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On the representation of H-invariants in the Selberg class

by

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1. Introduction. The *extended Selberg class* of functions, $\mathcal{S}^\sharp$, introduced by J. Kaczorowski and A. Perelli in [2], is a general class of Dirichlet series F such that

(i) the series
$$F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s},$$
converges absolutely for $\operatorname{Re} s > 1$,

(ii) there exists an integer $m \geq 0$ such that $(s-1)^m F(s)$ is an entire function of finite order,

(iii) F satisfies the functional equation
$$\Phi_F(s) = w\,\overline{\Phi_F(1-\bar{s})},$$
where
$$\Phi_F(s) = F(s) Q_F^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) = F(s)\gamma(s),$$
with $Q_F > 0$, $r \geq 0$, $\lambda_j > 0$, $|w| = 1$, $\operatorname{Re}\mu_j \geq 0$, $j = 1, \ldots, r$. The function $\gamma(s)$ is called the *gamma factor*.

The smallest integer $m \geq 0$ such that $(s-1)^m F(s)$ is entire is denoted by m_F and called the *polar order* of F. It is easy to see (due to the functional equation and the Stirling formula for the gamma function) that the function $(s-1)^{m_F} F(s)$ is actually an entire function of order one.

The *Selberg class* of functions (introduced by A. Selberg in [10]) consists of all $F \in \mathcal{S}^\sharp$ such that

(iv) for every $\epsilon > 0$, $a_F(n) \ll n^\epsilon$ (the Ramanujan conjecture),

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(v) there is an expansion

$$\log F(s) = \sum_{n=1}^{\infty} \frac{b_F(n)}{n^s}, \tag{1}$$

where $b_F(n) = 0$ for all $n \neq p^m$ with $m \geq 1$ and p prime, and $b_F(n) \ll n^{\theta}$ for some $\theta < 1/2$.

The last axiom is called the *Euler product axiom*, since it implies multiplicativity of the coefficients $a_F(n)$.

It is believed that the extended Selberg class contains all zeta and L-functions of number-theoretical interest, and that the Selberg class contains all zeta and L-functions having an Euler product. A panoramic view on the Selberg class can be found in the survey papers [1, 3, 8, 9].

The notion of invariant in the extended Selberg class arises from the fact that, due to the multiplication and factorial formulas for the gamma function, the data $(Q_F, \lambda, \mu, \omega)$ of the functional equation of F, where $\lambda = (\lambda_1, \dots, \lambda_r)$ and $\mu = (\mu_1, \dots, \mu_r)$, is not uniquely determined by F. Hence, an *invariant* (resp. a *numerical invariant*) of a function $F \in \mathcal{S}^\sharp$ is an expression (resp. a number) defined in terms of the data of F which is uniquely determined by F itself. In the series of papers on the structural problems in the Selberg class J. Kaczorowski and A. Perelli introduced the notion of invariant of the functional equation and proved a lot of important properties of invariants (see [4]–[6]). In particular, they proved that for an integer $n \geq 0$, the numbers $H_F(n)$ defined by

$$H_F(n) = 2 \sum_{j=1}^{r} \frac{B_n(\mu_j)}{\lambda_j^{n-1}},$$

where $B_n(x)$ is the nth Bernoulli polynomial, are (numerical) invariants. The numbers $H_F(n)$ are called the *H-invariants.*

The special cases

$$H_F(0) = 2 \sum_{j=1}^{r} \lambda_j = d_F$$

and

$$H_F(1) = 2 \sum_{j=1}^{r} \left(\mu_j - \frac{1}{2} \right) = \xi_F = \eta_F + i\theta_F$$

are particularly important. They are called the *degree* and the *ξ-invariant*, respectively. The real and imaginary parts of the ξ-invariant are called, respectively, the *parity* and the *shift* of $F \in \mathcal{S}^\sharp$.

Other important invariants for functions $F \in \mathcal{S}^\sharp$ are the *conductor* q_F and the *root number* ω_F^* defined as follows:

$$q_F = (2\pi)^{d_F} Q_F^2 \prod_{j=1}^{r} \lambda_j^{2\lambda_j},$$

$$\omega_F^* = \omega e^{-i\frac{\pi}{2}(\eta_F+1)} \left(\frac{q_F}{(2\pi)^{d_F}}\right)^{i\theta_F/d_F} \prod_{j=1}^{r} \lambda_j^{-2i\,\mathrm{Im}\,\mu_j}.$$

A set $\{I_j\}_{j\in J}$ of numerical invariants is called a *set of basic invariants* if $I_j(F_1) = I_j(F_2)$ for all $j \in J$ implies that $F_1(s)$ and $F_2(s)$ satisfy the same functional equation, for any $F_1, F_2 \in \mathcal{S}^\sharp$. In other words, a set of basic invariants characterizes the functional equation of every $F \in \mathcal{S}^\sharp$. More precisely, such a set is called a *global* set of basic invariants, as opposed to a *local* set of basic invariants, characterizing the functional equation of a given function $F \in \mathcal{S}^\sharp$.

Theorem A ([5, Th. 1], see also [8, Th. 4.3]). *The H-invariants $H_F(n)$, $n \geq 0$, the conductor q_F and the root number ω_F^* form a global set of basic invariants.*

In [5], J. Kaczorowski and A. Perelli obtained an interpretation of H-invariants and conductor as coefficients in a certain asymptotic expansion of the gamma factor of the functional equation and raised the problem of interpreting $H_F(n)$, $n \geq 2$, in terms of F alone, without explicit reference to the functional equation (see also [8, Problem 4.1]). The purpose of this paper is to give a solution of this problem.

