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# Difference sets and polynomials of prime variables

by

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**1. Introduction.** For a set $A$ of positive integers, define

$$\bar{d}(A) = \limsup_{x\to\infty} \frac{|A \cap [1,x]|}{x}.$$

Furstenberg [9, Theorem 1.2] and Sárközy [21] independently confirmed the following conjecture of Lovász:

Theorem 1.1. *Suppose that $A$ is a set of positive integers with $\bar{d}(A) > 0$. Then there exist $x, y \in A$ and a positive integer $z$ such that $x - y = z^2$.*

In fact, the $z^2$ in Theorem 1.1 can be replaced by an arbitrary integral-valued polynomial $f(z)$ with $f(0) = 0$. On the other hand, Sárközy [22] also solved a problem of Erdős:

Theorem 1.2. *Suppose that $A$ is a set of positive integers with $\bar{d}(A) > 0$. Then there exist $x, y \in A$ and a prime $p$ such that $x - y = p - 1$.*

For the further developments of Theorems 1.1 and 1.2, the readers are referred to [23], [18], [1], [11], [16], [17], [20]. In the present paper, we shall give a common generalization of Theorems 1.1 and 1.2. Define

$$\Lambda_{b,W} = \{x : Wx + b \text{ is prime}\}$$

for $1 \le b \le W$ with $(b, W) = 1$.

Theorem 1.3. *Let $\psi(x)$ be a polynomial with integral coefficients and zero constant term. Suppose that $A \subseteq \mathbb{Z}^+$ satisfies $\bar{d}(A) > 0$. Then there exist $x, y \in A$ and $z \in \Lambda_{1,W}$ such that $x - y = \psi(z)$.*

Corollary 1.1. *Let $\psi(x)$ be a polynomial with rational coefficients and zero constant term. Suppose that $A \subseteq \mathbb{Z}^+$ satisfies $\bar{d}(A) > 0$. Then there exist $x, y \in A$ and a prime $p$ such that $x - y = \psi(p - 1)$.*

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*Proof.* Let $W$ be the least common multiple of the denominators of the coefficients of $\psi$. Then the coefficients of $\psi^*(x) = \psi(Wx)$ are all integers. Hence by Theorem 1.3, there exist $x, y \in A$ and $z \in \Lambda_{1,W}$ such that

$$x - y = \psi^*(z) = \psi(p-1)$$

where $p = Wz + 1$. ∎

About one month after the first version of this paper was put on the arXiv server, in [3] Bergelson and Lesigne proved that the set

$$\{(\psi_1(p-1), \dots, \psi_m(p-1)) : p \text{ prime}\}$$

is an enhanced van der Corput set $\mathbb{Z}^m$, where $\psi_1, \dots, \psi_m$ are polynomials with integral coefficients and zero constant term. Of course, their result can be extended to the set $\{(\psi_1(z), \dots, \psi_m(z)) : z \in \Lambda_{1,W}\}$ without any special difficulty. On the other hand, Kamae and Mendès France [15] proved that any van der Corput set is also a set of 1-recurrence. Hence Bergelson and Lesigne's result also implies our Theorem 1.3 and Corollary 1.1. In fact, they showed that the set $\{\psi(p-1) : p \text{ prime}\}$ is not only a set of 1-recurrence, but also a set of strong 1-recurrence.

For two sets $A, X$ of positive integers, define

$$\bar{d}_X(A) = \limsup_{x \to \infty} \frac{|A \cap X \cap [1, x]|}{|X \cap [1, x]|}.$$

Let $\mathcal{P}$ denote the set of all primes. In [12], Green established a Roth-type extension of a result of van der Corput [6] on 3-term arithmetic progressions in primes:

*Let $P$ be a set of primes with $\bar{d}_{\mathcal{P}}(P) > 0$. Then there exists a non-trivial 3-term arithmetic progression contained in $P$.*

The key to Green's proof is a transference principle, which transfers a subset $P \subseteq \mathcal{P}$ to a subset $A \subseteq \mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$ with $|A|/N \ge \bar{d}_{\mathcal{P}}(P)/64$, where $N$ is a large prime. Using Green's methods, we show:

THEOREM 1.4. *Let $\psi(x)$ be a polynomial with integral coefficients and zero constant term. Suppose that $P \subseteq \mathcal{P}$ satisfies $\bar{d}_{\mathcal{P}}(P) > 0$. Then there exist $x, y \in P$ and $z \in \Lambda_{1,W}$ such that $x - y = \psi(z)$.*

Similarly, we have

COROLLARY 1.2. *Let $\psi(x)$ be a polynomial with rational coefficients and zero constant term. Suppose that $P \subseteq \mathcal{P}$ satisfies $\bar{d}_{\mathcal{P}}(P) > 0$. Then there exist $x, y \in P$ and a prime $p$ such that $x - y = \psi(p-1)$.*

On the other hand, the well-known Szemerédi theorem [24] asserts that for any set $A$ of positive integers with $\bar{d}(A) > 0$, there exist arbitrarily long arithmetic progressions contained in $A$. In [2], Bergelson and Leibman extended Theorem 1.1 and Szemerédi's theorem:

*Let* $\psi_1(x), \ldots, \psi_m(x)$ *be arbitrary integral-valued polynomials with* $\psi_1(0) = \cdots = \psi_m(0) = 0$. *Then for any set* $A$ *of positive integers with* $\bar{d}(A) > 0$, *there exist* $x \in A$ *and an integer* $z$ *such that* $x + \psi_1(z), \ldots, x + \psi_m(z)$ *are all contained in* $A$.

Recently, Tao and Ziegler [26] proved the following:

*Let* $\psi_1(x), \ldots, \psi_m(x)$ *be arbitrary integral-valued polynomials with* $\psi_1(0) = \cdots = \psi_m(0) = 0$. *Then for any set* $P$ *of primes with* $\bar{d}_{\mathcal{P}}(P) > 0$, *there exist* $x \in P$ *and an integer* $z$ *such that* $x + \psi_1(z), \ldots, x + \psi_m(z)$ *are all contained in* $P$.

This is a generalization of Green and Tao's celebrated result [13] that the primes contain arbitrarily long arithmetic progressions. Furthermore, with the help of a very deep result due to Green and Tao [14] on the Gowers norms [10], Frantzikinakis, Host and Kra [8] proved that if $\bar{d}(A) > 0$ then $A$ contains a 3-term arithmetic progression with difference $p - 1$, where $p$ is a prime. In fact, using the methods of Green and Tao [14], it is not difficult to replace $A$ by $P$ with $\bar{d}_{\mathcal{P}}(P) > 0$ in the result of Frantzikinakis, Host and Kra.

Motivated by the above results, here we propose two conjectures:

Conjecture 1.1. *Let* $\psi_1(x), \ldots, \psi_m(x)$ *be arbitrary polynomials with rational coefficients and zero constant terms. Then for any set* $A$ *of positive integers with* $\bar{d}(A) > 0$, *there exist* $x \in A$ *and a prime* $p$ *such that* $x + \psi_1(p-1), \ldots, x + \psi_m(p-1)$ *are all contained in* $A$.

Conjecture 1.2. *Let* $\psi_1(x), \ldots, \psi_m(x)$ *be arbitrary polynomials with rational coefficients and zero constant terms. Then for any set* $P$ *of primes with* $\bar{d}_{\mathcal{P}}(P) > 0$, *there exist* $x \in P$ *and a prime* $p$ *such that* $x + \psi_1(p-1), \ldots, x + \psi_m(p-1)$ *are all contained in* $P$.

The proofs of Theorems 1.3 and 1.4 will be given in Sections 3 and 4. Throughout this paper, without specific mention, the constants implied by $\ll$, $\gg$ and $O(\cdot)$ will only depend on the degree of $\psi$.

**2. Some lemmas on exponential sums.** Let $\mathbb{T}$ denote the torus $\mathbb{R}/\mathbb{Z}$. For any function $f$ over $\mathbb{Z}$, define $f^{\Delta}(x) = f(x+1) - f(x)$. Also, we abbreviate $e^{2\pi\sqrt{-1}x}$ to $e(x)$. Let

$$\psi(x) = a_1 x^k + \cdots + a_k x$$

be a polynomial with integral coefficients. In this section, we always assume that $W, |a_1|, \ldots, |a_k| \leq \log N$.

