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Sturm type theorem for Siegel modular forms of genus 2 modulo p

by

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1. Introduction and statement of main results. Despite extensive research on congruence properties of the Fourier coefficients of elliptic modular forms, surprisingly little is known about congruence properties of Siegel modular forms of higher genus. In this note, we consider congruences for the Fourier coefficients of Siegel modular forms of genus 2, studied by Sturm [15] in the case of an elliptic modular form.

Let $f = \sum_{n=0}^{\infty} a_f(n)q^n \in \mathcal{O}_L[[q]]$ be an elliptic modular form of weight k on a congruence subgroup $\Gamma^{(1)}$ of $\mathrm{SL}_2(\mathbb{Z})$, where $\mathcal{O}_L$ denotes the ring of integers of a number field L. Sturm proved in [14] that if β is a prime ideal of $\mathcal{O}_L$ for which $a_f(n) \equiv 0 \pmod{\beta}$ for $0 \le n \le \frac{k}{12}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma^{(1)}]$, then $a_f(n) \equiv 0 \pmod{\beta}$ for every $n \ge 0$. This result is called the *Sturm formula*. In fact, the Sturm formula implies that all the Fourier coefficients of f modulo β are determined by the first $\frac{k}{12}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma^{(1)}]$ coefficients. We note that, using the ring structure of weak Jacobi forms, the first two authors derived in [4] an analogue of the Sturm formula for Jacobi forms.

Our first main theorem is an analogue of the Sturm formula for Siegel modular forms of genus 2. Note that Poor and Yuen [12] gave a result in this direction which is more general, but different (see Remark 1.2(2)). Our result was obtained independently.

We fix some notations. Let $p \ge 5$ be a prime. Let Γ_2 be the full Siegel modular group $\Gamma_2 = \mathrm{Sp}_2(\mathbb{Z})$ and Γ be a congruence subgroup of Γ_2 with level N. Let $M_k(\Gamma)$ be the space of Siegel modular forms of weight k on Γ and let $S_k(\Gamma)$ be the space of cusp forms in $M_k(\Gamma)$. If all the Fourier coefficients of $F \in M_k(\Gamma)$ are p-integral rational and zero modulo p, then

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we write $F \equiv 0 \pmod{p}$. With these notations we state our analogue of the Sturm formula for a Siegel modular form of genus 2.

THEOREM 1.1. *Let k be an even positive integer. Suppose that F is a Siegel modular form in $M_k(\Gamma)$ with p-integral rational coefficients having the form*

$$F(\tau, z, \tau') = \sum_{\substack{n,m \in \frac{1}{N}\mathbb{Z},\, r \in \frac{1}{2N}\mathbb{Z} \\ n,m,nm-r^2 \geq 0}} A(n,r,m) q^n \xi^{2r} q'^m,$$

with $q = e^{2\pi i \tau}$, $\xi = e^{2\pi i z}$, $q' = e^{2\pi i \tau'}$ and $\left(\begin{smallmatrix} \tau & z \\ z & \tau' \end{smallmatrix}\right)$ in the Siegel upper half-plane $\mathcal{H}_2$ of degree 2, where $\mathcal{H}_2 := \left\{\left(\begin{smallmatrix} \tau & z \\ z & \tau' \end{smallmatrix}\right) \in M_2(\mathbb{C}) \;\middle|\; \mathrm{Im}\left(\begin{smallmatrix} \tau & z \\ z & \tau' \end{smallmatrix}\right) > 0\right\}$. If $A(n,r,m) \equiv 0 \pmod{p}$ for every n, m such that

$$0 \leq n \leq \frac{k}{10}[\Gamma_2 : \Gamma] \quad \text{and} \quad 0 \leq m \leq \frac{k}{10}[\Gamma_2 : \Gamma],$$

then $F \equiv 0 \pmod{p}$.

REMARK 1.2. (1) The bounds in Theorem 1.1 are sharp for $\Gamma = \Gamma_2$; this will be discussed in Section 3.1.

