mathcal{X} \to \operatorname{Spec}(O_K)$ is smooth near p. Denote by $T_p\mathcal{X}$ the restriction to p of the dual of that sheaf. For every place $v \in M_{\text{fin}}$, and couple of integers x and i, the O_v-module $H^0(p, H^{\otimes x} \otimes (T_p\mathcal{X})^{\otimes i})_v$ is equipped with the norm induced by the integral structure.

Let $p_K \in X_K(K)$ and $p : \operatorname{Spec}(O_K) \to \mathcal{X}$ the corresponding section. Let $f \in H^0(\hat{X}_p, E_{p,K}^\vee)$. Since we fixed an isomorphism of $E_p^\vee$ with $\mathcal{O}^m_{\hat{\mathcal{X}}_p}$, the section f can be written as m power series,

$$f = \Big(\sum_{i=1}^{\infty} a_i(1)Z^i, \ldots, \sum_{i=1}^{\infty} a_i(m)Z^i\Big)$$

with $a_i(j) \in K$.

4.1. Definition. Let α be a nonnegative real number. We will say that f is an *LG-germ* of type α if the following holds:

(a) For every place $v \in M_K$ and every $j = 1, \ldots, m$, the power series $\sum_i a_i(j)Z^i$ have positive radii of convergence.
(b) There is a finite set S of places such that if $v \notin S$ there is a constant C_v such that, for every $j = 1, \ldots, m$,

$$\|a_i(j)\|_v \leq \frac{C_v^i}{\|i!\|_v^\alpha}.$$

(c) $\prod_{v \notin S} C_v < \infty$.

Following the proofs of [Ga, §3], one may prove that:

• The notion of LG-germ of type α does not depend on the chosen isomorphisms; thus the notion depends only on the germ of section. However, the constants C_v may depend on the choices.

• If E is equipped with a connection which is nonsingular at p, then a formal horizontal section is an LG-germ of type 1.

• If moreover, for almost all $v \in M_K$, the connection has vanishing p-curvature, then the formal horizontal section is an LG-germ of type zero.

The last two statements are proved in [Bo, §3.4] (cf. also [A1] or [Bom]).

Fix s points $p_{1,K}, \ldots, p_{s,K}$ in $X_K(K)$. Denote by $p_j : \mathrm{Spec}(O_K) \to \mathcal{X}$ the corresponding sections. Suppose that for every point $p_{j,K}$ we have an LG-germ $f_j \in H^0(\hat{X}_{p_{j,K}}, E^\vee_{p_{j,K}})$ of type α. If we take a suitable blow up of $\mathcal{X}$, we may suppose that the p_j's extend to sections $p_j : \mathrm{Spec}(O_K) \to \mathcal{X}$ which do not intersect. For every j, the section f_j induces an O_K-linear map $\langle \cdot, f_j\rangle : G_x \to H^0(\hat{X}_{p_j}, H^{\otimes x}_{p_{j,K}})$, and by composition a map

$$\langle \cdot, f\rangle := (\langle \cdot, f_1\rangle, \ldots, \langle \cdot, f_s\rangle) : G_x \to \bigoplus_{j=1}^{s} H^0(\hat{X}_{p_j}, H^{\otimes x}_{p_{j,K}}).$$

Denote by $\mathrm{res}_i : \bigoplus_j H^0(\hat{X}_{p_j}, H^{\otimes x}_{p_{j,K}}) \to \bigoplus_j H^0(\hat{X}^i_{p_j}, H^{\otimes x}_{p_{j,i}}) \otimes K$ the restriction map, by $\langle \cdot, f\rangle_i$ the map obtained by composing $\langle \cdot, f\rangle$ with the res_i and by G^i_x the kernel of $\langle \cdot, f\rangle_i$.

The snake lemma applied to the exact sequence

$$0 \to \bigoplus_{j=1}^{s} H^0(p_j, H^{\otimes x} \otimes (T_{p_j}\mathcal{X})^{\otimes -i}) \to \bigoplus_{j=1}^{s} H^0(\hat{X}^{i+1}_{p_j}, H^{\otimes x}_{p_j,i+1}) \to \bigoplus_{j=1}^{s} H^0(\hat{X}^{i}_{p_j}, H^{\otimes x}_{p_j,i})$$

induces a canonical inclusion

$$\gamma^i_x : G^i_x/G^{i+1}_x \to \bigoplus_{j=1}^{s} H^0(p_j, H^{\otimes x} \otimes (T_{p_j}\mathcal{X})^{\otimes -i}) \otimes K.$$

For every $v \in M_{\mathrm{fin}}$ both $(G^i_x/G^{i+1}_x)_v$ and $\bigoplus_j H^0(p_j, H^{\otimes x} \otimes (T_{p_j}\mathcal{X})^{\otimes -i})_v$ are equipped with norms, induced by the integral structure. Observe that naturally the former has the sup norm. Thus we may compute the norm $\|\gamma^i_x\|_v$ of the operator γ^i_x.