Lemma 2.1. *Suppose that* $h(x)$ *is an arbitrary polynomial and* $0 < \nu < 1$. *Then for any* $\alpha \in \mathbb{T}$,

$$\sum_{x=1}^{N} h(x)e(\alpha\psi(x)) = \frac{1}{q}\sum_{r=1}^{q} e(a\psi(r)/q)\sum_{x=1}^{N} h(x)e((\alpha - a/q)\psi(x)) + O_{\deg h}(h(N)N^{\nu})$$

*provided that* $|\alpha q - a| \leq N^{\nu}/\psi(N)$ *with* $1 \leq a \leq q \leq N^{\nu}$.

*Proof.* Let $\theta = \alpha - a/q$. Then by partial summation, we have

$$\sum_{x=1}^{N} h(x)e(a\psi(x)/q)e(\theta\psi(x)) = h(N)e(\theta\psi(N))F_N(a/q) - \sum_{y=1}^{N-1}(h(y+1)e(\theta\psi(y+1)) - h(y)e(\theta\psi(y)))F_y(a/q),$$

where

$$F_y(a/q) := \sum_{x=1}^{y} e(a\psi(x)/q) = \frac{y}{q}\sum_{r=1}^{q} e(a\psi(r)/q) + O(q).$$

Clearly,

$$\begin{aligned} h(y+1)e(\theta\psi(y+1)) - h(y)e(\theta\psi(y)) &= (h(y+1) - h(y))e(\theta\psi(y+1)) \\ &\quad + h(y)e(\theta\psi(y))(e(\theta\psi^{\Delta}(y)) - 1) \\ &= O(h^{\Delta}(y)) + O(h(y)\theta\psi^{\Delta}(y)). \end{aligned}$$

This concludes that

$$\sum_{x=1}^{N} h(x)e(a\psi(x)/q)e(\theta\psi(x)) = \frac{1}{q}\sum_{r=1}^{q} e(a\psi(r)/q)\sum_{x=1}^{N} h(x)e(\theta\psi(x)) + O(\theta q N\psi^{\Delta}(N)h(N)) + O(qh^{\Delta}(N)N). \quad \blacksquare$$

Define

$$\lambda_{b,W}(x) = \begin{cases} \dfrac{\phi(W)}{W}\log(Wx+b) & \text{if } Wx+b \text{ is prime,} \\ 0 & \text{otherwise,} \end{cases}$$

where $\phi$ is the Euler totient function.

LEMMA 2.2. *Suppose that $h(x)$ is an arbitrary polynomial and $B > 1$. Then for any $\alpha \in \mathbb{T}$,*

$$\sum_{x=1}^{N} h(x)\lambda_{b,W}(x)e(\alpha\psi(x)) = \frac{\phi(W)}{\phi(Wq)} \sum_{\substack{1\le r\le q\\ (Wr+b,q)=1}} e(a\psi(r)/q)\sum_{x=1}^{N} h(x)e((\alpha-a/q)\psi(x)) + O_{\deg h}(h(N)Ne^{-c\sqrt{\log N}})$$

*provided that*

$$|\alpha q-a|\le (\log N)^B/\psi(N) \quad \textit{with} \quad 1\le a\le q\le (\log N)^B,$$

*where $c$ is a positive constant.*

*Proof.* Let

$$F_y(a/q) = \sum_{x=1}^{y} \lambda_{b,W}(x)e(a\psi(x)/q) = \sum_{\substack{1\le r\le Wq\\ (r,q)=1\\ r\equiv b \,(\mathrm{mod}\, W)}} e(a\psi((r-b)/W)/q) \sum_{\substack{x\in\Lambda_{r,Wq}\\ Wqx+r\le Wy+b}} \frac{\phi(W)q}{\phi(Wq)}\lambda_{r,Wq}(x).$$

The well-known Siegel–Walfisz theorem (cf. [7]) asserts that

$$\sum_{\substack{p\le y \text{ is prime}\\ p\equiv b\,(\mathrm{mod}\, q)}} \log p = \frac{y}{\phi(q)} + O(ye^{-c'\sqrt{\log y}})$$

provided that $q\le(\log y)^{c_1}$, where $c_1$, $c'$ are positive constants. Hence

$$\sum_{\substack{x\in\Lambda_{r,Wq}\\ Wqx+r\le Wy+b}} \lambda_{r,Wq}(x) = \frac{y}{q} + O(Wye^{-c'\sqrt{\log(Wy)}}).$$

It follows that

$$F_y(a/q) = \frac{\phi(W)y}{\phi(Wq)} \sum_{\substack{1\le r\le q\\ (Wr+b,q)=1}} e(a\psi(r)/q) + O(ye^{-c'\sqrt{\log y}/2}).$$

Let $\theta = \alpha - a/q$. Then

$$\begin{aligned}\sum_{x=1}^{N} h(x)\lambda_{b,W}(x)e(\alpha\psi(x)) &= h(N)e(\theta\psi(N))F_N(a/q) \\ &\quad - \sum_{y=1}^{N-1}(h(y+1)e(\theta\psi(y+1)) - h(y)e(\theta\psi(y)))F_y(a/q) \\ &= \frac{\phi(W)}{\phi(Wq)} \sum_{\substack{1\le r\le q\\ (Wr+b,q)=1}} e(a\psi(r)/q)\sum_{y=1}^{N} h(y)e(\theta\psi(y)) \\ &\quad + O_{\deg h}(h(N)Ne^{-c'\sqrt{\log N}/3})\end{aligned}$$

by noting that

$$\begin{aligned}h(y+1)e(\theta\psi(y+1)) &- h(y)e(\theta\psi(y)) \\ &= O(h^{\Delta}(y)) + O(h(y)\theta\psi^{\Delta}(y+1)). \quad \blacksquare\end{aligned}$$

Lemma 2.3. *For any $\theta \in \mathbb{T}$,*

$$\sum_{x=1}^{N} \psi^{\Delta}(x-1)e(\theta\psi(x)) = \sum_{x=1}^{\psi(N)} e(\theta x) + O(\theta\psi(N)\psi^{\Delta}(N)).$$

*Proof.* Clearly

$$\begin{aligned}\sum_{x=1}^{N} \psi^{\Delta}(x-1)e(\theta\psi(x)) - \sum_{x=1}^{\psi(N)} e(\theta x) &= \sum_{x=1}^{N} e(\theta\psi(x)) \sum_{y=0}^{\psi^{\Delta}(x-1)-1}(1-e(-\theta y)) \\ &= O\Big(\sum_{x=1}^{N}\sum_{y=0}^{\psi^{\Delta}(x-1)-1} \theta y\Big) \\ &= O(\theta\psi(N)\psi^{\Delta}(N)). \quad \blacksquare\end{aligned}$$

Lemma 2.4. *For any $\varepsilon > 0$,*

$$\sum_{x=1}^{N} e(\alpha\psi(x)) \ll_{\varepsilon} N^{1+\varepsilon}\Big(\frac{a_1}{q} + \frac{a_1}{N} + \frac{q}{N^k}\Big)^{2^{1-k}}$$

*provided that $|\alpha - a/q| \le q^{-2}$.*

*Proof.* We leave the proof as an exercise for the readers, since it is just a little modification of the proof of Weyl's inequality [27, Lemma 2.4]. $\blacksquare$

LEMMA 2.5 (Hua). *Suppose that* $(q, a_1, \ldots, a_k) = 1$. *Then*

$$\sum_{r=1}^{q} e(\psi(r)/q) \ll_\varepsilon q^{1-1/k+\varepsilon} \quad \textit{for any } \varepsilon > 0.$$

*Proof.* See [27, Theorem 7.1]. ■

LEMMA 2.6.

$$\int_{\mathbb{T}} \Big| \sum_{x=1}^{N} \psi^{\Delta}(x-1) e(\alpha\psi(x)) \Big|^{\rho} d\alpha \ll_\rho \gcd(\psi)\psi(N)^{\rho-1} \quad \textit{for } \rho \geq k\, 2^{k+2},$$

*where* $\gcd(\psi)$ *denotes the greatest common divisor of* $a_1, \ldots, a_k$.