(2) The results of Poor and Yuen [12] are stated in terms of the dyadic trace. For example, when $\Gamma = \Gamma_2$, their results imply the following:

THEOREM ([12]). *Let $F = \sum_T a(T) e^{\pi i \,\mathrm{tr}(TZ)} \in M_k(\Gamma_2)$ be a Fourier expansion of F. Here T runs over all even, symmetric and semi-positive definite 2×2 matrices. Suppose that $a(T) \in \mathbb{Z}_{(p)}$ and $a(T) \equiv 0 \pmod{p}$ for all T with dyadic trace $w(T) \leq k/3$ (note that the bound will be $k/6$ if T is taken as the form of a half integral matrix). Then $F \equiv 0 \pmod{p}$.*

(3) As mentioned above, our bounds are sharp for $\Gamma = \Gamma_2$. However, since the statements in Theorem 1.1 and [12] are different, it seems hard to tell which gives a better bound. Now, we compare the number of Fourier coefficients required to confirm $F \equiv 0 \pmod{p}$, based on the following examples:

Number of $T = \left(\begin{smallmatrix} 2n & r \\ r & 2m \end{smallmatrix}\right) \geq 0$ $(n \leq m)$

k	$w(T) \leq k/3$	$2n, 2m \leq k/10$	k	$w(T) \leq k/3$	$2n, 2m \leq k/10$
2	1	1	14	12	7
4	1	1	16	18	7
6	4	1	18	30	7
8	4	1	20	30	7
10	6	7	22	42	22
12	12	7	24	61	22

From this table we can see that our bounds are the same or better than theirs when the level is one and the weight k is less than 24 except $k = 10$.

For elliptic modular forms, congruences involving Atkin's $U(p)$-operator were studied with important applications in the context of traces of singular moduli and class equations (see Ahlgren and Ono [1], Elkies, Ono, and Yang [6], and Chapter 7 of Ono [11]). Recently the first two authors and Richter [5] investigated congruences involving an analogue of Atkin's $U(p)$-operator for a Siegel modular form. As an application of Theorem 1.1 we improve the results in [5] by removing a condition on F.

Before stating our second main theorem we introduce further notations. Let

$$\widetilde{M_k} := \Big\{F \pmod p : F(Z) = \sum a(T)e^{\pi i\,\mathrm{tr}(TZ)} \in M_k(\Gamma),\ a(T) \in \mathbb{Z}_{(p)}\Big\},$$

where $\mathbb{Z}_{(p)} := \mathbb{Z}_p \cap \mathbb{Q}$. For Siegel modular forms with p-integral rational coefficients, we define their filtrations modulo p by

$$\omega(F) := \inf\{k : F \pmod p \in \widetilde{M_k}\}.$$

Let $Z := \left(\begin{smallmatrix} \tau & z \\ z & \tau' \end{smallmatrix}\right)$. The differential operator $\mathbb{D} = (2\pi)^{-2}\big(4\frac{\partial}{\partial\tau}\frac{\partial}{\partial\tau'} - \frac{\partial^2}{\partial z^2}\big)$ acts on the Fourier expansions of Siegel modular forms as

$$\mathbb{D}\Big(\sum a(T)e^{\pi i\ \mathrm{tr}(TZ)}\Big) = \sum \det(T)a(T)e^{\pi i\,\mathrm{tr}(TZ)}.$$

Let

$$F(Z)|U(p) := \sum_{\substack{T={}^tT\geq 0\\ T\,\mathrm{even}\\ p|\det T}} a(T)e^{\pi i\,\mathrm{tr}(TZ)}$$

be an analogue of Atkin's $U(p)$-operator for a Siegel modular form

$$F(Z) = \sum_{\substack{T={}^tT\geq 0\\ T\,\mathrm{even}}} a(T)e^{\pi i\ \mathrm{tr}(TZ)}.$$

Now we are ready to state the theorem on congruences involving an analogue of Atkin's $U(p)$-operator for a Siegel modular form.