The solution can be briefly explained as follows. First, we notice that the "superzeta" functions $\mathbf{Z}_F(s,z)$ from trivial zeros of $F \in \mathcal{S}^\sharp$, introduced in Section 3, may be easily written in terms of Hurwitz zeta functions, hence, by the standard properties of Hurwitz zeta functions, the invariants $H_F(n)$ for $F \in \mathcal{S}^\sharp$ and $n \geq 1$ can be expressed in terms of $\mathbf{Z}_F(1-n,0)$. Moreover, by Voros' theory of generalized zeta functions and zeta-regularized products [16], the function $\mathbf{Z}_F(s,z)$ is related to the "superzeta" function $\mathcal{Z}_F(s,z)$ from the non-trivial zeros of $F \in \mathcal{S}$. Such a link becomes very simple for integer s and allows us to express $H_F(n)$ in terms of $\mathcal{Z}_F(1-n,0)$, $n \geq 1$.

2. Zeta-regularized products. Let $\{y_k\}_{k\in\mathbb{N}}$ be the sequence of zeros of an entire function H of order 1, repeated according to their multiplicities. Then the series

$$Z(s,z) = \sum_{k=1}^{\infty} (z - y_k)^{-s} \tag{2}$$

converges absolutely for $\operatorname{Re} s > 1$ and a fixed complex z such that $z-y_k \notin \mathbb{R}^-$ for all k. Here and throughout, we assume that $0 \in \mathbb{R}^-$ and define the function $z \mapsto z^{-s}$ in a standard way, using the principal branch of the logarithm with $\arg z \in (-\pi, \pi)$ in the slit plane $\mathbb{C} \setminus (-\infty, 0]$.

The series $Z(s, z)$ is called the *zeta function associated to the zeros of* H, or the *"superzeta" function from the zeros* $\{y_k\}_{k\in\mathbb{N}}$.

According to A. Voros [16], this kind of series was first considered by Hj. Mellin in [7] (see also [16, Appendix D] for an English translation of [7], with comments). An informative summary of previous results on "superzeta" functions can be found in [16, Section 5.5].

In [12]–[16], A. Voros considered "superzeta" functions in different settings (geometric, arithmetical and algebraic). In order to make our exposition more explicit, we will summarize the results of [12], [14], [15], and [16] needed later by stating them as a proposition, and briefly indicate its proof, referring to the corresponding results of Voros. We may assume that $y_k \neq 0$ for all k, since, as pointed out in [14, p. 355], a basic feature of the construction of zeta-regularized products is their full invariance under translations $\{-y_k\}_{k\in\mathbb{N}} \mapsto \{z - y_k\}_{k\in\mathbb{N}}$, $z \in \mathbb{C}$. (The numbers x_k in the notation of [14, Section 1.1] and [16, Chapter 2] correspond to our $-y_k$.) Let us note here that the function $F \in \mathcal{S}$ satisfies the main assumptions imposed on the "primary functions $L(x)$" by A. Voros in [15, Section 1.1] and [16, Chapter 10], the only significant difference being the functional equation that relates values of F at s with values of the "conjugate" function $\overline{F}(s) = \overline{F(\overline{s})}$ at $1 - s$ in our setting (in contrast to the equation that relates $L(s)$ to $L(1 - s)$ in the setting of Voros). Therefore, in our proposition below, we will impose the same assumptions on the entire function $\varDelta$ of order $\mu_0 = 1$, as in [15, Section 2.1] and [16, Chapter 2].

Proposition B ([12], [14], [15], [16]). *Let $\{y_k\}_{k\in\mathbb{N}}$ be the sequence of zeros of an entire function $\varDelta$ of order* 1 *and let*

$$\varDelta(z) = e^{B_1 z + B_0} \prod_{k=1}^{\infty} \left(1 - \frac{z}{y_k}\right) e^{z/y_k}$$

be the corresponding Hadamard product. Assume that $\varDelta(z)$ has the asymptotic expansion

$$\log \varDelta(z) \sim \widetilde{a}_1 z(\log z - 1) + b_1 z + \widetilde{a}_0 \log z + b_0 + \sum_{\{\mu_k\}\setminus\{0\}} a_k z^{\mu_k} \tag{3}$$

as $|z| \to \infty$ in the sector $|\arg z| < \theta < \pi$ $(\theta > 0)$, for some sequence $1 > \mu_1 > \cdots > \mu_n \downarrow -\infty$, such that the series on the right-hand side of (3) *can be repeatedly differentiated term by term.*

Then for all $z \in \mathbb{C}$ such that $z - y_k \notin \mathbb{R}^-$ for all k, the zeta function (2) *has a meromorphic continuation to the half-plane* $\operatorname{Re} s < 2$, *regular at $s = 0$ obtained through the continuation of the Mellin-transform representation*

$$Z(s,z) = \frac{\sin \pi s}{\pi(1-s)} I(s,z) = \frac{\sin \pi s}{\pi(1-s)} \int_0^\infty Z(2, z+y) y^{1-s}\, dy, \tag{4}$$

valid for $1 < \operatorname{Re} s < 2$ *and all $z \in \mathbb{C}$ such that $z - y_k \notin \mathbb{R}^-$ for all k.*

Furthermore, the zeta-regularized product $D(z)$ associated to $Z(s,z)$, defined as $D(z) := e^{-Z'(0,z)}$, where $'$ denotes differentiation with respect to the first variable, is related to $\Delta(z)$ through the formula

$$D(z) = e^{-(b_1 z + b_0)} \Delta(z). \tag{5}$$

Proof. Formula (4) is a special case of [16, formula (2.28), p. 15] for $1 < \operatorname{Re}\, s < 2$. Continuation of the integral $I(s,z)$ further to the left is obtained using (3), by repeated integration by parts as explained in [13, Appendix A], [12, pp. 442–443] and [16, Section 1.5].