*Proof.* Notice that

$$\int_0^1 \Big| \sum_{x=1}^{N} (a\psi)^{\Delta}(x-1) e(\alpha a\psi(x)) \Big|^{\rho} d\alpha = a^{\rho-1} \int_0^a \Big| \sum_{x=1}^{N} \psi^{\Delta}(x-1) e(\alpha\psi(x)) \Big|^{\rho} d\alpha$$
$$= a^{\rho} \int_0^1 \Big| \sum_{x=1}^{N} \psi^{\Delta}(x-1) e(\alpha\psi(x)) \Big|^{\rho} d\alpha.$$

So without loss of generality, we may assume that $\gcd(\psi) = 1$. Let $\nu = 1/5$ and $\varepsilon = 2^{-k}\nu - k/(2\rho)$. Let

$$\mathfrak{M}_{a,q} = \{\alpha \in \mathbb{T} : |\alpha q - a| \leq N^{\nu}/\psi(N)\}, \quad \mathfrak{M} = \bigcup_{\substack{1 \leq a \leq q \leq N^{\nu} \\ (a,q)=1}} \mathfrak{M}_{a,q}$$

and $\mathfrak{m} = \mathbb{T} \setminus \mathfrak{M}$. Clearly $\operatorname{mes}(\mathfrak{M}) \leq 2N^{3\nu}/\psi(N)$, where mes denotes the Lebesgue measure.

If $\alpha \in \mathfrak{m}$, then by Lemma 2.4 we have

$$\sum_{x=1}^{N} \psi^{\Delta}(x-1) e(\alpha\psi(x))$$
$$= \psi^{\Delta}(N-1) \sum_{x=1}^{N} e(\alpha\psi(x)) - \sum_{y=1}^{N-1} (\psi^{\Delta}(y) - \psi^{\Delta}(y-1)) \sum_{x=1}^{y} e(\alpha\psi(x))$$
$$\ll_\varepsilon \psi^{\Delta}(N) N^{1+\varepsilon-2^{1-k}\nu}.$$

Hence

$$\int_{\mathfrak{m}} \Big| \sum_{x=1}^{N} \psi^{\Delta}(x-1) e(\alpha\psi(x)) \Big|^{\rho} d\alpha \ll_\varepsilon \psi(N)^{\rho} N^{\rho(\varepsilon-2^{1-k}\nu)} = o(\psi(N)^{\rho-1}).$$

On the other hand, if $\alpha \in \mathfrak{M}$, then by Lemmas 2.1 and 2.3,

$$\sum_{x=1}^{N} \psi^{\Delta}(x-1)e(\alpha\psi(x)) = \frac{1}{q}\sum_{r=1}^{q} e(a\psi(r)/q) \sum_{x=1}^{\psi(N)} e((\alpha - a/q)x) + O(\psi^{\Delta}(N)N^{\nu}).$$

Let $L = \lfloor \rho/2 \rfloor$. Obviously

$$\int_{\mathfrak{M}} \Big|\sum_{x=1}^{N} \psi^{\Delta}(x-1)e(\alpha\psi(x))\Big|^{\rho} d\alpha \ll \psi(N)^{\rho-2L} \int_{\mathfrak{M}} \Big|\sum_{x=1}^{N} \psi^{\Delta}(x-1)e(\alpha\psi(x))\Big|^{2L} d\alpha.$$

So it suffices to show that

$$\int_{\mathfrak{M}} \Big|\sum_{x=1}^{N} \psi^{\Delta}(x-1)e(\alpha\psi(x))\Big|^{2L} d\alpha \ll_L \psi(N)^{2L-1}.$$

Now

$$\Big|\sum_{x=1}^{N} \psi^{\Delta}(x-1)e(\alpha\psi(x))\Big|^{2L} = \Big|\frac{1}{q}\sum_{r=1}^{q} e(a\psi(r)/q) \sum_{x=1}^{\psi(N)} e((\alpha - a/q)x)\Big|^{2L} + O(\psi(N)^{2L-1}\psi^{\Delta}(N)N^{\nu}).$$

Hence

$$\int_{\mathfrak{M}} \Big|\sum_{x=1}^{N} \psi^{\Delta}(x-1)e(\alpha\psi(x))\Big|^{2L} d\alpha = \sum_{\substack{1\le a\le q\le N^{\nu}\\ (a,q)=1}} \int_{\mathfrak{M}_{a,q}} \Big|\frac{1}{q}\sum_{r=1}^{q} e(a\psi(r)/q) \sum_{x=1}^{\psi(N)} e((\alpha - a/q)x)\Big|^{2L} d\alpha + O(\psi(N)^{2L-1}\psi^{\Delta}(N)N^{\nu}\mathrm{mes}(\mathfrak{M})).$$

Clearly

$$\int_{\mathfrak{M}_{a,q}} \Big|\sum_{x=1}^{\psi(N)} e((\alpha - a/q)x)\Big|^{2L} d\alpha \le \int_{\mathbb{T}} \Big|\sum_{x=1}^{\psi(N)} e((\alpha - a/q)x)\Big|^{2L} d\alpha = \sum_{\substack{1\le x_1,\dots,x_{2L}\le\psi(N)\\ x_1+\dots+x_L=x_{L+1}+\dots+x_{2L}}} 1 \le \psi(N)^{2L-1}.$$

And by Lemma 2.5,

$$\sum_{\substack{1\le a\le q\le N^\nu\\(a,q)=1}}\bigg|\frac{1}{q}\sum_{r=1}^{q}e(a\psi(r)/q)\bigg|^{2L}\ll_\varepsilon\sum_{\substack{1\le a\le q\le N^\nu\\(a,q)=1}}q^{-2L(1/k-\varepsilon)}\le\sum_{1\le q\le N^\nu}q^{1-2L(1/k-\varepsilon)}$$
$$=O_L(1)$$

since $L>(1/k-\varepsilon)^{-1}$. We are done. ∎

LEMMA 2.7. *Supposing that* $(a,q)=1$, *we have*

$$\sum_{\substack{1\le r\le q\\(Wr+b,q)=1}}e(a\psi(r)/q)\ll_\varepsilon\gcd(\psi)q^{1-1/k(k+1)+\varepsilon}.$$

*Proof.* Clearly

$$\sum_{\substack{1\le r\le q\\(Wr+b,q)=1}}e(a\psi(r)/q)=\sum_{r=1}^{q}e(a\psi(r)/q)\sum_{d\mid(Wr+b,q)}\mu(d),$$

where $\mu$ is the Möbius function. Note that $d\mid(Wr+b)\Rightarrow(d,W)=1$ since $(W,b)=1$. Hence

$$\sum_{\substack{1\le r\le q\\(Wr+b,q)=1}}e(a\psi(r)/q)=\sum_{\substack{d\mid q\\b_d\text{ exists}}}\mu(d)\sum_{\substack{1\le r\le q\\r\equiv b_d\ (\mathrm{mod}\ d)}}e(a\psi(r)/q),$$

where $1\le b_d\le d$ is the integer such that $Wb_d+b\equiv0\pmod d$.

For those $d\le q^{1/k(k+1)}$ for which $b_d$ exists, we have

$$\sum_{\substack{1\le r\le q\\r\equiv b_d\ (\mathrm{mod}\ d)}}e(a\psi(r)/q)=\sum_{r=0}^{q/d-1}e(a\psi(dr+b_d)/q).$$

Write

$$\psi(dr+b_d)=\sum_{i=1}^{k}a_{k-i+1}\sum_{j=0}^{i}\binom{i}{j}d^jr^jb_d^{i-j}=\sum_{j=0}^{k}d^jr^j\sum_{i=j}^{k}\binom{i}{j}a_{k-i+1}b_d^{i-j}$$
$$=a_1'r^k+a_2'r^{k-1}+\cdots+a_k'r+a_{k+1}'.$$

Notice that

$$(q,a_1',\dots,a_k')=(q,d^ka_1,a_2',\dots,a_k')\le d^k(q,a_1,a_2',\dots,a_k').$$

Also

$$a_2'=d^{k-1}(a_2+ka_1b_d).$$

Therefore

$$(q,a_1,a_2',\dots,a_k')=(q,a_1,d^{k-1}a_2,a_3',\dots,a_k')\le d^{k-1}(q,a_1,a_2,a_3',\dots,a_k').$$