When f is an LG-germ, one can bound the norms at all finite places of the γ^i_x's.

4.2. Theorem. *With the notations as above, suppose that f is an LG-germ of type α. Then there exists a constant C such that*

$$\sum_{v \in M_{\mathrm{fin}}} \log \|\gamma^i_x\|_v \leq [K:\mathbb{Q}]\alpha i \log(i) + C(i+x).$$

Proof. We first remark the following general statement: Let k be a normed field and $\varphi : V^1_k \to V^2_k$ be a linear map between finite-dimensional normed vector spaces over k. Let $V^i \subset V^i_k$ be the set of elements of norm less than or equal to one. Suppose that there exists a constant $A \in k^*$ such that $\varphi(Av) \subset V^2$ for every $v \in V^1$. Then $\|\varphi\| \leq 1/\|A\|$.

For every $j \in \{1, \dots, s\}$, let $\mathrm{pr}_j : \bigoplus_j H^0(p_j, H^{\otimes x} \otimes (T_{p_j}\mathcal{X})^{\otimes -i}) \to H^0(p_j, H^{\otimes x} \otimes (T_{p_j}\mathcal{X})^{\otimes -i})$ be the projection. Fix one of the p_j. Let $v \notin S$. The restriction of $\hat{\mathcal{X}}_{p_j}$ to $\mathrm{Spec}(O_v)$ is isomorphic to $\mathrm{Spf}(O_v[\![Z]\!])$ (via an isomorphism fixed as above).

Suppose that $P \in (G^i_x)_{O_v}$. Since we fixed an isomorphism of E_{p_j} with the trivial vector bundle of rank m, the restriction of P to $(\hat{\mathcal{X}}_{p_j})_v$ is represented by $(g_1, \dots, g_m)$ with $g_i \in O_v[\![Z]\!]$. By definition $\langle P, f_j\rangle = \sum_{s=1}^m g_s \sum_\ell a_\ell(s) Z^\ell = h_j(Z)$. Since $P \in (G^i_x)_{O_v}$, we have $h_j(Z) = \sum_{\ell=i}^\infty h_\ell Z^\ell$ and $\mathrm{pr}_j \circ \gamma^i_x(P) = h_i$. Since f_j is an LG-germ of type α, we have

$$\frac{\|i!\|^\alpha_v}{C^i_v}\|h_i\|_v \leq 1.$$

The norm on $\bigoplus_j H^0(p_j, H^{\otimes x} \otimes (T_{p_j}\mathcal{X})^{\otimes -i})$ is the sup of the norms on each factor, so for every $v \notin S$ we have

$$\frac{\|i!\|_v^\alpha}{C_v^i}\|\gamma_x^i(P)\|_v \leq 1.$$

The conclusion follows from the remark at the beginning of this proof, the Stirling formula and the standard Cauchy inequality at places in S.

5. E-sections of arithmetic type. In this section we will introduce the concept of E-sections of arithmetic type of a vector bundle over an affine curve. These are analytic sections of an algebraic vector bundle whose order of growth is the inverse of the type of their formal development at a fixed algebraic point. The main examples of E-sections of arithmetic type are the E-functions of the theory of Siegel–Shidlovskiĭ or the more recent "arithmetic series of order s" introduced by André [A1].

We fix the same notations as in the previous section. Moreover we denote by M_∞ the set of complex embeddings of K. If $\sigma \in M_\infty$ we will denote by $\overline{\sigma}$ the conjugate embedding. A subset $S_K \subseteq M_\infty$ is said to be *regular* if $\sigma \in S$ implies that $\overline{\sigma} \in S$.

Let X_K be a smooth projective curve over a number field K. Fix s points $p_{1,K}, \ldots, p_{s,K} \in X_K(K)$.

For every $\sigma \in M_\infty$, we will denote by X_σ, E_σ, H_σ etc. the restriction of X, E, H etc. to $\mathbb{C}$ via σ.

Let E_K be a vector bundle over X_K of rank m.

5.1. Definition. Let $S_K \subseteq M_\infty$ be a nonempty regular subset of cardinality a, and α a nonnegative real number. An *E-section $\tilde{f}$ of arithmetic type* of E with respect to S_K, α and the points $p_{1,K}, \ldots, p_{s,K}$ is the following data:

(a) for every $j = 1, \ldots, s$, a germ of section $f_j \in E_{p_{j,K}}$ which is an LG-germ of type α;

(b) for every $\sigma \in S_K$, an affine open subset U_σ of X_σ containing $p_{j,\sigma} = \sigma(p_j)$ and an analytic section $f_\sigma \in H^0(U_\sigma, E_\sigma)$ such that:

  (b.1) for every j, the germ of f_σ at $p_{j,\sigma}$ is $\sigma(f_j)$;

  (b.2) the section f_σ has finite order of growth ρ_σ and $\alpha\rho_\sigma = as/[K:\mathbb{Q}]$ (in particular, if $\alpha = 0$, then it suffices that the order of growth is finite).