Since $Z(s,z)$ is regular at $s = 0$, $D(z)$ is well defined. Finally, formula (5) is stated in [15, formula (2.3), p. 177] and holds true under our assumptions. The proof is complete.

3. Zeta functions built over zeros of a function $F \in \mathcal{S}^\sharp$. We will consider two "superzeta" functions arising from zeros of a function $F \in \mathcal{S}^\sharp$: the function $\mathcal{Z}_F(s,z)$ built over the non-trivial zeros of F, and the function $\mathbf{Z}_F(s,z)$ built over the trivial zeros. The *non-trivial zeros* of F are zeros of the function Φ_F, and the *trivial zeros* are the ones arising from the poles of the factor of the functional equation.

The functional equation for $F \in \mathcal{S}$ can be written as

$$\Phi_F^c(z) = w\, \overline{\Phi_F^c(1-\overline{z})}, \tag{6}$$

where

$$\Phi_F^c(z) = (z-1)^{m_F} z^{m_F} \Phi_F(z) = (z-1)^{m_F} z^{m_F} F(z) G^{-1}(z) \tag{7}$$

and

$$G^{-1}(z) = Q_F^z \prod_{j=1}^r \Gamma(\lambda_j z + \mu_j).$$

Therefore, the trivial zeros of F coincide with the zeros of $z^{-m_F} G(z)$. The zero $\rho = 0$, if present, usually requires special attention, since it may arise as both a trivial and a non-trivial zero (in the case when $m_F = 0$).

Set $A_F = \{j \in \{1, \dots, r\} : \mu_j = 0\}$, and let a_F denote the number of elements in A_F. Then $G^{-1}(z)$ has a pole at $z = 0$ of order a_F, hence $\rho = 0$ may arise as a trivial zero of F only in the case when $a_F > m_F$ and in that case the order of the trivial zero $\rho = 0$ is equal to $a_F - m_F$. Actually, the inequality $a_F \geq m_F$ always holds true. Namely, if $a_F < m_F$, then $m_F \geq 1$, hence $\rho = 1$ is not a zero of $(z-1)^{m_F} F(z)$, by the definition of m_F. Since $\operatorname{Re}(\lambda_j + \mu_j) > 0$, we conclude that $\rho = 1$ is not a zero of $\Phi_F^c(z)$. On the other hand, $\rho = 0$ is not a pole of F, hence it is a zero of $z^{m_F} F(z) G^{-1}(z)$ of order greater than or equal to $m_F - a_F$. Therefore, $\rho = 0$ is a zero of $\Phi_F^c(z)$ of order at least $m_F - a_F$. By the functional equation (6), $\rho = 1$ is also a zero of $\Phi_F^c(z)$ of the same order, a contradiction.

We will consider the following two "superzeta" functions:

$$\mathcal{Z}_F(s,z) = \sum_{\rho} (z-\rho)^{-s} \qquad (\operatorname{Re} s > 1),$$

where the sum is taken over the all non-trivial zeros ρ (counted with multiplicities) of the function F, $z \in X = \{z \in \mathbb{C} : z - \rho \notin \mathbb{R}^- \text{ for all } \rho\}$, and

$$\mathbf{Z}_F(s,z) = \sum_{\eta_k} (z-\eta_k)^{-s} - m_F z^{-s} \qquad (\operatorname{Re} s > 1), \tag{8}$$

where the sum is taken over the zeros $\eta_k = \eta_{n,j} = -(n+\mu_j)/\lambda_j$, $n = 0, 1, \dots$, $j = 1, \dots, r$, of G, counted with multiplicities, and $z \in X_1 = \{z \in \mathbb{C} : z - \eta_k \notin \mathbb{R}^- \text{ for all } k\}$.

Since $z^{-m_F} G(z)$ is an entire function of order 1, $\mathbf{Z}_F(s,z)$ is well defined for $F \in \mathcal{S}^\sharp$, $z \in X_1$, $\operatorname{Re} s > 1$. If $A_F \neq \emptyset$ the term z^{-s} appears in the sum on the right-hand side of (8) a_F times, hence

$$\mathbf{Z}_F(s,z) = \sum_{\eta_k \neq 0} (z-\eta_k)^{-s} + (a_F - m_F) z^{-s} \qquad (\operatorname{Re} s > 1).$$

This shows that $\mathbf{Z}_F(s,z)$ is equal to the sum $\sum_\kappa (z-\kappa)^{-s}$ over all trivial zeros κ of F (including the zero $\kappa = 0$, if present).