Similarly, we obtain

$$(q,a_1',\dots,a_k')\le d^{k(k+1)/2}(q,a_1,\dots,a_k).$$

Thus by Lemma 2.5,

$$\begin{aligned}\sum_{r=0}^{q/d-1} e(a\psi(dr+b_d)/q) &\ll_\varepsilon (q/d,a_1',\dots,a_k')\Big(\frac{q/d}{(q/d,a_1',\dots,a_k')}\Big)^{1-1/k+\varepsilon/k}\\ &\le (q,a_1',\dots,a_k')^{\frac{1-\varepsilon}{k}} d^{\frac{1-\varepsilon}{k}-1} q^{1-\frac{1-\varepsilon}{k}}\\ &\le (a_1,\dots,a_k)^{\frac{1-\varepsilon}{k}} d^{(\frac{k+1}{2}+\frac{1}{k})(1-\varepsilon)-1} q^{1-\frac{1-\varepsilon}{k}}.\end{aligned}$$

On the other hand, clearly

$$\Big|\sum_{r=0}^{q/d-1} e(a\psi(dr+b_d)/q)\Big|\le\frac{q}{d}<q^{1-1/k(k+1)}$$

when $d>q^{1/k(k+1)}$. Thus

$$\begin{aligned}\Big|\sum_{\substack{1\le r\le q\\ (Wr+b,q)=1}} e(a\psi(r)/q)\Big| &\le \sum_{\substack{d\mid q,\, d\le q^{1/k(k+1)}\\ \text{and } b_d \text{ exists}}}\Big|\sum_{\substack{1\le r\le q\\ r\equiv b_d\ (\mathrm{mod}\ d)}} e(a\psi(r)/q)\Big|\\ &\quad+\sum_{\substack{d\mid q,\, d> q^{1/k(k+1)}\\ \text{and } b_d \text{ exists}}}\Big|\sum_{\substack{1\le r\le q\\ r\equiv b_d\ (\mathrm{mod}\ d)}} e(a\psi(r)/q)\Big|\\ &\ll_\varepsilon d(q)(\gcd(\psi)^{\frac{1-\varepsilon}{k}} q^{1-\frac{1-\varepsilon}{k}+\frac{1-\varepsilon}{k+1}}+q^{1-\frac{1}{k(k+1)}})\\ &\ll_\varepsilon \gcd(\psi) q^{1-\frac{1}{k(k+1)}+\varepsilon},\end{aligned}$$

where $d(q)$ is the divisor function. ∎

Lemma 2.8. *For any $A>0$, there is a $B=B(A,k)>0$ such that*

$$\sum_{x=1}^{N}\lambda_{b,W}(x)e(\alpha\psi(x))\ll_B N(\log N)^{-A}$$

*provided that $|\alpha-a/q|\le q^{-2}$ with $1\le a\le q$, $(a,q)=1$ and $(\log N)^B\le q\le \psi(N)(\log N)^{-B}$.*

*Proof.* Vinogradov dealt with the case $\psi(x)=x^k$ and $W=1$ in [28]. The general proof is standard but long, so we omit it. ∎

Lemma 2.9.

$$\int_{\mathbb{T}}\Big|\sum_{x=1}^{N}\psi^{\Delta}(x-1)\lambda_{b,W}(x)e(\alpha\psi(x))\Big|^{\rho}d\alpha\ll_\rho \gcd(\psi)\psi(N)^{\rho-1}$$

*for $\rho\ge k\,2^{k+2}+1$.*

*Proof.* Without loss of generality, we assume that $\gcd(\psi) = 1$. Let $B > 2\rho$ be a sufficiently large integer satisfying the requirement of Lemma 2.8 for $A = 2\rho$. Let

$$\mathfrak{M}_{a,q} = \{\alpha \in \mathbb{T} : |\alpha q - a| \le (\log N)^{2B}/\psi(N)\},$$

$$\mathfrak{M} = \bigcup_{\substack{1\le a\le q\le (\log N)^{2B}\\ (a,q)=1}} \mathfrak{M}_{a,q}$$

and $\mathfrak{m} = \mathbb{T} \setminus \mathfrak{M}$.

If $\alpha \in \mathfrak{m}$, then there exist $(\log N)^{2B} \le q \le \psi(N)(\log N)^{-2B}$ and $1 \le a \le q$ with $(a, q) = 1$ such that $|\alpha - a/q| \le q^{-2}$. By Lemma 2.8,

$$\sum_{x=1}^{y} \lambda_{b,W}(x) e(\alpha\psi(x)) \ll_B y(\log y)^{-2\rho}$$

for $N(\log N)^{-B/k} \le y \le N$. Therefore

$$\begin{aligned}
&\Big|\sum_{x=1}^{N} \psi^{\Delta}(x-1)\lambda_{b,W}(x)e(\alpha\psi(x))\Big| \\
&= \Big|\psi^{\Delta}(N-1)\sum_{x=1}^{N} e(\alpha\psi(x))\lambda_{b,W}(x) - \sum_{y=1}^{N-1}(\psi^{\Delta})^{\Delta}(y-1)\sum_{x=1}^{y} e(\alpha\psi(x))\lambda_{b,W}(x)\Big| \\
&\le \psi^{\Delta}(N-1)\Big|\sum_{x=1}^{N} e(\alpha\psi(x))\lambda_{b,W}(x)\Big| + \sum_{1\le y<N(\log N)^{-B/k}} |y(\psi^{\Delta})^{\Delta}(y-1)| \\
&\quad + \sum_{N(\log N)^{-B/k}\le y<N} (\psi^{\Delta})^{\Delta}(y-1)\Big|\sum_{x=1}^{y} e(\alpha\psi(x))\lambda_{b,W}(x)\Big| \\
&\ll_B \psi(N)(\log N)^{-2\rho}.
\end{aligned}$$

Let $L = \lfloor(\rho-1)/2\rfloor$. Then we have

$$\begin{aligned}
&\int_{\mathfrak{m}} \Big|\sum_{x=1}^{N} \psi^{\Delta}(x-1)\lambda_{b,W}(x)e(\alpha\psi(x))\Big|^{\rho} d\alpha \\
&\qquad \ll_B (\psi(N)(\log N)^{-2\rho})^{\rho-2L} \int_{\mathfrak{m}} \Big|\sum_{x=1}^{N} \psi^{\Delta}(x-1)\lambda_{b,W}(x)e(\alpha\psi(x))\Big|^{2L} d\alpha \\
&\qquad \ll_L \psi(N)^{\rho-2L}(\log N)^{-2\rho} \int_{\mathbb{T}} \Big|\sum_{x=1}^{N} \psi^{\Delta}(x-1)\lambda_{b,W}(x)e(\alpha\psi(x))\Big|^{2L} d\alpha.
\end{aligned}$$

Noting that

$$\begin{aligned}
&\int_{\mathbb{T}} \Big|\sum_{x=1}^{N} \psi^{\Delta}(x-1)\lambda_{b,W}(x)e(\alpha\psi(x))\Big|^{2L} d\alpha \\
&\qquad = \sum_{\substack{1\le x_1,\dots,x_{2L}\le N\\ \psi(x_1)+\cdots+\psi(x_L)=\psi(x_{L+1})+\cdots+\psi(x_{2L})}} \prod_{j=1}^{2L} \psi^{\Delta}(x_j-1)\lambda_{b,W}(x_j) \\
&\qquad \le (\log(WN+b))^{2L} \sum_{\substack{1\le x_1,\dots,x_{2L}\le N\\ \psi(x_1)+\cdots+\psi(x_L)=\psi(x_{L+1})+\cdots+\psi(x_{2L})}} \prod_{j=1}^{2L} \psi^{\Delta}(x_j-1) \\
&\qquad \ll_L (\log N)^{2L} \int_{\mathbb{T}} \Big|\sum_{x\le N} \psi^{\Delta}(x-1)e(\alpha\psi(x))\Big|^{2L} d\alpha,
\end{aligned}$$

so using Lemma 2.6 we have

$$\int_{\mathfrak{m}} \Big|\sum_{x\le N} \psi^{\Delta}(x-1)\lambda_{b,W}(x)e(\alpha\psi(x))\Big|^{\rho} d\alpha \ll_L \psi(N)^{\rho-1}(\log N)^{-\rho}.$$

If $\alpha \in \mathfrak{M}_{a,q}$, then by Lemma 2.2,

$$\begin{aligned}
&\Big|\sum_{x\le N} \psi^{\Delta}(x-1)\lambda_{b,W}(x)e(\alpha\psi(x))\Big|^{\rho} \\
&\quad = \Big|\frac{\phi(W)}{\phi(Wq)} \sum_{\substack{1\le r\le q\\ (Wr+b,q)=1}} e(a\psi(r)/q) \sum_{x\le N} \psi^{\Delta}(x-1)e((\alpha-a/q)\psi(x))\Big|^{\rho} \\
&\qquad + O(\psi(N)^{\rho}(\log N)^{-7B}).
\end{aligned}$$