5.2. Remark. (a) An E-function in the sense of Siegel–Shidlovskiĭ is an E-section of arithmetic type; in this case, we have only one point, $S_K = M_\infty$ and the α involved is one.

(b) An "arithmetic Gevrey series of order $s < 0$" in the sense of André [A1] is an E-section of arithmetic type; again we only have one point, $S_K = M_\infty$ and the α involved is $-s$.

(c) If L/K is a finite extension and $f \in E_{p,K}$ is an E-section of type α over K, then $f \in E_{p,L}$ is an E-section of arithmetic type: take as S_L the set of τ such that τ/σ for $\sigma \in S_K$.

(d) Notice that, on the projective line, the main differences between E-functions and E-sections of arithmetic type are: (1) E-sections of arithmetic type may have order of growth not one; (2) (more important) E-sections of arithmetic type may have more than one essential singularity, whereas E-functions are always entire functions.

(e) An interesting example of E-section of arithmetic type (and our main theorem will concern that example) is given by a horizontal section of a fiber bundle with a meromorphic connection having order of growth ρ and assuming rational values at ρ nonsingular rational points; at each of the rational points the section will be an LG-germ of type 1. In the introduction we gave some way to construct some of these examples. Unfortunately we do not know examples which do not come from some covering construction and with more than one rational value at a rational point.

(f) When $K = \mathbb{Q}$, by Corollary 1.2 the order of growth cannot be less than s. It is not difficult to find a similar lower bound for arbitrary number fields (because Theorem 1.1 of [Ga] holds in general).

In this section we show that, given an E-section of arithmetic type, it is possible to construct sections with high order of vanishing and bounded sup norm.

First of all we have to fix integral structures (we refer to [BGS] for precise definitions and properties). As in the previous section, we suppose that $\mathcal{X} \to \mathrm{Spec}(O_K)$ is a regular projective model of X_K, and E_K extends to a vector bundle E over $\mathcal{X}$. We also suppose that H is a relatively ample line bundle on $\mathcal{X}$. For every place $\sigma \in M_\infty$, we suppose that E_σ and H_σ are equipped with smooth metrics (and the metric on H is sufficiently positive). We also fix metrics on X_σ. Thus, for every integer x, the vector bundle $E^\vee(x)$ is an hermitian vector bundle over $\mathcal{X}$.

For every integer x, the O_K-module $H^0(\mathcal{X}, E^\vee(x))$ is equipped with a structure of *hermitian O_K-module*: for every $\sigma \in M_\infty$, $H^0(X_\sigma, E^\vee(x)_\sigma)$ is equipped with the L^2 metric (notice that the L^2 norm and the sup norm are comparable by for instance [Bo, §4.1]). As in the previous section, we will denote this module by G_x.

Fix an E-section $\tilde{f}$ of arithmetic type with respect to the points $p_{1,K}, \ldots, p_{s,K}$.

With the notations of the previous section, for every positive integer i, we obtain a natural O_K-linear map $G_x \to \bigoplus_j H^0(\hat{X}^i_{p_j}, H^{\otimes x}_{p_{j,i}}) \otimes K$. Again denote by G^i_x its kernel. Put $c := \deg(H_K)$. We want to prove that, under these conditions, for every $\epsilon \in (0,1)$ there is a nonvanishing section of bounded norm in $G_x^{x\frac{c}{s}m(1-\epsilon)}$.

5.3. Theorem. *Suppose that the above hypotheses hold. Fix $\epsilon \in (0,1)$. Then we can find a constant c_1, depending only on ϵ, α and the points p_j, but independent of the vector bundle E, and a constant c_2, for which the following holds: for every sufficiently large positive integer x there is a nonzero section $P \in G_x^{x\frac{c}{s}m(1-\epsilon)}$ such that*

$$\sup_{\sigma\in M_\infty} \log \|P\|_\sigma \le c_1 x \log(x) + c_2 x.$$

Before we start the proof, we need to recall some classical tools from Arakelov geometry (cf. for instance [BGS]).

• If M is an hermitian line bundle over $\mathrm{Spec}(O_K)$ we will define its *Arakelov degree* by

$$\widehat{\deg}(M) := \log \mathrm{Card}(M/s\cdot O_K) - \sum_{\sigma\in M_\infty} \log \|s\|_\sigma.$$

for any $s \in M \setminus \{0\}$. This quantity is well defined because of the product formula.

• If E is an arbitrary hermitian vector bundle over $\mathrm{Spec}(O_K)$ then the line bundle $\bigwedge^{\max} E$ is canonically equipped with an hermitian metric; consequently, we can define the hermitian line bundle $\bigwedge^{\max} E$. We then define $\widehat{\deg}(E) := \deg(\bigwedge^{\max}(E))$.