THEOREM 1.3. *Let F be a Siegel modular form of degree 2, even weight k, level 1, and with p-integral rational coefficients, where $p > k$ is a prime. Suppose that $F \not\equiv 0 \pmod p$. If $p > 2k-5$, then $F|U(p) \not\equiv 0 \pmod p$. If $k < p < 2k-5$, then*

$$\omega(\mathbb{D}^{(3p+3)/2-k}(F)) = \begin{cases} 3p-k+3 & \text{if } F|U(p) \not\equiv 0 \pmod p, \\ 2p-k+4 & \text{if } F|U(p) \equiv 0 \pmod p. \end{cases}$$

2. Proofs of main theorems. Let $E_4^{(2)}, E_6^{(2)}, \chi_{10}$ and χ_{12} denote the generators (introduced by Igusa [7]) of $M_k(\Gamma_2)$ of weights 4, 6, 10 and 12, respectively. Here $E_k^{(2)}$ $(k = 4, 6)$ is the normalized Eisenstein series of weight k and genus 2 $\big($i.e. $a\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) = 1\big)$ and the cusp forms χ_k $(k = 10, 12)$

are normalized by $a\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right) = 1$. Then all the Fourier coefficients of these four generators are rational integers (cf. [8]). Nagaoka [9] and Böcherer and Nagaoka [2] investigated Siegel modular forms modulo p.

Consider the Witt operator $W : M_k(\Gamma_2) \to M_k(\mathrm{SL}_2(\mathbb{Z})) \otimes M_k(\mathrm{SL}_2(\mathbb{Z}))$ defined by

$$W(F)(\tau, \tau') := F\left(\left(\begin{smallmatrix} \tau & 0 \\ 0 & \tau' \end{smallmatrix}\right)\right), \qquad (\tau, \tau') \in \mathbb{H} \times \mathbb{H},$$

where $M_k(\mathrm{SL}_2(\mathbb{Z}))$ is the space of modular forms of weight k on $\mathrm{SL}_2(\mathbb{Z})$. For example, note that (see [9])

$$\begin{aligned} W(E_4^{(2)})(\tau, \tau') &= E_4(\tau)E_4(\tau'), \quad & W(\chi_{10})(\tau, \tau') &= 0, \\ W(E_6^{(2)})(\tau, \tau') &= E_6(\tau)E_6(\tau'), \quad & W(\chi_{12})(\tau, \tau') &= \Delta(\tau)\Delta(\tau'), \end{aligned}$$

where E_k is the normalized Eisenstein series of weight k on $\mathrm{SL}_2(\mathbb{Z})$ and Δ is the unique normalized cusp form of weight 12 on $\mathrm{SL}_2(\mathbb{Z})$.

Lemma 2.1 (Corollary 4.2 in [9]). *Let $M_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$ be a $\mathbb{Z}$-module generated by all Siegel modular forms in $M_k(\Gamma_2)$ whose Fourier coefficients are rational and p-integral. If $F \in M_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$ satisfies $W(F) \equiv 0$, then $F/\chi_{10} \in M_{k-10}(\Gamma_2)_{\mathbb{Z}_{(p)}}$.*

With the above lemmas, we prove Theorem 1.1 for the case of level 1.

Proposition 2.2. *Suppose that $\Gamma = \Gamma_2$. Then the conclusion of Theorem* 1.1 *is true.*

Proof. By assumption, $A(n, r, m) \equiv 0 \pmod p$ for every n, m such that $0 \le n, m \le k/10$. Thus $A(n, r, m) \equiv 0 \pmod p$ for every n, m such that $0 \le n \le \frac{1}{12}(k + 2m)$ and $0 \le m \le k/10$.

Let $F(\tau, z, \tau') = \sum_{m=0}^{\infty} \phi_m(\tau, z) q'^m$ be the Fourier–Jacobi expansion of F, where ϕ_m is a Jacobi form of weight k and index m. For each m such that $0 \le m \le k/10$, we have $\phi_m \equiv 0 \pmod p$ by the analogue of the Sturm formula for Jacobi forms (see Theorem 1.3 in [4]) since $A(n, r, m) \equiv 0 \pmod p$ for n, $0 \le n \le \frac{1}{12}(k + 2m)$. Thus

$$A(n, r, m) \equiv 0 \pmod p \quad \text{if } 0 \le m \le k/10. \tag{2.1}$$

Let $W(F)$ be the Witt operator defined as

$$W(F(Z)) = F(\tau, 0, \tau').$$

Then

$$W(F(Z)) = \sum_{\alpha=1}^{\beta} f_\alpha(\tau) g_\alpha(\tau') := \sum_{n,m \ge 0} B(n, m) q^n q'^m,$$

where f_α and g_α are modular forms of weight k on $\mathrm{SL}_2(\mathbb{Z})$. From the ring structure of modular forms on $\mathrm{SL}_2(\mathbb{Z})$, we have