In view of Lemma 2.7, letting $\varepsilon=(k+2)^{-4}$, we have

$$\begin{aligned}
&\sum_{\substack{1\le a\le q\le(\log N)^B\\ (a,q)=1}} \Big|\frac{\phi(W)}{\phi(Wq)} \sum_{\substack{1\le r\le q\\ (Wr+b,q)=1}} e(a\psi(r)/q)\Big|^{\rho} \\
&\qquad\qquad \ll_{\varepsilon} \sum_{1\le q\le(\log N)^B} q^{1-\rho(\frac{1}{k(k+1)}-2\varepsilon)} = O_{\rho,\varepsilon}(1).
\end{aligned}$$

Applying Lemma 2.6, we conclude that

$$\begin{aligned}
&\int_{\mathfrak{M}} \Big|\sum_{x\le N} \psi^{\Delta}(x-1)\lambda_{b,W}(x)e(\alpha\psi(x))\Big|^{\rho}\,d\alpha\\
&\quad= \sum_{\substack{1\le a\le q\le (\log N)^B\\ (a,q)=1}} \Big|\frac{\phi(W)}{\phi(Wq)} \sum_{\substack{1\le r\le q\\ (Wr+b,q)=1}} e(a\psi(r)/q)\Big|^{\rho}\\
&\qquad\times \int_{\mathfrak{M}_{a,q}} \Big|\sum_{x\le N} \psi^{\Delta}(x-1)e((\alpha-a/q)\psi(x))\Big|^{\rho}\,d\alpha\\
&\qquad+ O(\mathrm{mes}(\mathfrak{M})\psi(N)^{\rho}(\log N)^{-7B})\\
&\quad\le \Big(\sum_{\substack{1\le a\le q\le (\log N)^B\\ (a,q)=1}} \Big|\frac{\phi(W)}{\phi(Wq)} \sum_{\substack{1\le r\le q\\ (Wr+b,q)=1}} e(a\psi(r)/q)\Big|^{\rho}\Big)\\
&\qquad\times \int_{\mathbb{T}} \Big|\sum_{x\le N} \psi^{\Delta}(x-1)e(\alpha\psi(x))\Big|^{\rho}\,d\alpha + O(\psi(N)^{\rho-1}(\log N)^{-B})\\
&\quad\ll_{\rho,\varepsilon} \psi(N)^{\rho-1}. \ \blacksquare
\end{aligned}$$

LEMMA 2.10. *Suppose that $\psi$ is positive and strictly increasing on $[1,N]$. Let $p\ge\psi(N)$ be a prime. Then*

$$\frac{1}{p}\sum_{r=1}^{p}\Big|\sum_{z=1}^{N} \psi^{\Delta}(z-1)\lambda_{b,W}(z)e(-r\psi(z)/p)\Big|^{\rho} \ll_{\rho} \gcd(\psi)\psi(N)^{\rho-1}$$

*for $\rho\ge k\,2^{k+2}+1$.*

*Proof.* We require a well-known result of Marcinkiewicz and Zygmund (cf. [12, Lemma 6.5]):

$$\sum_{r\in\mathbb{Z}_p}\Big|\sum_{x=1}^{p} f(x)e(-xr/p)\Big|^{\rho} \ll_{\rho} p\int_{\mathbb{T}} |\widehat{f}(\theta)|^{\rho}\,d\theta$$

for any function $f:\mathbb{Z}_p=\mathbb{Z}/p\mathbb{Z}\to\mathbb{C}$, where

$$\widehat{f}(\theta)=\sum_{x=1}^{p} f(x)e(-\theta x).$$

Define

$$f(x)=\begin{cases}\psi^{\Delta}(z-1)\lambda_{b,W}(z) & \text{if } x=\psi(z) \text{ where } 1\le z\le N,\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$\sum_{r\in\mathbb{Z}_p}\Big|\sum_{z=1}^{N}\psi^{\Delta}(z-1)\lambda_{b,W}(z)e(-\psi(z)r/p)\Big|^{\rho}$$

$$=\sum_{r\in\mathbb{Z}_p}\Big|\sum_{x=1}^{p}f(x)e(-xr/p)\Big|^{\rho}\ll_{\rho}p\int_{\mathbb{T}}\Big|\sum_{x=1}^{p}f(x)e(-x\theta)\Big|^{\rho}d\theta$$

$$=p\int_{\mathbb{T}}\Big|\sum_{z=1}^{N}\psi^{\Delta}(z-1)\lambda_{b,W}(z)e(-\psi(z)\theta)\Big|^{\rho}d\theta\ll_{\rho}\gcd(\psi)p\psi(N)^{\rho-1},$$

where Lemma 2.9 is applied in the last inequality. ■

**3. Proof of Theorem 1.3.** Clearly Theorem 1.3 is a consequence of the following theorem:

THEOREM 3.1. *Suppose that $k\ge t\ge 1$ are integers, $a_{k-t+1}$ is a non-zero integer and $0<\delta\le 1$. Let $\psi(x)=a_1x^k+a_2x^{k-1}+\cdots+a_{k-t+1}x^t$ be an arbitrary polynomial with integral coefficients and positive leading coefficient. Then for any positive integer $W$, there exist $N(\delta,W,\psi)$ and $c(\delta,a_{k-t+1})>0$ satisfying*

$$\min_{\substack{A\subseteq\{1,\dots,n\}\\|A|\ge\delta n}}|\{(x,y,z):x,y\in A,\ z\in\Lambda_{1,W},\ x-y=\psi(z)\}|$$

$$\ge c(\delta,a_{k-t+1})\,\frac{Wn^{1+1/k}a_1^{-1/k}}{\phi(W)\log n}$$

*if $n\ge N(\delta,W,\psi)$.*

REMARK. We emphasize that in Theorem 3.1 the constant $c(\delta,a_{k-t+1})$ only depends on $k,\delta,a_{k-t+1}$. As we will see later, this fact is important in the proof of Theorem 1.4.

*Proof.* Similarly to Tao's arguments [25] on Roth's theorem [19], we apply induction on $\delta$. Suppose that $P(\delta)$ is a proposition on $0<\delta\le 1$. Assume that $P(\delta)$ satisfies the following conditions:

(i) There exists $0<\delta_0<1$ such that $P(\delta)$ holds for any $\delta_0\le\delta\le 1$.
(ii) There exists a continuous function $\varepsilon(\delta)>0$ such that $\delta+\varepsilon(\delta)\le 1$ for any $0<\delta\le\delta_0$ and $P(\delta+\varepsilon(\delta))\Rightarrow P(\delta)$.
(iii) If $0<\delta'<\delta\le 1$, then $P(\delta')\Rightarrow P(\delta)$.

Then we claim that $P(\delta)$ holds for any $0<\delta\le 1$. In fact, suppose on the contrary that there exists $0<\delta\le 1$ such that $P(\delta)$ does not hold. Let

$$\delta^*=\limsup_{\substack{0<\delta\le 1\\ P(\delta)\text{ does not hold}}}\delta.$$

From (i), we know that $\delta^* \le \delta_0$. Since $\delta + \varepsilon(\delta)$ is continuous, there exists $0 < \delta_1 < \delta^*$ such that

$$|(\delta^* + \varepsilon(\delta^*)) - (\delta_1 + \varepsilon(\delta_1))| < \frac{1}{2}\,\varepsilon(\delta^*),$$

i.e., $0 < \delta_1 < \delta^* < \delta_1 + \varepsilon(\delta_1) \le 1$. Hence $P(\delta_1 + \varepsilon(\delta_1))$ holds but $P(\delta_1)$ does not by the definition of $\delta^*$. This obviously contradicts (ii) and (iii).