• Suppose that E_1 is an hermitian O_K-module and $L_1,\ldots,L_s$ are hermitian line bundles over $\mathcal{O}_K$. Let $\varphi : E_1 \to \bigoplus_j L_j \otimes K$ be an injective linear map. For every place $v \in M_K$ (finite or infinite) we denote by $\|\varphi\|_v$ the norm of φ. One easily finds that if φ is nonzero, then

$$\widehat{\deg}(E_1) \le \mathrm{rk}(E_1)\Big(\sup_j \widehat{\deg}(L_j) + \sum_{v\in M_K} \log\|\varphi\|_v\Big).$$

• There exists a constant $\chi(K)$ depending only on K such that the following holds: Suppose that E is an hermitian O_K-module with $\widehat{\deg}(E) \ge A$. Then there exists a *nonzero* element $x \in E$ such that

$$\sup_{\sigma\in M_\infty} \|x\|_\sigma \le -\frac{A}{\mathrm{rk}(E)} + \log(\mathrm{rk}(E)) + \chi(K)$$

(cf. [BGS, Thm. 5.2.4 and below]).

• If x is sufficiently large, we may suppose that $\widehat{\deg}(G_x) \ge 0$.

Proof of Theorem 5.3. As in the previous section, for every integer i, we have an injective map

$$\gamma_x^i : G_x^i / G_x^{i+1} \to \bigoplus_{j=1}^{s} H^0(p, H^{\otimes x} \otimes (T_{p_j}\mathcal{X})^{\otimes -i}) \otimes K. \tag{5.3.1}$$

Note that since the G_x^i's are submodules of G_x, they are naturally hermitian O_K-modules. From the properties listed above we find that we can find a constant A depending only on p and H such that

$$\widehat{\deg}(G_x^{i+1}) \geq \widehat{\deg}(G_x^i) - \mathrm{rk}(G_x^i/G_x^{i+1})\Big(A(x+i) + \sum_{v \in M_K} \log \|\gamma_x^i\|_v\Big).$$

Since $\tilde{f}$ is an E-section of arithmetic type, by Theorem 4.2, there are constants c_3 and α such that

$$\sum_{v \in M_{\mathrm{fin}}} \log \|\gamma_x^i\|_v \leq [K:\mathbb{Q}]\alpha i \log(i) + c_3(i+x).$$

Let S be the set of infinite places involved in the definition of E-section of arithmetic type. By the classical Cauchy inequality there is a constant C such that if σ is an infinite place not contained in S, then

$$\log \|\gamma_x^i\|_\sigma \leq C(x+i).$$

In order to estimate the norm at places of S we need a refinement of Theorem 3.9.

Let $\sigma \in S$. Let $j \in \{1, \ldots, s\}$. We equip the line bundle $\mathcal{O}_{U_\sigma}(p_j)$ with the following metric: Let $\mathbb{I}_{p_j}$ be the canonical section of $\mathcal{O}_{U_\sigma}(p_j)$. We define $\|\mathbb{I}_{p_j}\|(z) = \exp(\frac{1}{2} g_{p_j}(z))$. By adjunction, this defines a norm on $T_{p_j}\overline{X}_\sigma$. Let $s \in (G_i)_\sigma$; then $f_\sigma(s)\prod_j \mathbb{I}_{p_j}^{-i}$ is a holomorphic section $\tilde{F}$ of $(H^{\otimes x}(-\sum_j ip_j))_\sigma$. To compute the norm of γ_x^i at places in S_K we have to compare $\|\tilde{F}\|(p_j)$, for every j, with $\|s\|_\infty$.

By the Stokes formula we find, for every real number r,

$$\int_{U_\sigma} \log \|\tilde{F}\| \cdot dd^c g_p^r = \int_{U_\sigma} dd^c \log \|\tilde{F}\| \cdot g_p^r.$$

By Proposition 3.3, for $r \gg 0$, we may suppose that if $g_{p_j}(z) \geq r$ then $g_{p_{j_i}}(z) = g_{p_j}(z)(1+\epsilon(z))$ with $|\epsilon(z)| \leq \epsilon$, where ϵ is a fixed constant. Thus, by the property of the Green functions (cf. §3), the definition of the norm on $\mathcal{O}(p_j)$, the Cauchy–Schwarz inequality and the Poincaré–Lelong formula we find

$$\log \|s\|_\infty + \int_{S(r)} \log \|f_\sigma\|\, d^c g_p - is(1-\epsilon)\log(r) - \log \|\tilde{F}\|(p) \geq -x(H, U_\sigma)(r).$$

Thus, we can find constants C and $\epsilon > 0$, depending only on H, and a constant λ_σ depending on f, such that, as soon as $r \gg 0$,

$$\log \|\gamma_x^i\|_\sigma \leq -is(1-\epsilon)\log(r) + xC\log(r) + \lambda_\sigma r^{\frac{as}{[K:\mathbb{Q}]\alpha}(1+\epsilon)}.$$