Let $X_2 = \{z \in X_1 : \operatorname{Re}(\lambda_j z + \mu_j) > 0,\ j = 1, \dots, r\}$. For $F \in \mathcal{S}^\sharp$ and all $z \in X_2$, the function $\mathbf{Z}_F(s,z)$ can also be written as

$$\mathbf{Z}_F(s,z) = \sum_{j=1}^{r} \lambda_j^s \zeta(s, \lambda_j z + \mu_j) - m_F z^{-s} \qquad (\operatorname{Re} s > 1), \tag{9}$$

where $\zeta(s,w)$ denotes the Hurwitz zeta function. By [16, Section 3.6], the function $\zeta(s,w)$ has a meromorphic continuation (in the s variable, for $\operatorname{Re} w > 0$) to the whole complex plane, with a single pole at $s = 1$, simple and of residue 1. Therefore, the right-hand side of (9) provides a meromorphic continuation of $\mathbf{Z}_F(s,z)$ to the whole s-plane (in the given range of z), with a single pole at $s = 1$, simple and of residue $\sum_{j=1}^r \lambda_j = \frac{1}{2} H_F(0)$.

Since $\zeta(-n,w) = -B_{n+1}(w)/(n+1)$ for all $n \in \mathbb{N} \cup \{0\}$ and $\operatorname{Re} w > 0$ (see, e.g., [14, p. 353] or [16, formula (3.37) on p. 29]), we obtain

$$\mathbf{Z}_F(-n,z) = -\sum_{j=1}^{r} \frac{B_{n+1}(\lambda_j z+\mu_j)}{\lambda_j^n(n+1)} - m_F z^n$$

for every non-negative integer n and all $z \in X_2$. The property

$$B_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} B_k(x) y^{n-k}$$

of Bernoulli polynomials and the definition of H-invariants implies that

$$\begin{aligned}\mathbf{Z}_F(-n,z) = {} & \frac{-1}{2(n+1)} H_F(n+1) \\ & - \frac{1}{2(n+1)} \sum_{k=0}^{n} \binom{n+1}{k} H_F(k) z^{n+1-k} - m_F z^n \end{aligned} \tag{10}$$

for $n \in \mathbb{N} \cup \{0\}$, in the given range of z.

The function on the right-hand side of (10) provides analytic continuation of the function $\mathbf{Z}_F(-n,z)$, $n \in \mathbb{N}\cup\{0\}$, to the whole z-plane. Putting $n=0$ one also gets $\mathbf{Z}_F(0,z) = -\frac{1}{2} H_F(0)\, z - \frac{1}{2} H_F(1) - m_F$.

This proves the following proposition.

Proposition 3.1. *Let $F \in \mathcal{S}^\sharp$ and $z \in X_2$. Then*

$$H_F(n) = -2n\mathbf{Z}_F(1-n,0) \quad \textit{for } n\in\mathbb{N},\ n\ge 2,$$
$$H_F(1) = -2(\mathbf{Z}_F(0,0)+m_F), \qquad H_F(0) = 2\,\mathrm{Res}_{s=1}\,\mathbf{Z}_F(s,z).$$

The above proposition shows that the H-invariants may be interpreted as special values of (a meromorphic continuation of) the “superzeta” function $\mathbf{Z}_F(s,z)$ from the trivial zeros of $F \in \mathcal{S}^\sharp$. This result may be regarded as a partial solution to [8, Problem 4.1], since $\mathbf{Z}_F(s,z)$ depends directly on the factor of the functional equation.

In the last statement of Proposition 3.1 there is a variable z appearing only on the right-hand side of the equation. This is not surprising, as it follows from the fact that $\mathrm{Res}_{s=1}\,\zeta(s,w) = 1$, independently of w in the half-plane $\mathrm{Re}\, w > 0$.

4. The main result. In this section we will prove that H-invariants may be represented in terms of certain special values of the “superzeta” function $\mathcal{Z}_F(s,z)$ from the non-trivial zeros of $F \in \mathcal{S}$. Firstly, we will obtain a meromorphic continuation formula for $\mathcal{Z}_F(s,z)$ in the half-plane $\mathrm{Re}\, s \le 1$. Then we will consider special values of $\mathcal{Z}_F(s,z)$ when s is a negative integer (or zero) and prove that they are related to values of $H_F(n)$.

Let us recall that the admissible sets X and X_2 are defined by $X = \{z\in\mathbb{C} : z-\rho \notin \mathbb{R}^- \text{ for all } \rho\}$ and $X_2 = \{z\in\mathbb{C} : z-\eta_k \notin \mathbb{R}^- \text{ for all } k \text{ and } \mathrm{Re}(\lambda_j z+\mu_j) > 0,\ j=1,\dots,r\}$.

Theorem 4.1. *Let $F \in \mathcal{S}$. Then $\mathcal{Z}_F(s,z)$ has the integral representation*

$$\mathcal{Z}_F(s,z) = -\mathbf{Z}_F(s,z) + \frac{m_F}{(z-1)^s} + \frac{\sin \pi s}{\pi}\mathcal{J}_F(s,z), \tag{11}$$

valid for $\operatorname{Re} s < 1$ *and* $z \in X \cap X_2 \setminus (-\infty, 1]$, *where*

$$\mathcal{J}_F(s,z) = \int_0^\infty \frac{F'}{F}(z+y) y^{-s}\, dy \tag{12}$$

is a holomorphic function in the half-plane $\operatorname{Re} s < 1$.

Proof. We only sketch the proof, since it follows the lines of the proof of the analytic continuation formula from [14, Sec. 2.1], [15, Sec. 2.2] and [16, Section 10.3] (with $t + 1/2$ replaced by z).