Suppose that $A \subset \{1, \dots, n\}$ with $|A| \ge \delta n$. Firstly, we shall show that the conclusion of Theorem 3.1 holds for $\delta \ge 3/4$. Define

$$r_{W,\psi}(A) = |\{(x, y, z) : x, y \in A,\ z \in \Lambda_{1,W},\ x - y = \psi(z)\}|.$$

Clearly

$$|\{z \in \Lambda_{1,W} : 1 \le \psi(z) \le n/3\}| \ge \frac{1}{4k}\,\frac{Wn^{1/k}a_1^{-1/k}}{\phi(W)\log n}$$

whenever $n$ is sufficiently large (depending on the coefficients of $\psi$). Moreover, for any $1 \le z \le n/3$,

$$\begin{aligned}|\{(x, y) &: x, y \in A,\ x - y = z\}| \\ &= |A \cap (z + A)| = 2|A| - |A \cup (z + A)| \ge \frac{2 \cdot 3n}{4} - \frac{4n}{3} = \frac{n}{6}.\end{aligned}$$

Hence

$$r_{W,\psi}(A) \ge \frac{1}{24k}\,\frac{Wn^{1+1/k}a_1^{-1/k}}{\phi(W)\log n}.$$

Now we assume that $\delta < 3/4$. Let $\varepsilon = \varepsilon(\delta, a_{k-t+1})$ be a small positive real number and $Q = Q(\delta, a_{k-t+1})$ be a large integer to be chosen later. We shall show that if the assertion of Theorem 3.1 holds for $\delta + \varepsilon$, it also holds for $\delta$. Define

$$\psi_q(x) = \psi(qx)/q^t = a_1 q^{k-t} x^k + \dots + a_{k-t+1} x^t.$$

By the induction hypothesis on $\delta + \varepsilon$, for any $1 \le q \le Q$,

$$\min_{\substack{A \subseteq \{1,\dots,n\} \\ |A| \ge (\delta+\varepsilon)n}} r_{Wq,\psi_q}(A) \ge \frac{c(\delta + \varepsilon, a_{k-t+1})}{2}\,\frac{Wq}{\phi(Wq)}\,\frac{n^{1+1/k}(a_1 q^{k-t})^{-1/k}}{\log n}$$

provided that

$$n \ge \max_{1 \le q \le Q} N(\delta + \varepsilon, Wq, \psi_q).$$

Let $\mathbb{A}_m(b, d)$ denote the arithmetic progression $\{b, b+d, \dots, b+(m-1)d\}$. Suppose that

$$n \ge \max\{e^{k(|a_1|+\dots+|a_{k-t+1}|)Q^{k-t}}, 10^4 \varepsilon^{-1} Q^t \max_{1 \le q \le Q} N(\delta + \varepsilon, Wq, \psi_q)\}$$

and $A \subseteq \{1, \dots, n\}$ with $|A| = \delta n$. Let $m = \lfloor 10^{-2}\varepsilon Q^{-t} n\rfloor$. Observe that $|\{b : x, y \in \mathbb{A}_m(b, q^t)\}| \le m$ for every pair $(x, y)$. Let

$$A_{b,q^t} = \{1 + (x - b)/q^t : x \in A \cap \mathbb{A}_m(b, q^t)\} \subseteq \{1, \dots, m\}.$$

Clearly if $x', y' \in A_{b,q^t}$ and $z' \in \Lambda_{1,Wq}$ satisfy that $x' - y' = \psi_q(z')$, then

$$x = b + (x' - 1)q^t, \quad y = b + (y' - 1)q^t \in A, \quad z = z'q \in \Lambda_{1,W}$$

and $x - y = \psi(z)$. So if there exists $1 \le q \le Q$ such that

$$|\{1 \le b \le n - mq^t : |A_{b,q^t}| \ge (\delta + \varepsilon)m\}| \ge \varepsilon n,$$

then

$$\begin{aligned} r_{W,\psi}(A) &\ge \frac{1}{m} \sum_{1 \le b \le n - mq^t} r_{Wq,\psi_q}(A_{b,q^t}) \\ &\ge \varepsilon n \, \frac{c(\delta + \varepsilon, a_{k-t+1})}{2} \, \frac{Wq}{\phi(Wq)} \, \frac{m^{1/k}(a_1 q^{k-t})^{-1/k}}{\log m} \\ &\ge \frac{c(\delta + \varepsilon, a_{k-t+1})\varepsilon^{1+1/k}}{400Q} \, \frac{W n^{1+1/k} a_1^{-1/k}}{\phi(W) \log n}. \end{aligned}$$

So we may assume that

$$|\{1 \le b \le n - mq^t : |A \cap \mathbb{A}_m(b, q^t)| \ge (\delta + \varepsilon)m\}| < \varepsilon n \tag{3.1}$$

for each $1 \le q \le Q$. Let

$$M = \max\{x \in \mathbb{Z} : \psi(x) \le n\}.$$

Clearly $M = n^{1/k} a_1^{-1/k}(1 + o(1))$. We shall show that

$$\int_{\mathbb{T}} \Big(\Big|\sum_{x \in A \cap [1,n]} e(\alpha x)\Big|^2 - \delta^2 \Big|\sum_{x \le n} e(\alpha x)\Big|^2\Big)\Big(\sum_{z \le M} \psi^{\Delta}(z - 1)\lambda_{1,W}(z) e(\alpha\psi(z))\Big)\, d\alpha$$

is relatively small.

For $1 \le q \le Q$, define

$$\mathfrak{M}_{a,q} = \Big\{\alpha : |\alpha - a/q| \le \frac{1}{2} q^{-t} m^{-1}\Big\}.$$

Let

$$\mathfrak{M} = \bigcup_{\substack{1 \le a \le q \le Q \\ (a,q)=1}} \mathfrak{M}_{a,q}, \quad \mathfrak{m} = \mathbb{T} \setminus \mathfrak{M}.$$

Let $B$ be a sufficiently large integer. For $1 \le q \le (\log M)^B$, define

$$\mathfrak{M}^*_{a,q} = \{\alpha : |\alpha q - a| \le (\log M)^B/\psi(M)\}.$$

Let

$$\mathfrak{M}^* = \bigcup_{\substack{1 \le a \le q \le (\log M)^B \\ (a,q)=1}} \mathfrak{M}^*_{a,q}, \quad \mathfrak{m}^* = \mathbb{T} \setminus \mathfrak{M}^*.$$

Suppose that $\alpha \in \mathfrak{m}$. We know

$$|\alpha q - a| \le (\log M)^B/\psi(M)$$

for some $1 \le a \le q < \psi(M)(\log M)^{-B}$ with $(a, q) = 1$. If $\alpha \in \mathfrak{m}^*$, i.e., $q > (\log M)^B$, then

$$|\alpha - a/q| \le q^{-2} \quad \text{and} \quad (\log y)^{B/2} \le \psi(y)(\log y)^{-B/2}$$

for any $M(\log M)^{-B/(2k)} \le y \le M$. So applying Lemma 2.8 and partial summation, we have

$$\sum_{z \le M} \psi^{\Delta}(z-1)\lambda_{1,W}(z)e(\alpha\psi(z)) \ll_B \psi(M)(\log M)^{-1} \le n(\log M)^{-1}$$

whenever $B$ is sufficiently large.

Now suppose that $q < (\log M)^B$, i.e., $\alpha \in \mathfrak{M}^*$. Applying Lemmas 2.2 and 2.3, we have

$$\begin{aligned}
&\sum_{z \le M} \psi^{\Delta}(z-1)\lambda_{1,W}(z)e(\alpha\psi(z)) \\
&= \frac{\phi(W)}{\phi(Wq)} \sum_{\substack{1 \le r \le q \\ (Wr+1,q)=1}} e(a\psi(r)/q) \sum_{z \le M} \psi^{\Delta}(z-1)e((\alpha - a/q)\psi(z)) \\
&\quad + O(\psi^{\Delta}(M)M(\log M)^{-4B}) \\
&= \frac{\phi(W)}{\phi(Wq)} \sum_{\substack{1 \le r \le q \\ (Wr+1,q)=1}} e(a\psi(r)/q)\sum_{z \le n} e((\alpha - a/q)z) + O(\psi^{\Delta}(M)M(\log M)^{-4B}).
\end{aligned}$$

Since $\alpha \in \mathfrak{m}$, either $q > Q$ or $|\alpha - a/q| > \frac{1}{2}q^{-t}m^{-1}$. If $q > Q$, then in light of Lemma 2.7,

$$\begin{aligned}
\left|\frac{\phi(W)}{\phi(Wq)} \sum_{\substack{1 \le r \le q \\ (Wr+1,q)=1}} e(a\psi(r)/q)\right| &\le \frac{1}{\phi(q)}\left| \sum_{\substack{1 \le r \le q \\ (Wr+1,q)=1}} e(a\psi(r)/q)\right| \\
&\le C_1|a_{k-t+1}|q^{-1/k(k+2)}.
\end{aligned}$$