For each i we put $r = i^{\alpha[K:\mathbb{Q}]/(as(1+\epsilon))}$ and we deduce that there are constants C_1, C_2 and ϵ_1, with C_1 depending only on α and H, and ϵ_1 as small as we want, in particular independent of E, i and x, such that

$$\sum_{v\in M_K} \log \|\gamma_x^i\|_v \le xC_1 \log(i) + \epsilon_1 i \log(i) + C_2(i+x).$$

Observe that $\mathrm{rk}(G_x^i/G_x^{i+1}) \le s$, so we can find constants C_i, with C_4 depending only on H, s and α, such that, summing all together, we obtain

$$\widehat{\deg}(G_x^{x\frac{c}{s}m(1-\epsilon)}) \ge C_3x^2 - C_4\Big(\sum_{i=1}^{x\frac{c}{s}m(1-\epsilon)} x\log(i) + \epsilon_1 i\log(i) + C_2(i+x)\Big).$$

Thus, since we can take ϵ_1 very small compared to m^2, there are constants C_6 and C_7, with C_6 independent of E, and C_7 depending on m, f and H, such that

$$\widehat{\deg}(G_x^{x\frac{c}{s}m(1-\epsilon)}) \ge -(C_6mx^2\log(x) + C_7x^2).$$

By the Riemann–Roch theorem, $\mathrm{rk}(G_x)$ is about cmx. By the filtration (5.3.1), the rank of $G_x^{x\frac{c}{s}m(1-\epsilon)}$ is greater than ϵmx. Consequently, there is a *nonzero section* P of $G_x^{x\frac{c}{s}m(1-\epsilon)}$ such that

$$\sup_{\sigma\in M_\infty} \log\|P\|_\sigma \le \frac{mC_8}{m\epsilon} x\log(x) + Cx.$$

The conclusion follows, since C_8 depends only on the p_j, H and is independent of E.

Fix s rational points $p_{1,K},\dots,p_{s,K}$ of $X_K(K)$ and a fiber bundle (E,∇) with a meromorphic connection as above. Fix an E-section $\tilde{f}$ of arithmetic type with respect to these points which is horizontal for the connection. Theorem 5.3 gives rise to an integral global section P. We may take derivatives of P with respect to the connection and we obtain other sections with the same properties:

First of all we have to ensure that the derivative of an integral section is again an integral section. We may extend the connection $\nabla : E^\vee(x)_K \to E^\vee(x)_K \otimes \Omega^1_{\hat{X}}(D_K)$ to an *integral connection*

$$\nabla : E(x) \to E \otimes \omega_{\mathcal{X}/O_K}(D+V)$$

where V is a vertical divisor. We also fix an integral element $\partial \in T_{\mathcal{X}/O_K}(D)$ which, generically, does not vanish at the p_i's. By construction, if $P \in H^0(\mathcal{X}, E^\vee(x))$, then $\nabla_\partial(P) \in H^0(\mathcal{X}, E^\vee(V)(x+2))$; in particular, $\nabla_\partial(P)$ is a section over the model $\mathcal{X}$. For every point $p_{j,K}$, the order of vanishing of $\langle\nabla_\partial(P), f_j\rangle$ at $p_{j,K}$ is one less than the order of vanishing of $\langle P, f_j\rangle$ at $p_{j,K}$. Moreover a straightforward application of the classical Cauchy–Schwarz inequality implies that, for every complex embedding σ, the linear map $(\nabla_\partial)_\sigma : E^\vee(x)_\sigma \to (E^\vee_\sigma(x+2))_\sigma$ has bounded norm. Thus we proved:

5.4. Proposition. *There is a constant A depending only on (E,∇) and ∂ such that the following holds: if $P \in H^0(\mathcal{X}, E^\vee(x))$ is an integral section such that* $\sup_\sigma \log \|P\|_\sigma \le C$ *and* $\mathrm{ord}_{p_j}(\langle P, f_j\rangle) \ge C_1$ *for every j then:*

(i) $\nabla_\partial(P)$ *is a section of* $E^\vee(V)(x+2)$ *over* $\mathcal{X}$ *such that*

$$\sup_\sigma \log \|\nabla_\partial(P)\|_\sigma \le C + A;$$

(ii) $\mathrm{ord}_{p_j}(\langle \nabla_\partial(P), f_j\rangle) \ge C_1 - 1$ *for every j.*

6. Proof of the main theorem. In this section we will show how to generalize the Siegel–Shidlovskiĭ theory to an arbitrary curve and to a connection with arbitrary meromorphic singularities and E-sections of arithmetic type over an arbitrary set of points.