$$W(F(Z)) = \sum_{\substack{12i+4j+6t=k \\ t=0,1}} f_{(i)}(\tau)\Delta(\tau')^i E_4(\tau')^j E_6(\tau')^t,$$

where $f_{(i)}$ is also a modular form of weight k on $\mathrm{SL}_2(\mathbb{Z})$. Since

$$E_k(\tau) = 1 + O(q) \quad \text{and} \quad \Delta(\tau) = q + O(q^2),$$

the q-expansion of $\Delta(\tau')^i E_4(\tau')^j E_6(\tau')^t$ has the form

$$\Delta(\tau')^i E_4(\tau')^j E_6(\tau')^t = (q')^i + \cdots .$$

The numbers j and t are uniquely determined by choosing a value of i.

From (2.1) we find that if $m \le k/10$, then $B(n,m) \equiv 0 \pmod p$. This implies that $f_{(i)} \equiv 0 \pmod p$ for $i \le k/10$. Note that indeed $i \le k/10$ since $12i + 4j + 6t = k$. Thus we have $W(F) \equiv 0 \pmod p$, and there is a Siegel modular form $F' \in M_k(\Gamma^2)_{\mathbb{Z}_{(p)}}$ such that $W(F') = (1/p)W(F)$. Applying Lemma 2.1 to $F - pF'$, we have

$$F \equiv F_{(1)}(Z)\chi_{10}(Z) \pmod p,$$

where $F_{(1)}(Z)$ is a Siegel modular form of weight $k-10$ and genus 2 on Γ_2.

Let $F_{(1)}(\tau, z, \tau') = \sum_{m=0}^{\infty} \phi_m^{(1)}(\tau, z)q'^m$ be the Fourier–Jacobi expansion of $F_{(1)}$. Then $\chi_{10}(Z)$ has the Fourier–Jacobi expansion $\sum_{m=1}^{\infty} \psi_m(\tau, z)q'^m$ such that $\psi_1(\tau, z) \not\equiv 0 \pmod p$. Thus for every m with $0 \le m \le k/10 - 1$,

$$\phi_m \equiv 0 \pmod p.$$

Hence, following the previous argument, we have

$$F_{(1)} \equiv F_{(2)}(Z)\chi_{10}(Z) \pmod p,$$

where $F_{(2)}(Z)$ is a Siegel modular form of weight $k-20$ and genus 2.

Repeating this argument, we obtain

$$F_{(t)} \equiv F_{(t+1)}(Z)\chi_{10}(Z) \pmod p$$

if $1 \le t < t_0$ and $t_0 = [k/10]$. Thus

$$F \equiv F_{(t_0)}\chi_{10}(Z)^{t_0} \pmod p.$$

We will complete the proof by showing that $F_{(t_0)} \equiv 0 \pmod p$. Let $\sum_{m=0}^{\infty} \phi_m^{(t_0)}(\tau, z)q'^m$ be the Fourier–Jacobi expansion of $F_{(t_0)}$. Since $k/10 - t_0 \ge 0$, we have $\phi_0^{(t_0)} \equiv 0 \pmod p$. Moreover, the weight of $F_{(t_0)}$ is less than 8, and $\dim_{\mathbb{C}} M_k(\Gamma_2) \le 1$ for $k \le 10$. Thus, $F_{(t_0)} \equiv 0 \pmod p$. ∎

For a subring $R \subset \mathbb{C}$, we denote by $M_k(\Gamma)_R$ the space of all $f \in M_k(\Gamma)$ whose Fourier coefficients are in R. Let p be a prime. We denote by v_p the normalized additive valuation on $\mathbb{Q}$ (i.e. $v_p(p)=1$). For

$$F(\tau,z,\tau')=\sum_{\substack{n,m\in\frac{1}{N}\mathbb{Z},\, r\in\frac{1}{2N}\mathbb{Z}\\ n,m,nm-r^2\geq 0}} A(n,r,m)q^n\xi^{2r}q'^m\in M_k(\Gamma)_{\mathbb{Q}},$$

we define

$$v_p(F):=\inf\Big\{v_p(A(n,r,m))\ \Big|\ n,m\in\frac{1}{N}\mathbb{Z},\ r\in\frac{1}{2N}\mathbb{Z},\ n,m,4nm-r^2\geq 0\Big\}.$$