And if $|\alpha - a/q| > \frac{1}{2}q^{-t}m^{-1}$, then

$$\left|\sum_{z=1}^{n} e((\alpha - a/q)z)\right| = \left|\frac{1 - e((\alpha - a/q)n)}{1 - e(\alpha - a/q)}\right| \le 4\pi q^t m.$$

Hence for $\alpha \in \mathfrak{m}$,

$$\begin{aligned}
\sum_{z \le M} \psi^{\Delta}(z-1)\lambda_{1,W}(z)e(\alpha\psi(z)) \le C_1|a_{k-t+1}|Q^{-1/k(k+2)}n + 4\pi m Q^t& \\
+ O(n(\log n)^{-1}).&
\end{aligned}$$

Suppose that $\alpha \in \mathfrak{M}$. Let $\tau = \mathbf{1}_A - \delta$ where $\mathbf{1}_A(x) = 1$ or $0$ according to whether $x \in A$ or not. Let

$$S(\alpha) = \sum_{c=0}^{m-1} e(\alpha c) \quad \text{and} \quad T(\alpha) = \sum_{b=1}^{n} \tau(b) e(\alpha b).$$

Then

$$\begin{aligned} S(\alpha q^t)T(\alpha) &= \sum_{b=1}^{n} \tau(b) \sum_{c=0}^{m-1} e(\alpha(b + cq^t)) \\ &= \sum_{b=1}^{n-mq^t} e(\alpha(b + (m-1)q^t)) \sum_{c=0}^{m-1} \tau(b + cq^t) + R(\alpha), \end{aligned}$$

where $|R(\alpha)| \le 2m^2 q^t$. When $|\alpha q^t - aq^{t-1}| \le \frac{1}{2} m^{-1}$,

$$|S(\alpha q^t)| = |S(\alpha q^t - aq^{t-1})| = \left| \frac{1 - e(m(\alpha q^t - aq^{t-1}))}{1 - e(\alpha q^t - aq^{t-1})} \right| \ge \frac{m}{\pi}.$$

Hence for $\alpha \in \mathfrak{M}_{a,q}$,

$$\begin{aligned} m|T(\alpha)| &\le \pi |S(\alpha q^t) T(\alpha)| \\ &\le \pi \Big| \sum_{b=1}^{n-mq^t} e(\alpha(b + (m-1)q^t)) \sum_{c=0}^{m-1} \tau(b + cq^t) \Big| + \pi |R(\alpha)|. \end{aligned}$$

Notice that $|\{1 \le b \le n - mq^t : x \in \mathbb{A}_m(b, q^t)\}| \le m$, and the equality holds if $1 + (m-1)q^t \le x \le n - mq^t$. It follows that

$$\begin{aligned} m|A| \ge \sum_{b=1}^{n-mq^t} |A \cap \mathbb{A}_m(b, q^t)| &= \sum_{x \in A} \sum_{b=1}^{n-mq^t} \mathbf{1}_{\mathbb{A}_m(b,q^t)}(x) \\ &\ge m|A| - 2m^2 q^t, \end{aligned}$$

whence

$$\Big| \sum_{b=1}^{n-mq^t} (|A \cap \mathbb{A}_m(b, q^t)| - (\delta + \varepsilon)m) \Big| \le \varepsilon nm + (2 + \delta + \varepsilon) m^2 q^t.$$

By the assumption (3.1), we have

$$\sum_{\substack{1 \le b \le n - mq^t \\ |A \cap \mathbb{A}_m(b,q^t)| \ge (\delta+\varepsilon)m}} (|A \cap \mathbb{A}_m(b, q^t)| - (\delta + \varepsilon)m) \le \varepsilon n(1 - \delta)m.$$

It follows that

$$\begin{aligned}\sum_{b=1}^{n-mq^t} \big|\,|A\cap\mathbb{A}_m(b,q^t)|-\delta m\big| &\le \sum_{b=1}^{n-mq^t} \big|\,|A\cap\mathbb{A}_m(b,q^t)|-(\delta+\varepsilon)m\big|+\varepsilon nm\\ &\le 2\sum_{\substack{1\le b\le n-mq^t\\ |A\cap\mathbb{A}_m(b,q^t)|\ge(\delta+\varepsilon)m}} (|A\cap\mathbb{A}_m(b,q^t)|-(\delta+\varepsilon)m)\\ &\quad+\Big|\sum_{b=1}^{n-mq^t}(|A\cap\mathbb{A}_m(b,q^t)|-(\delta+\varepsilon)m)\Big|+\varepsilon nm\\ &\le 4\varepsilon nm+4m^2q^t.\end{aligned}$$

Thus for any $\alpha\in\mathfrak{M}$,

$$\begin{aligned}|T(\alpha)| &\le \frac{\pi}{m}\Big(\Big|\sum_{b=1}^{n-mq^t} e(\alpha(b+(m-1)q^t))\sum_{c=0}^{m-1}\tau(b+cq^t)\Big|+2m^2q^t\Big)\\ &\le \frac{\pi}{m}\Big(\sum_{b=1}^{n-mq^t}\big|\,|A\cap\mathbb{A}_m(b,q^t)|-\delta m\big|+2m^2q^t\Big)\\ &\le 4\pi\varepsilon n+6\pi mQ^t,\end{aligned}$$

i.e.,

$$\Big|\sum_{x=1}^{n}\mathbf{1}_A(x)e(\alpha x)-\delta\sum_{x=1}^{n}e(\alpha x)\Big|\le 16\varepsilon n.$$

It is easy to see that

$$\begin{aligned}\big|\,|x|^2-|y|^2\big| &\le \big|\,|x|-|y|\,\big|^{2/\rho}(|x|+|y|)^{2-2/\rho}\\ &\le 4|x-y|^{2/\rho}(|x|^{2-2/\rho}+|y|^{2-2/\rho})\end{aligned}$$

for any $\rho\ge 2$. Let $\rho=k\,2^{k+3}$. Then

$$\begin{aligned}&\Big|\int_{\mathfrak{M}}\Big(\Big|\sum_{x=1}^{n}\mathbf{1}_A(x)e(\alpha x)\Big|^2-\delta^2\Big|\sum_{x=1}^{n}e(\alpha x)\Big|^2\Big)\Big(\sum_{z=1}^{M}\psi^{\Delta}(z-1)\lambda_{1,W}(z)e(\alpha\psi(z))\Big)\,d\alpha\Big|\\ &\qquad\le 4(16\varepsilon n)^{2/\rho}\int_{\mathfrak{M}}\Big(\Big|\sum_{x=1}^{n}\mathbf{1}_A(x)e(\alpha x)\Big|^{2-2/\rho}+\delta^{2-2/\rho}\Big|\sum_{x=1}^{n}e(\alpha x)\Big|^{2-2/\rho}\Big)\\ &\qquad\quad\times\Big|\sum_{z=1}^{M}\psi^{\Delta}(z-1)\lambda_{1,W}(z)e(\alpha\psi(z))\Big|\,d\alpha.\end{aligned}$$

By the Hölder inequality,

$$\int_{\mathfrak{M}} \Big|\sum_{x=1}^{n} \mathbf{1}_A(x)e(\alpha x)\Big|^{2-2/\rho} \Big|\sum_{z=1}^{M} \psi^{\Delta}(z-1)\lambda_{1,W}(z)e(\alpha\psi(z))\Big|\, d\alpha$$
$$\le \Big(\int_{\mathbb{T}} \Big|\sum_{x=1}^{n} \mathbf{1}_A(x)e(\alpha x)\Big|^{2} d\alpha\Big)^{1-1/\rho} \Big(\int_{\mathbb{T}} \Big|\sum_{z=1}^{M} \psi^{\Delta}(z-1)\lambda_{1,W}(z)e(\alpha\psi(z))\Big|^{\rho} d\alpha\Big)^{1/\rho}.$$

Lemma 2.9 yields

$$\int_{\mathbb{T}} \Big|\sum_{z=1}^{M} \psi^{\Delta}(z-1)\lambda_{1,W}(z)e(\alpha\psi(z))\Big|^{\rho} d\alpha \le C_2|a_{k-t+1}|\psi(M)^{\rho-1}.$$