6.1. Theorem. *Let X_K be a smooth projective curve defined over the number field K. Let D_K be an effective divisor on X_K and (E_K, ∇_K) be a vector bundle of rank $m > 1$ with connection with meromorphic singularities on D_K. Let $p_{1,K}, \ldots, p_{s,K} \in X(K)$ be rational points. Let $\tilde f$ be a* Zariski dense horizontal section *which is an E-section of arithmetic type with respect to the $p_{j,K}$'s, some α and some regular subset $S_K \subseteq M_\infty$. Let $\sigma \in S_K$. Suppose that $q \in X_K(K) \setminus \{D, p_1, \ldots, p_s\}$. Then*

$$\mathrm{Trdeg}_K(K(f_\sigma(\sigma(q)))) = m.$$

6.2. Remark. (a) By "$\tilde f$ is a horizontal section" we mean that all the local sections $f_1, \ldots, f_s$ (or, equivalently, all the sections f_σ, $\sigma \in S_K$) of E, involved in the definition of E-section $\tilde f$ of arithmetic type, are formal (analytic) horizontal for the connection ∇^E.

(b) $\tilde f$ being Zariski dense means that the image of none of the formal local sections $\sigma(f_1), \ldots, \sigma(f_s)$ is included in a proper Zariski closed subset of E_σ. This is equivalent to requiring that the image of the section f_σ is Zariski dense.

(c) If we apply the theorem to $\mathbb{P}^1$ with $s = 1$ and we suppose that the horizontal section is an E-function, we find the classical theorem of Siegel and Shidlovskiĭ (cf. [La]).

Before we give the proof of the theorem we will produce a metrical criterion which implies that the coordinates of an element of a complex vector space are algebraically independent. This criterion is a version in metric language of a classical trick by Siegel.

A criterion for transcendence. As before, K will be a number field and O_K will be its ring of integers. We fix an embedding $\sigma : K \to \mathbb{C}$. Let E be an hermitian O_K-module of rank m. If V is an hermitian O_K-module, denote by V_K the K-vector space $V \otimes_{O_K} K$ and by $V_\mathbb{C}$ the $\mathbb{C}$-vector space $V \otimes \mathbb{C}$ ($\mathbb{C}$ is an O_K-module via σ). We describe here a criterion which implies that the coordinates of an element $f \in E_{\mathbb{C}}$ satisfying it are algebraically independent over K.

For every integer n, we denote by E_n the hermitian O_K-module $\mathrm{Sym}^n(E)$ and by m_n its rank.

Let $f \in E_{\mathbb{C}}$. Denote by f_n the image of $f^{\otimes n}$ in $(E_n)_{\mathbb{C}}$. For every n denote by V_n the smallest K-subspace containing f_n, and by r_n its dimension.

6.3. Definition. We will say that f is *algebraically independent* over K if $V_n = E_n$ for every positive integer n.

6.4. Remark. If we fix a basis of E_K, then f is algebraically independent over K if its coordinates are transcendental numbers algebraically independent over $\mathbb{Q}$.

We fix an hermitian line bundle H over $\mathrm{Spec}(O_K)$. If M is an hermitian O_K-module, for every integer x we denote by $M(x)$ the hermitian vector bundle $F \otimes H^{\otimes x}$. For $P_i \in E^{\vee}(x)$ we denote by F_i the vector $\langle P_i, f\rangle \in H_{\mathbb{C}}^{\otimes x}$.

The criterion we want to prove is the following:

6.5. Proposition. *Let $f \in E_{\mathbb{C}}$ be as above. Suppose that we can find positive constants c_i for which the following holds: For every n we can find $x_0(n)$ such that for all $x \geq x_0(n)$ there exist sections $P_1^n(x), \dots, P_{m_n}^n(x) \in E_n(x)$ and constants $b_i = b_i(n)$ such that:*

- *The $P_i^n(x)$ are linearly independent.*
- $\sup_{\sigma \in M_\infty} \log \|P_i^n(x)\| \leq c_1 x \log(x) + b_1 x.$
- *Denoting $\langle P_i^n(x), f_n\rangle \in H_{\mathbb{C}}^{\otimes x}$ by $F_i^n(x)$, we have*

$$\sup_{\sigma \in M_\infty} \log \|F_i^n(x)\| \leq c_1 x \log(x) - c_2 m_n x \log(x) + b_2 x.$$

Then f is algebraically independent over K.

First of all we observe the following trivial fact:

• Suppose that V_1 and V_2 are vector spaces and $\dim(V_2) < \dim(V_1)$. Then

$$\lim_{n \to \infty} \frac{\dim(\mathrm{Sym}^n(V_2))}{\dim(\mathrm{Sym}^n(V_1))} = 0.$$

The proof is trivial and left to the reader.

6.6. Lemma. *The vector f is algebraically independent over K if and only if there is a constant $c > 0$ such that for every n,*

$$\frac{\dim(V_n)}{\dim(E_n)} \geq c.$$

Proof. If f is algebraically independent over K then, by definition we may take $c = 1$.