Assume that $v_p(F)\geq 0$, in other words $F\in M_k(\Gamma)_{\mathbb{Z}_{(p)}}$. Then we define the order of F by

$$\begin{aligned}\mathrm{ord}_p(F):=&\ \min\Big\{m\ \Big|\ \phi_m(\tau,z)=\sum_{n,r}A(n,r,m)\xi^{2r}q^n\not\equiv 0 \pmod p\Big\}\\ =&\ \min\{m\mid v_p(A(n,r,m))=0 \text{ for some } n,r\}.\end{aligned}$$

If $v_p(F)\geq 1$, then we define $\mathrm{ord}_p(F):=\infty$. Note that

$$\mathrm{ord}_p(FG)=\mathrm{ord}_p(F)+\mathrm{ord}_p(G). \tag{2.2}$$

Following the argument of Sturm [14], we prove Theorem 1.1 with the help of Proposition 2.2.

Proof of Theorem 1.1. Let $i:=[\Gamma_2:\Gamma]$. Suppose that $A(n,r,m)\equiv 0 \pmod p$ for every n, m such that $0\leq n\leq\frac{1}{10}ki$ and $0\leq m\leq\frac{1}{10}ki$. Then $A(n,r,m)\equiv 0 \pmod p$ for every n, m such that $0\leq n\leq\frac{1}{12}(ki+2m)$ and $0\leq m\leq\frac{1}{10}ki$. Thus $\mathrm{ord}_p(F)>\frac{1}{10}ki$ by the analogue of the Sturm formula for Jacobi forms (see Theorem 1.2 in [4]).

We decompose Γ_2 as $\Gamma_2=\bigcup_{j=1}^{i}\Gamma\gamma_j$, where $\gamma_1=1_2$. Let

$$\Psi:=F\prod_{j=2}^{i}F|_k\gamma_j\in M_{ki}(\Gamma_2)_{\mathbb{Q}(\mu_N)}\qquad(\mu_N:=e^{2\pi i/N}).$$

Then all Fourier coefficients of Ψ are in $\mathbb{Q}(\mu_N)$ according to Shimura's result [13]. We take a constant $\lambda\in\mathbb{Q}(\mu_N)$ such that at least one of the non-zero Fourier coefficients of $\lambda\Psi$ is in $\mathbb{Q}$. For example, we may take $\lambda:=A_\Psi(n,r,m)^{-1}$ for a nonzero Fourier coefficient $A_\Psi(n,r,m)$ of Ψ.

Moreover, we consider

$$\begin{aligned}\Phi:=&\sum_{\sigma\in\mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})}(\lambda\Psi)^\sigma=\sum_{\sigma\in\mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})}\Big(\lambda F\prod_{j=2}^{i}F|_k\gamma_j\Big)^\sigma\\ =&\,F\sum_{\sigma\in\mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})}\Big(\lambda\prod_{j=2}^{i}F|_k\gamma_j\Big)^\sigma,\end{aligned}$$

where σ runs over all elements of $\mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$ and f^σ is defined by applying σ to each Fourier coefficient of f when $f\in M_k(\Gamma)_{\mathbb{Q}(\mu_N)}$. Then

$\Phi \in M_{ki}(\Gamma_2)_{\mathbb{Q}}$. This follows from Sturm's result [15, p. 344]. The choice of λ implies that $\Phi \neq 0$. Hence we can take a suitable constant $C \in \mathbb{Q}$ such that

$$v_p\Big(C \sum_{\sigma\in \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})} \Big(\lambda \prod_{j=2}^{i} F|_k\gamma_j\Big)^{\sigma}\Big) = 0.$$

This means $C\Phi \in M_{ki}(\Gamma_2)_{\mathbb{Z}_{(p)}}$. Using (2.2), we obtain

$$\mathrm{ord}_p(C\Phi) = \mathrm{ord}_p(F) + \mathrm{ord}_p\Big(C \sum_{\sigma\in \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})} \Big(\lambda \prod_{j=2}^{i} F|_k\gamma_j\Big)^{\sigma}\Big).$$