We start with the asymptotic expansion of the entire function $z^{-m_F}G(z)$ of order one as

$$\begin{aligned}(13)\qquad &\log G(z) - m_F \log z \\ &= -\frac{1}{2}H_F(0) z(\log z - 1) - \frac{1}{2}(\log q_F - H_F(0)\log 2\pi)z \\ &\quad - \left(\frac{1}{2}H_F(1) + m_F\right)\log z - \sum_{j=1}^{r}\left(\mu_j - \frac{1}{2}\right)\log \lambda_j - r\log\sqrt{2\pi} \\ &\quad - \frac{1}{2}\sum_{n=1}^{N}\frac{(-1)^{n+1}}{n(n+1)}H_F(n+1)z^{-n} + O(|z|^{-N-1})\end{aligned}$$

for all $N \in \mathbb{N}$, as $z \to \infty$ with $|\arg z| < \pi$, proved by J. Kaczorowski and A. Perelli in [5, formula (2.8)], and apply Proposition B with $\Delta(z) = z^{-m_F}G(z)$, $\mu_k = -k$ ($k \geq 1$), $Z(s,z) = \mathbf{Z}_F(s,z)$ to obtain

$$\mathbf{Z}_F(s,z) = \frac{\sin \pi s}{\pi(1-s)} I_F(s,z) = \frac{\sin \pi s}{\pi(1-s)} \int_0^\infty \mathbf{Z}_F(2, z+y) y^{1-s}\, dy \tag{14}$$

for $1 < \operatorname{Re} s < 2$ and all $z \in X_2$.

Moreover, the zeta-regularized product $\mathbf{D}_F(z) := e^{-\mathbf{Z}'_F(0,z)}$, associated to the sequence of trivial zeros of F, can be expressed as

$$\mathbf{D}_F(z) = e^{b_1 z + b_0} z^{-m_F} G(z), \tag{15}$$

with $-b_0$ being the constant term in the expansion (13) and

$$b_1 = \frac{1}{2}(\log q_F - H_F(0)\log 2\pi).$$

The Euler product axiom (especially, the fact that $b_F(1) = 0$) yields

$$(\log F(z))^{(n)} = O(|z|^{-N-1}) \tag{16}$$

for all $N, n \in \mathbb{N}$, as $|z| \to \infty$ with $|\arg z| < \pi/2$. This together with (7) yields the asymptotic expansion

$$\log \Phi_F^c(z) \sim \widetilde{a}_1 z(\log z - 1) + b_1 z + \widetilde{a}_0 \log z + b_0 + \sum_{n=1}^{\infty} a_n z^{-n}, \tag{17}$$

as $|z| \to \infty$ with $|\arg z| < \pi/2$, repeatedly differentiable term by term. Applying Proposition B with $\Delta(z) = \Phi_F^c(z)$, $\mu_k = -k$ $(k \geq 1)$, $\theta = \pi/2$ and $Z(s,z) = \mathcal{Z}_F(s,z)$ we conclude that the zeta function $\mathcal{Z}_F(s,z)$ for $z \in X$ has a meromorphic continuation to the half-plane $\operatorname{Re} s < 2$, regular at $s = 0$, that is obtained through a meromorphic continuation of the representation

$$\mathcal{Z}_F(s,z) = \frac{\sin \pi s}{\pi(1-s)} \mathcal{I}_F(s,z) = \frac{\sin \pi s}{\pi(1-s)} \int_0^{\infty} \mathcal{Z}_F(2, z+y) y^{1-s}\, dy \tag{18}$$

valid for $1 < \operatorname{Re} s < 2$ and all $z \in X$. Furthermore, the zeta-regularized product $\mathcal{D}_F(z) := e^{-\mathcal{Z}_F'(0,z)}$ built over the non-trivial zeros of F is well defined and equal to $e^{-(b_1 z + b_0)} \Phi_F^c(z)$.

Now $(z-1)^{m_F} F(z) = \mathbf{D}_F(z) \mathcal{D}_F(z)$, hence

$$\begin{aligned} \mathbf{Z}_F(2,z) + \mathcal{Z}_F(2,z) &= -(\log \mathbf{D}_F(z))'' - (\log \mathcal{D}_F(z))'' \\ &= -\frac{m_F}{(z-1)^2} + \left(\frac{F'}{F}(z)\right)'. \end{aligned}$$

This together with (14) and (18) yields the representation

$$\begin{aligned} &\mathbf{Z}_F(s,z) + \mathcal{Z}_F(s,z) \\ &\quad = \frac{\sin \pi s}{\pi(1-s)} \int_0^{\infty} \left(\frac{m_F}{(z+y-1)^2} - \left(\frac{F'}{F}(z+y) \right)' \right) y^{1-s}\, dy \end{aligned} \tag{19}$$

valid for $1 < \operatorname{Re} s < 2$ and $z \in (X \cap X_1) \setminus (-\infty, 1]$.

Now, we use (16) and proceed as in [15, Section 2.2] and [14, Section 2] to deduce (by repeated integration by parts) that $\mathcal{J}_F(s,z)$ is holomorphic in the half-plane $\operatorname{Re} s < 1$. The proof is complete.

Our main result is the following theorem.