Conversely, suppose that f is *not* algebraically independent over K; then there is an n and a nontrivial subspace $V_n \subsetneq E_n$ containing f_n. Thus for every integer m, $f_{nm} \in \mathrm{Sym}^m(V_n) \subsetneq E_{nm}$. Consequently, there is a subsequence n_m such that

$$\lim_{m\to\infty} \frac{\dim(V_{n_m})}{\dim(E_{n_m})} = 0.$$

The conclusion follows.

Proposition 6.5 will be a consequence of Lemma 6.6 above and the following proposition applied to $f_n \in (E_n)_{\mathbb{C}}$:

6.7. Proposition. *Let E be an hermitian O_K-module and $f \in E_{\mathbb{C}}$ as above. Suppose that we can find constants c_i and b_j for which the following holds: For every x sufficiently large, there exist $P_1, \dots, P_m \in E^\vee(x)$ such that:*

- *$P_1, \dots, P_m$ are linearly independent.*
- *$\sup_{\sigma\in M_\infty} \log \|P_i\|_\sigma \le c_1 x \log(x) + b_1 x$.*
- *$\sup_{\sigma\in M_\infty} \log \|F_i\|_\sigma \le c_1 x \log(x) - c_2 m x \log(x) + b_2 x$.*

Then there are constants C_i depending only on the c_i's *such that*

$$r_1 \ge C_1 m + C_2.$$

Proof. Denote by $V_K \hookrightarrow E_K$ the minimal K-subspace containing f. Let $V := V_K \cap E$; then $r_1 = \mathrm{rk}(V)$. For every positive integer x, denote by $\tilde{P}_i$ the image of P_i in $V^\vee(x)$. Observe that there are constants d_j such that $\widehat{\deg}(V^\vee(x)) = d_1 + d_2 x$. We can find r_1 elements among the $\tilde{P}_i$ which are linearly independent; we may suppose that they are $\tilde{P}_1, \dots, \tilde{P}_{r_1}$. The isomorphism of the Cramer rule, which is an isometry, gives rise to the equality

$$(\tilde{P}_1 \wedge \cdots \wedge \tilde{P}_{r_1}) \otimes f = \sum_i (-1)^i (\tilde{P}_1 \wedge \cdots \wedge \hat{\tilde{P}}_i \wedge \cdots \wedge \tilde{P}_{r_1}) \otimes F_i.$$

Since $\tilde{P}_1 \wedge \cdots \wedge \tilde{P}_{r_1}$ is an *integral* section of $V^\vee(x)$, we have

$$\log \|\tilde{P}_1 \wedge \cdots \wedge \tilde{P}_{r_1}\|_\sigma \ge d_1 + d_2 x - ([K:\mathbb{Q}] - 1)(r_1 c_1 x \log(x) + b_1 x).$$

Thus we find

$$\begin{aligned} &d_1 + d_2 x - ([K:\mathbb{Q}] - 1)(r_1 c_1 x \log(x) + b_1 x) + d_3 \\ &\quad \le (r_1 - 1) c_1 x \log(x) + c_1 x \log(x) - m c_2 x \log(x) + c_3 x \log(x) + b_2 x + d_4, \end{aligned}$$

where the constants b_i, c_i and d_i are independent of x. We divide everything by $x\log(x)$ and let x tend to infinity to obtain

$$r_1 c_1 [K:\mathbb{Q}] - m c_2 + c_3 \ge 0.$$

The conclusion follows.

Proof of the theorem. We fix models $\mathcal{X}$ of X_K, D of D_K and (E,∇) of (E_K,∇_K) as in the previous sections. We also fix a positive metric on the ample line $H := \mathcal{O}(D)$. Let c be the degree of H_K; adding some points to D if necessary we may suppose that c is much greater than s. We eventually fix an integral derivation $\partial \in H^0(\mathcal{X}, (\omega^1_{\mathcal{X}/O_K})^\vee(D))$ which does not vanish at the points p_j and q; notice that this can be done once we suppose that c much greater compared to s. Finally, we fix a section $s' \in H^0(\mathcal{X}, H)$ not vanishing at $p_{i,K}$ or q.

In order to apply 6.5, it suffices to replace E by one of its symmetric products, f by its symmetric power, and construct sections satisfying the hypothesis of Proposition 6.7. Thus we need m linearly independent sections of $E^\vee(x)_q$ satisfying the hypotheses of 6.7.

By Theorem 5.3, for every $x \gg 0$ we may construct $P_1 \in H^0(\mathcal{X}, E^\vee(x))$ such that, denoting by $F_1 \in \bigoplus_j H^0(X_{p_{j,K}}, \mathcal{O}(x))$ the section $\langle P_1, f_{p_{j,K}}\rangle_j$, for every j we have $\mathrm{ord}_{p_j}(F_1) \geq x\frac{c}{s}m(1-\epsilon)$ and $\sup_\sigma \log \|P_1\|_\sigma \leq ax\log(x)+c_1x$.