Note that $\mathrm{ord}_p(C\Phi) \geq \mathrm{ord}_p(F) > \frac{1}{10}ki$. Thus, by Proposition 2.2 we have

$$\mathrm{ord}_p(C\Phi) = \infty.$$

This implies

$$\mathrm{ord}_p(F) + \mathrm{ord}_p\Big(C \sum_{\sigma\in \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})} \Big(\lambda \prod_{j=2}^{i} F|_k\gamma_j\Big)^{\sigma}\Big) = \infty.$$

The second part is finite and hence we have $\mathrm{ord}_p(F) = \infty$, so $F \equiv 0 \pmod p$. This completes the proof of Theorem 1.1. ∎

Proof of Theorem 1.3. By Theorem 1 in [5], it is enough to show that if $A(n,r,m) \equiv 0 \pmod p$ for every (n,r,m) such that $p \nmid nm$, then $F \equiv 0 \pmod p$. To prove it we consider the Fourier–Jacobi expansion

$$F(\tau, z, \tau') = \sum_{m=0}^{\infty} \phi_m(\tau, z) e^{2\pi i m\tau'}.$$

Suppose that $A(n,r,m) \equiv 0 \pmod p$ whenever $p \nmid nm$. Then $A(n,r,m) \equiv 0 \pmod p$ for $m < p$ and $n < p$. Note that $k/10 < p$. Applying Theorem 1.1 completes the proof. ∎

3. Examples

3.1. Examples for level 1. Let $t(k) = [k/10]$. Consider a Siegel modular form

$$G_k(Z) = \sum_{\substack{n,m\in\mathbb{Z},\, r\in\frac{1}{2}\mathbb{Z} \\ n,m,nm-r^2\geq 0}} A_G(n,r,m) q^n \xi^{2r} q'^m$$

of weight k and genus 2 defined as

$$G_k(Z) := \begin{cases} E_4^{(2)}(Z)^i E_6^{(2)}(Z)^j \chi_{10}^{t(k)}(Z) \ (i + j + t(k) = k,\ j = 0 \text{ or } 1) \\ \qquad\qquad\qquad\qquad\qquad \text{if } k \not\equiv 2 \pmod{10}, \\ \chi_{10}^{t(k)-1}(Z)\chi_{12}(Z) \quad \text{if } k \equiv 2 \pmod{10}. \end{cases}$$

Recall that

$$\begin{aligned} E_4^{(2)}(Z) &= 1 + 240q + 240q' + \cdots, \\ E_6^{(2)}(Z) &= 1 - 504q - 504q' + \cdots, \\ \chi_{10}(Z) &= (\xi^{-1} - 2 + \xi)qq' + \cdots, \\ \chi_{12}(Z) &= (\xi^{-1} + 10 + \xi)qq' + \cdots. \end{aligned}$$

Thus we have

$$\begin{aligned} &G_k(Z) \\ &\quad = \begin{cases} (\xi^{-1} - 2 + \xi)^{t(k)} q^{t(k)} q'^{t(k)} + \cdots & \text{if } k \not\equiv 2 \pmod{10}, \\ (\xi^{-1} + 10 + \xi)(\xi^{-1} - 2 + \xi)^{t(k)-1} q^{t(k)} q'^{t(k)} + \cdots & \text{if } k \equiv 2 \pmod{10}. \end{cases} \end{aligned}$$

The coefficients $A_G(n, r, m)$ are integral and $A_G(n, r, m) \equiv 0 \pmod{p}$ for $n \le k/10 - 1$ and $m \le k/10 - 1$. Thus, when the level is one, the bounds in Theorem 1.1 are sharp since $G_k \not\equiv 0 \pmod{p}$.

3.2. Examples for level 11 and 19. It is known that we can construct a cusp form of $\Gamma_0^{(2)}(11)$ of weight 2 by Yoshida lift (cf. [16]). For the matrices

$$S_1^{(11)} := \begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 3 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 3 \end{pmatrix}, \quad S_2^{(11)} := \begin{pmatrix} 2 & 0 & 1 & \frac{1}{2} \\ 0 & 2 & \frac{1}{2} & -1 \\ 1 & \frac{1}{2} & 2 & 0 \\ \frac{1}{2} & -1 & 0 & 2 \end{pmatrix},$$