THEOREM 4.2. *Let $F \in \mathcal{S}$. Then*

(a) *For $n \in \mathbb{N}$ and $z \in (X \cap X_2) \setminus (-\infty, 1]$ one has*

$$\begin{aligned} \mathcal{Z}_F(-n,z) &= \frac{1}{2(n+1)} H_F(n+1) \\ &\quad + \frac{1}{2(n+1)} \sum_{k=0}^{n} \binom{n+1}{k} H_F(k) z^{n+1-k} + m_F (z-1)^n + m_F z^n. \end{aligned} \tag{20}$$

(b) *For a fixed integer $n \geq 0$, the function $\mathcal{Z}_F(-n,z)$ has an analytic continuation to the whole z-plane and*

$$H_F(n) = 2n(\mathcal{Z}_F(1-n,0) + (-1)^n m_F) \quad \textit{for } n \geq 2,$$
$$H_F(1) = 2(\mathcal{Z}_F(0,0) - 2m_F).$$

Proof. (a) Theorem 4.1 together with (10) implies that for $z \in (X \cap X_2) \setminus (-\infty, 1]$ one has

$$\mathcal{Z}_F(-n,z) = -\mathbf{Z}_F(-n,z) + \frac{m_F}{(z-1)^{-n}} = \frac{1}{2(n+1)} H_F(n+1) + \frac{1}{2(n+1)} \sum_{k=0}^{n} \binom{n+1}{k} H_F(k) z^{n+1-k} + m_F(z-1)^n + m_F z^n$$

and the proof is complete.

(b) The right-hand side of (20) is a polynomial in z, hence it provides the analytic continuation of $\mathcal{Z}_F(-n,z)$ (as a function of z) to the whole complex plane. Putting $z=0$ we get

$$\mathcal{Z}_F(-n,0) = \frac{H_F(n+1)}{2(n+1)} + (-1)^n m_F$$

for $n \geq 1$. This proves the first part of (b).

Furthermore, since $\zeta(0,a) = 1/2 - a$ for $\operatorname{Re} a > 0$, (9) and (11) imply that

$$\mathcal{Z}_F(0,z) = \frac{1}{2} H_F(1) + \frac{d_F}{2} z + 2m_F.$$

The right-hand side of the above equation yields the analytic continuation of $\mathcal{Z}_F(0,z)$ to the complex z-plane and completes the proof.

By repeated integration by parts in (12), having in mind (16), it is easy to check that $\mathcal{J}_F(s,z)$ is meromorphic in the whole s-plane with simple poles at $s=n$, $n \in \mathbb{N}$, and corresponding residues

$$\operatorname{Res}_{s=n} \mathcal{J}_F(s,z) = -\frac{1}{(n-1)!} (\log F(z))^{(n)} \quad (z \neq 1).$$

Therefore, the function $\frac{\sin \pi s}{\pi} \mathcal{J}_F(s,z)$ is entire, hence, by (11) the function $\mathcal{Z}_F(s,z)$ (as function of complex s, for a fixed, admissible z) has the same polar structure as $-\mathbf{Z}_F(s,z)$. Since $-\mathbf{Z}_F(s,z)$ (in the given range of z) has a simple pole at $s=1$ with residue $-\sum_{j=1}^{r} \lambda_j = -\frac{1}{2} H_F(0)$, it follows that $\mathcal{Z}_F(s,z)$ has a simple pole at $s=1$, of residue $-\frac{1}{2} H_F(0)$. Therefore $H_F(0) = -2 \operatorname{Res}_{s=1} \mathcal{Z}_F(s,z)$. The right-hand side of the last equation is independent of z, due to the last statement of Proposition 3.1.

On the other hand, by [11, Th. 3.4],

$$\lim_{T \to \infty} \sum_{|\operatorname{Im} \rho| \leq T} \frac{1}{z-\rho} = \frac{(\Phi_F^c)'}{\Phi_F^c}(z)$$

for all $z \in X$. Putting $\mathcal{Z}_F^*(1,z) := \frac{(\Phi_F^c)'}{\Phi_F^c}(z)$, applying (13), (17) and [15, display (2.26), p. 181] we get

$$b_1 = \frac{1}{2}(\log q_F - H_F(0)\log 2\pi) = \mathcal{Z}_F^*(1,z) - \mathrm{FP}_{s=1}\,\mathcal{Z}_F(s,z),$$

where $\mathrm{FP}_{s=1}\,\mathcal{Z}_F(s,z)$ denotes the constant term in the Laurent series expansion of $\mathcal{Z}_F(s,z)$ at the pole $s=1$. This proves the following corollary:

COROLLARY 4.3. *For* $z \in (X \cap X_2) \setminus (-\infty, 1]$ *one has*

(a) $H_F(0) = -2\,\mathrm{Res}_{s=1}\,\mathcal{Z}_F(s,z)$,
(b) $\log q_F = 2[\mathcal{Z}_F^*(1,z) - \log 2\pi \cdot \mathrm{Res}_{s=1}\,\mathcal{Z}_F(s,z) - \mathrm{FP}_{s=1}\,\mathcal{Z}_F(s,z)]$.

5. Concluding remarks. In this section we will give some further comments on extension of our results to a larger class of functions and possible construction of other "superzeta" functions of Voros presented in [15] and [16] in the setting of the Selberg class. We will also give an alternative proof of our main results.