Let $P_i := \nabla_\partial(P_{i-1})$. Applying Proposition 5.4, we then construct m integral sections $P_1,\ldots,P_m$ such that $\sup_\sigma \log\|P_i\|_\sigma \leq ax\log(x)+c_2x$ and $\mathrm{ord}_p(F_i) \geq x\frac{c}{s}m(1-\epsilon)-m$ (where again $F_i = \langle P_i, f_{p_{j,K}}\rangle_j$).

Since we suppose that $c > s$ and $x \gg 0$, we may apply the Zero Lemma 2.5 to deduce that $P_1,\ldots,P_m$ are linearly independent over $K(X_K)$. As in the previous sections, we denote by $\tilde{P}_i$ the sections $P_i \otimes (s')^{\otimes 2(m-i)}$. Observe that $\tilde{P}_i \in H^0(\mathcal{X}, E^\vee(x+2m+V))$ for some fixed vertical divisor V. Consequently, there is a constant c_3 such that

$$\deg(\tilde{P}_1 \wedge \cdots \wedge \tilde{P}_m) = mcx + c_3.$$

By the Cramer rule,

$$(\tilde{P}_1 \wedge \cdots \wedge \tilde{P}_m) \otimes f = \sum_i (-1)^i (\tilde{P}_1 \wedge \cdots \wedge \hat{\tilde{P}}_i \wedge \cdots \wedge \tilde{P}_m) \otimes F_i;$$

thus, for every j, $\mathrm{ord}_{p_j}(\tilde{P}_1 \wedge \cdots \wedge \tilde{P}_m) \geq x\frac{c}{s}m(1-\epsilon)-m$; consequently, there are constants ϵ_1 and c_4 such that

$$\mathrm{ord}_q(\tilde{P}_1 \wedge \cdots \wedge \tilde{P}_m) \leq cx\epsilon_1 + c_4.$$

Fix c, ϵ_1 and c_4 as above and denote by $c(x)$ the function $c\epsilon_1 x + c_4$. For $P \in H^0(\mathcal{X}, E^\vee(x))$ with $x \gg 0$ as above, we construct a sequence $\overline{P}_1 = P \otimes (s')^{c(x)}$ and $\overline{P}_{i+1} := \nabla_\partial(P_i) \otimes (s')^{\otimes c(x)-2(i+1)}$ with $0 \leq 2i \leq c(x)-1$. Observe that $\overline{P}_1,\ldots,\overline{P}_{c(x)}$ are global sections in $H^0(\mathcal{X}, E^\vee(c(x)+x+V))$.

6.8. Lemma. *With the notations as above, there are constants a_i independent of E (in particular independent of m) and constants b_j for which the following holds: For every $x \gg 0$ there exist m indices $\ell_1 < \cdots < \ell_m$ with $\ell_m \leq c(x)$ such that:*

(a) $\overline{P}_{\ell_1} \wedge \cdots \wedge \overline{P}_{\ell_m}$ *is an integral section* not vanishing at q*;*
(b) $\sup_\sigma \log \|\overline{P}_{\ell_i}\|_\sigma \leq a_1 x \log(x) + b_1$*;*
(c) $\mathrm{ord}_{p_j}(F_{\ell_i}) \geq a_2 x(m-1) - b_2$ *for every* j*.*

The lemma implies the theorem: indeed, by Theorem 3.9, the $\overline{P}_{\ell_i}$ satisfy the hypotheses of Proposition 6.7.

Proof of Lemma 6.8. The only thing we have to prove is (a); indeed, (b) and (c) are consequences of Proposition 5.4.

Let v be the order of vanishing of $\tilde{P}_1 \wedge \cdots \wedge \tilde{P}_m$ at q. We know that $v \leq c(x)$. By induction we see that if $h_1 < \cdots < h_m$, then

$$\nabla_\partial(P_{h_1} \wedge \cdots \wedge P_{h_m}) = \sum_{s_1 < \cdots < s_m} t_s(\overline{P}_{s_1} \wedge \cdots \wedge \overline{P}_{s_m})$$

with $s_i \leq h_m + 1$ and the functions t_s vanishing at q. Consequently, denoting by $\nabla_\partial^{\circ v}(\cdot)$ the operator $\nabla_\partial \circ \cdots \circ \nabla_\partial(\cdot)$ (v times), we find that

$$0 \neq \nabla_\partial^{\circ v}(\tilde{P}_1 \wedge \cdots \wedge \tilde{P}_m)|_q = \sum_{s_1 < \cdots < s_m} a_s(\overline{P}_{s_1} \wedge \cdots \wedge \overline{P}_{s_m})|_q$$

with $s_m \leq c(x)$. Thus there exist $\ell_1 < \cdots < \ell_m \leq m + cx + a$ such that

$$(\overline{P}_{\ell_1} \wedge \cdots \wedge \overline{P}_{\ell_m})|_q \neq 0.$$

The conclusion follows.

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