$$S_3^{(11)} := \begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 4 & 2 & \frac{3}{2} \\ \frac{1}{2} & 2 & 4 & \frac{7}{2} \\ 0 & \frac{3}{2} & \frac{7}{2} & 4 \end{pmatrix},$$

if we put

$$F_2^{(11)} := \tfrac{1}{24}(3\theta_{S_1^{(11)}} - \theta_{S_2^{(11)}} - 2\theta_{S_3^{(11)}}),$$

then $F_2^{(11)} \in S_2(\Gamma_0^{(2)}(11))_{\mathbb{Z}_{(p)}}$, where θ_{S_j} is defined by

$$\theta_{S_j}(Z) := \sum_{X \in M_{4,2}(\mathbb{Z})} e^{2\pi i \,\mathrm{tr}(S_j[X]Z)}.$$

We give one more example of Yoshida lift. For the matrices

$$S_1^{(19)} := \begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 5 & 0 \\ 0 & \frac{1}{2} & 0 & 5 \end{pmatrix}, \qquad S_2^{(19)} := \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 2 & 0 & 1 \\ \frac{1}{2} & 0 & 3 & \frac{3}{2} \\ \frac{1}{2} & 1 & \frac{3}{2} & 6 \end{pmatrix},$$

$$S_3^{(19)} := \begin{pmatrix} 2 & 0 & 1 & \frac{1}{2} \\ 0 & 2 & \frac{1}{2} & 1 \\ 1 & \frac{1}{2} & 3 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & 3 \end{pmatrix},$$

if we put

$$F_2^{(19)} := \tfrac{1}{8}(\theta_{S_1^{(19)}} - 2\theta_{S_2^{(19)}} + \theta_{S_3^{(19)}}),$$

then $F_2^{(19)} \in S_2(\Gamma_0^{(2)}(19))_{\mathbb{Z}_{(p)}}$.

Let $E_4 = E_4^{(2)}$, $E_6 = E_6^{(2)}$, and let χ_{10} and χ_{12} be Igusa's generators as introduced in Section 2. Set

$$\chi_{20} := 11E_4E_6\chi_{10} + 4\chi_{10}^2 + 8E_4^2\chi_{12}$$

and denote the Fourier expansion of F by

$$F = \sum_T a_F(T)e^{2\pi i \mathrm{tr}(TZ)}.$$

Here T is a half-integral matrix. Then the following proposition is known.

Proposition 3.1 (Nagaoka–Nakamura [10]). *The following hold:*

(1) $a_{F_2^{(11)}}(T) \equiv a_{-\chi_{12}}(T) \pmod{11}$ *for all* T *with* $\mathrm{tr}(T) \le 5$.
(2) $a_{F_2^{(19)}}(T) \equiv a_{\chi_{20}}(T) \pmod{19}$ *for all* T *with* $\mathrm{tr}(T) \le 4$.

Now we can show the following congruences.

Proposition 3.2.
(1) $F_2^{(11)} \equiv -\chi_{12} \pmod{11}$,
(2) $F_2^{(19)} \equiv \chi_{20} \pmod{19}$.

Proof. Since the proof of (2) is similar to the proof of (1), we will show (1) only. It is known by Böcherer–Nagaoka [3] that there exists a modular form $G_{12} \in M_{12}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ such that $F_2^{(11)} \equiv G_{12} \pmod{11}$. From Proposition 3.1 we have

$$a_{G_{12}}(T) \equiv a_{-\chi_{12}}(T) \pmod{11} \quad \text{for all } T \text{ with } \mathrm{tr}(T) \le 5.$$

Applying Theorem 1.1 to G_{12} and $-\chi_{12}$, we have $G_{12} \equiv -\chi_{12} \pmod{11}$. In fact, $k/10 = 12/10$. Hence it suffices to check the congruence of the Fourier coefficients for all T with $\mathrm{tr}(T) \leq 2$. Then $F_2^{(11)} \equiv G_{12} \equiv -\chi_{12} \pmod{11}$. This completes the proof of (1). ∎

REMARK 3.3. In [3], Böcherer–Nagaoka proved that, under a mild assumption, for each $F_2 \in M_2(\Gamma_0^{(2)}(p))_{\mathbb{Z}_{(p)}}$ there exists a Siegel modular form $G_{p+1} \in M_{p+1}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ such that $F_2 \equiv G_{p+1} \pmod{p}$.

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