REMARK 5.1 (Extension of main results to a larger class of functions). It is easy to see that Theorem 4.1, as well as Theorem 4.2, remain valid for all $F \in \mathcal{S}^\sharp$ such that $\log F(z)$ has a Dirichlet series representation

$$\log F(z) = \sum_{n=2}^{\infty} \frac{b_F(n)}{n^z}, \tag{21}$$

converging in a certain half-plane $\mathrm{Re}\, z > \sigma \geq 1$, without additional assumptions on the growth of the coefficients $b_F(n)$. Namely, the Ramanujan conjecture was not obviously needed in the proof of Theorem 4.1. Furthermore, convergence of the series (21) in the half-plane $\mathrm{Re}\, z > \sigma$ is sufficient to deduce that $\log F(z)$ and all its derivatives decay as $2^{-\mathrm{Re}\, z}$ as $\mathrm{Re}\, z \to +\infty$ with $|\arg z| < \pi/2$. This implies the bound (16), sufficient for the proof of Theorem 4.1. Therefore, our main results hold true for all $F \in \mathcal{S}^\sharp$ having an Euler product (21) convergent in some half-plane $\mathrm{Re}\, z > \sigma \geq 1$, without additional bounds on the coefficients $b_F(n)$. (Needless to say, the convergence of (21) in the half-plane $\mathrm{Re}\, z > \sigma > 0$ implies that $b_F(n) = o(n^\sigma)$, but σ may be greater than 1/2.)

Results of Section 4 may not extend to the class $\mathcal{S}^\sharp$, since representation (21) is essential in order to deduce (17) and to prove that $\mathcal{J}_F(s,z)$ is holomorphic for $\mathrm{Re}\, s < 1$.

REMARK 5.2 (On further applicability of Voros' theory of "superzeta" functions to the Selberg class). The absence of central symmetry $\rho \leftrightarrow 1-\rho$ in the set of zeros of a function $F \in \mathcal{S}$ implies that the zeros of $F \in \mathcal{S}$ do not necessarily come in pairs $\rho = 1/2 \pm i\tau_k$ with $\mathrm{Re}\, \tau_k > 0$. That is the main reason why it is not possible to define the Selberg class analogues of "superzeta" functions of the second and third kind, introduced in [16, p. 41] (see also [16, Sections 5.2, 5.3, 10.4 and 10.5]).

However, if the coefficients $a_F(n)$ of the Dirichlet series representation of $F \in \mathcal{S}$ are real numbers, then for all $n \in \mathbb{N}$, by the reflection principle, the zeros of F are symmetric with respect to the real line. (Actually, if ρ is a zero, then $\overline{\rho}$, $1-\rho$ and $1-\overline{\rho}$ are zeros of F). Therefore, the zeros of F come in pairs $\rho = 1/2 \pm i\tau_k$ with $\operatorname{Re}\tau_k > 0$, and results of [15, Sections 3 and 4] and of [16, Sections 10.4 and 10.5] may be easily generalized to yield properties of two new "superzeta" functions built over zeros of such $F \in \mathcal{S}$.

The only results of [16, Section 10] that may not easily be generalized in this case are the ones using the assumption that F is non-vanishing on the real interval $[0,1]$.

Remark 5.3 (A different proof of main results).

(i) The representation (15) of the zeta-regularized product $\mathbf{D}_F(z)$ may be obtained directly by differentiating equation (9) with respect to the s variable and taking $s=0$ to get

$$\mathbf{Z}'_F(0,z) = \sum_{j=1}^{r}[\log\lambda_j \cdot \zeta(0,\lambda_j z+\mu_j) + \zeta'(0,\lambda_j z+\mu_j)] + m_F \log z$$

for all $z \in X_2$. Using the formulas

$$\zeta(0,a) = 1/2 - a \quad \text{and} \quad \zeta'(0,a) = \log(\Gamma(a)/\sqrt{2\pi})$$

(see [16, Section 3.6]), we immediately obtain

$$\begin{aligned}\mathbf{D}_F(z) = \exp(-\mathbf{Z}'_F(0,z)) &= \exp\Big(\sum_{j=1}^{r}\log\lambda_j(\mu_j - 1/2) + \frac{r}{2}\log 2\pi\Big)\\ &\quad\cdot \exp\Big(\Big(\sum_{j=1}^{r}\lambda_j\log\lambda_j\Big)z\Big)\cdot z^{-m_F}\cdot\Big(\prod_{j=1}^{r}\Gamma(\lambda_j z+\mu_j)\Big)^{-1}\\ &= e^{b_0}\cdot z^{-m_F}\cdot G(z)e^{z\log Q_F}\exp\Big(\Big(\sum_{j=1}^{r}\lambda_j\log\lambda_j\Big)z\Big).\end{aligned}$$

Simple calculations show that

$$\log Q_F + \sum_{j=1}^{r}\lambda_j\log\lambda_j = \frac{1}{2}(\log q_F - H_F(0)\log 2\pi) = b_1$$

and (15) is proved.

(ii) Theorem 4.1, and hence Theorem 4.2, may be proved in a different way, without referring to results of Voros. The main reason for that is the special expression (9) of the zeta function $\mathbf{Z}_F(s,z)$ that yields its meromorphic continuation based on the properties of the Hurwitz zeta function. Here, we briefly explain how to obtain formula (19) directly.

Since the zeros of $z^{-m_F}G(z)$ coincide with the trivial zeros of F, having in mind that $z^{-m_F}G(z)$ is entire of order one, representing this function as a Hadamard product over its zeros, we can easily see that

$$\mathbf{Z}_F(2,z) = -[\log(z^{-m_F}G(z))]'' = \sum_{\kappa} \frac{1}{(z-\kappa)^2}$$

for $z \in X_2$. Analogously,

$$\mathcal{Z}_F(2,z) = -[\log(\Phi_F^c(z))]'' = \sum_{\rho} \frac{1}{(z-\rho)^2}$$

for $z \in X$. It is easy to see that for $1 < \operatorname{Re} s < 2$ and $z \in (X \cap X_2) \setminus (-\infty, 1]$ the series $\sum_\kappa y^{1-s}/(z-\kappa)^s$ and $\sum_\rho y^{1-s}/(z-\rho)^s$ may be integrated term by term to obtain the representation

$$\begin{aligned}&\mathbf{Z}_F(s,z) + \mathcal{Z}_F(s,z)\\ &\qquad = -\frac{\sin \pi s}{\pi(1-s)} \int_0^\infty ((\log(z^{-m_F}G(z)))'' + (\log(\Phi_F^c(z)))'')y^{1-s}\, dy,\end{aligned}$$

equivalent to (19), by (7). The analytic continuation of the above integral is obtained in the same way as in the proof of Theorem 4.1.

We have first given a longer proof of Theorem 4.1, using Proposition B, because in that proof we have also proved that the zeta-regularized products $\mathbf{D}_F(z)$ and $\mathcal{D}_F(z)$ built over the trivial and non-trivial zeros of F are well defined, and we have obtained their representation in terms of the functions $z^{-m_F}G(z)$ and $\Phi_F^c(z)$, respectively.

(iii) Corollary 4.3(b) may also be obtained directly from (11), without referring to results of [15]. Namely, since $\mathrm{FP}_{s=1}\,\zeta(s,w) = -\frac{\Gamma'}{\Gamma}(w)$ for $\operatorname{Re} w > 0$, we immediately see that $\mathrm{FP}_{s=1}\,\lambda^s\zeta(s,w) = \lambda\big(\log\lambda - \frac{\Gamma'}{\Gamma}(w)\big)$, hence

$$\begin{aligned}\mathrm{FP}_{s=1}\,\mathbf{Z}_F(s,z) &= \sum_{j=1}^{r} \lambda_j \left(\log\lambda_j - \frac{\Gamma'}{\Gamma}(\lambda_j z + \mu_j)\right) - \frac{m_F}{z}\\ &= \log Q_F + \sum_{j=1}^{r} \lambda_j \log\lambda_j - \frac{m_F}{z} + \frac{G'}{G}(z) = b_1 - \frac{m_F}{z} + \frac{G'}{G}(z)\end{aligned}$$

for $z \in X_2$. Since $\operatorname{Res}_{s=1} \mathcal{J}_F(s,z) = -\frac{F'}{F}(z)$, and hence $\mathrm{FP}_{s=1} \frac{\sin \pi s}{\pi} \mathcal{J}_F(s,z) = \frac{F'}{F}(z)$, from (11) we get

$$\mathrm{FP}_{s=1}\,\mathcal{Z}_F(s,z) = -b_1 + \frac{m_F}{z} + \frac{m_F}{z-1} + \frac{F'}{F}(z) - \frac{G'}{G}(z) = -b_1 + \mathcal{Z}_F^*(1,z),$$

and the proof is complete.

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References

[1] J. Kaczorowski, *Axiomatic theory of L functions: the Selberg class*, in: Analytic Number Theory (Cetraro, 2002), A. Perelli and C. Viola (eds.), Lecture Notes in Math. 1891, Springer, 2006, 133–209.
[2] J. Kaczorowski and A. Perelli, *On the structure of the Selberg class, I:* $0 \le d \le 1$, Acta Math. 182 (1999), 207–241.
[3] —, —, *The Selberg class: a survey*, in: Number Theory in Progress, in Honor of A. Schinzel, K. Győry et al. (eds.), de Gruyter, 1999, 953–992.
[4] —, —, *On the structure of the Selberg class, II: invariants and conjectures*, J. Reine Angew. Math. 524 (2000), 73–96.
[5] —, —, *On the structure of the Selberg class, IV: basic invariants*, Acta Arith. 104 (2002), 97–116.
[6] —, —, *A measure-theoretic approach to the invariants of the Selberg class*, ibid. 135 (2008), 19–30.
[7] Hj. Mellin, *Über die Nullstellen der Zetafunktion*, Ann. Acad. Sci. Fenn. Ser. A 10 (1917), no. 11.
[8] A. Perelli, *A survey of the Selberg class of L-functions, I*, Milan J. Math. 73 (2005), 19–52.
[9] —, *A survey of the Selberg class of L-functions, II*, Riv. Mat. Univ. Parma (7) 3* (2004), 83–118.
[10] A. Selberg, *Old and new conjectures and results about a class of Dirichlet series*, in: Proc. Amalfi Conf. Analytic Number Theory, E. Bombieri et al. (eds.), Università di Salerno, 1992, 367–385.
[11] L. Smajlović, *On Li's criterion for the Riemann hypothesis for the Selberg class*, J. Number Theory 130 (2010), 828–851.
[12] A. Voros, *Spectral functions, special functions and the Selberg zeta functions*, Comm. Math. Phys. 110 (1987), 439–465.
[13] —, *Zeta functions for the Riemann zeros*, Ann. Inst. Fourier (Grenoble) 53 (2003), 665–699.
[14] —, *More zeta functions for the Riemann zeros I*, in: Frontiers in Number Theory, Physics and Geometry, Springer, Berlin, 2006, 349–363.
[15] —, *Zeta functions over zeros of general zeta and L-functions*, in: T. Aoki et al. (eds.), Zeta Functions, Topology and Quantum Physics (Osaka, 2003), Developments Math. 14, Springer, New York, 2005, 171–196.
[16] —, *Zeta Functions over Zeros of Zeta Functions*, Lecture Notes Uni. Mat. Ital. 8, Springer, 2010.

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