Therefore

$$\int_{\mathfrak{M}} \Big|\sum_{x=1}^{n} \mathbf{1}_A(x)e(\alpha x)\Big|^{2-2/\rho} \Big|\sum_{z=1}^{M} \psi^{\Delta}(z-1)\lambda_{1,W}(z)e(\alpha\psi(z))\Big|\, d\alpha$$
$$\le C_2^{1/\rho}|a_{k-t+1}|^{1/\rho}(\delta n)^{1-1/\rho}n^{1-1/\rho}.$$

Similarly,

$$\int_{\mathfrak{M}} \Big|\sum_{x=1}^{n} e(\alpha x)\Big|^{2-2/\rho} \Big|\sum_{z=1}^{M} \psi^{\Delta}(z-1)\lambda_{1,W}(z)e(\alpha\psi(z))\Big|\, d\alpha$$
$$\le C_2^{1/\rho}|a_{k-t+1}|^{1/\rho}n^{2-2/\rho}.$$

We conclude that

$$\Big|\int_{\mathfrak{M}} \Big(\Big|\sum_{x=1}^{n} \mathbf{1}_A(x)e(\alpha x)\Big|^{2} - \delta^2\Big|\sum_{x=1}^{n} e(\alpha x)\Big|^{2}\Big) \Big(\sum_{z=1}^{M} \psi^{\Delta}(z-1)\lambda_{1,W}(z)e(\alpha\psi(z))\Big)\, d\alpha\Big|$$
$$\le 8C_2^{1/\rho}|a_{k-t+1}|^{1/\rho}\varepsilon^{2/\rho}(\delta^{1-1/\rho}+\delta^{2-2/\rho})n^2.$$

Now we have shown that

$$\Big|\int_{\mathbb{T}} \Big(\Big|\sum_{x\le n} \mathbf{1}_A e(\alpha x)\Big|^{2} - \delta^2\Big|\sum_{x\le n} e(\alpha x)\Big|^{2}\Big) \Big(\sum_{z\le M} \psi^{\Delta}(z-1)\lambda_{1,W}(z)e(\alpha\psi(z))\Big)\, d\alpha\Big|$$
$$\le (2C_1|a_{k-t+1}|Q^{-1/k(k+2)}n + 5\pi m Q^t)$$
$$\times \int_{\mathbb{T}} \Big(\Big|\sum_{x\le n} \mathbf{1}_A e(\alpha x)\Big|^{2} + \delta^2\Big|\sum_{x\le n} e(\alpha x)\Big|^{2}\Big)\, d\alpha$$
$$+ 8C_2^{1/\rho}|a_{k-t+1}|^{1/\rho}\varepsilon^{2/\rho}(\delta^{1-1/\rho}+\delta^{2-2/\rho})n^2$$
$$\le 4C_1|a_{k-t+1}|Q^{-1/k(k+2)}\delta n^2 + \varepsilon\delta n^2 + 16C_2^{1/\rho}|a_{k-t+1}|^{1/\rho}\varepsilon^{2/\rho}\delta^{1-1/\rho}n^2.$$

On the other hand, we have

$$\begin{aligned}&\int_{\mathbb{T}}\Big|\sum_{x=1}^{n}e(\alpha x)\Big|^2\Big(\sum_{z=1}^{M}\psi^{\Delta}(z-1)\lambda_{1,W}(z)e(\alpha\psi(z))\Big)\,d\alpha\\&\quad=\sum_{\substack{1\le x,y\le n\\1\le z\le M\\x-y=\psi(z)}}\psi^{\Delta}(z-1)\lambda_{1,W}(z)\ge\sum_{\substack{1\le x,y\le n\\M/4+1\le z\le M/2\\x-y=\psi(z)}}\psi^{\Delta}(z-1)\lambda_{1,W}(z)\\&\quad\ge\frac{M}{8}\,(n-\psi(M/2))\psi^{\Delta}(M/4).\end{aligned}$$

It follows that

$$\begin{aligned}&\int_{\mathbb{T}}\Big|\sum_{x=1}^{n}\mathbf{1}_A(x)e(\alpha x)\Big|^2\Big(\sum_{z=1}^{M}\psi^{\Delta}(z-1)\lambda_{1,W}(z)e(\alpha\psi(z))\Big)\,d\alpha\\&\quad\ge\delta^2\int_{\mathbb{T}}\Big|\sum_{x=1}^{n}e(\alpha x)\Big|^2\Big(\sum_{z=1}^{M}\psi^{\Delta}(z-1)\lambda_{1,W}(z)e(\alpha\psi(z))\Big)\,d\alpha\\&\qquad-4C_1|a_{k-t+1}|Q^{-1/k(k+2)}\delta n^2-\varepsilon\delta n^2-16C_2^{1/\rho}|a_{k-t+1}|^{1/\rho}\varepsilon^{2/\rho}\delta^{1-1/\rho}n^2\\&\quad\ge\frac{k\delta^2n^2}{4^{k+1}}-4C_1|a_{k-t+1}|Q^{-1/k(k+2)}\delta n^2-\varepsilon\delta n^2\\&\qquad-16C_2^{1/\rho}|a_{k-t+1}|^{1/\rho}\varepsilon^{2/\rho}\delta^{1-1/\rho}n^2.\end{aligned}$$

Let $\varepsilon=4^{-(k+2)\rho}\delta^{(\rho+1)/2}C_2^{-1/2}|a_{k-t+1}|^{-1/2}$ and

$$Q=4^{(k+1)^4}\delta^{-2k(k+2)}C_1^{k(k+2)}|a_{k-t+1}|^{k(k+2)}.$$

Therefore

$$\begin{aligned}&|\{(x,y,z):x,y\in A,\ z\in\Lambda_{1,W},\ x-y=\psi(z)\}|\\&\quad\ge\frac{W/\phi(W)}{\psi^{\Delta}(M)\log(WM+1)}\\&\qquad\times\int_{\mathbb{T}}\Big|\sum_{x=1}^{n}\mathbf{1}_A(x)e(\alpha x)\Big|^2\Big(\sum_{z=1}^{M}\psi^{\Delta}(z-1)\lambda_{1,W}(z)e(\alpha\psi(z))\Big)\,d\alpha\\&\quad\ge\frac{W\delta^2}{4^{k+2}k\phi(W)}\,\frac{n^{1+1/k}a_1^{-1/k}}{\log n}.\end{aligned}$$

This yields the desired result. ■

Finally, let us briefly discuss the bound in Theorem 1.3. Let $R_{W,\psi}(\delta)$ be the least integer $n$ such that for any $A\subseteq\{1,\dots,n\}$, there exist $x,y\in A$ and $z\in\Lambda_{1,W}$ satisfying $x-y=\psi(z)$. In our proof, we choose $\varepsilon=\varepsilon(\delta)=O_{|a_{k-t}|}(\delta^{O_k(1)})$ and $Q=Q(\delta)=O_{|a_{k-t}|}(\delta^{-O_k(1)})$. So the iteration process

$\delta \to \delta + \varepsilon(\delta)$ will end after $O_{|a_{k-t}|}(\delta^{-O_k(1)})$ steps. Also, clearly for $\delta > 3/4$,

$$R_{W,\psi}(\delta) \ll (|a_1| + \cdots + |a_{k-t}|)(\min\{p : p \in \Lambda_{1,W}\})^k.$$

Notice that when the iteration process ends, $W$ becomes $WQ^{O_{|a_{k-t}|}(\delta^{-O_k(1)})}$ and $a_i$ becomes $a_i Q^{O_{|a_{k-t}|}(\delta^{-O_k(1)})}$. Hence we have

$$R_{W,\psi}(\delta) \le \exp(O_{W,a_1,\dots,a_{k-t}}(\delta^{-O_{|a_{k-t}|}(\delta^{-O_k(1)})})),$$

since $\min\{p : p \in \Lambda_{1,W}\} \le e^{O(W)}$. In other words, if a subset $A \subseteq \{1, \dots, n\}$ satisfies $|A| \ge O_{W,a_1,\dots,a_{k-t}}(n/\log\log\log n)$, then there exist $x, y \in A$ and $z \in \Lambda_{1,W}$ such that $x - y = \psi(z)$. Of course, this bound is very rough. We believe that it could be improved using some more refined estimations (e.g. [18], [1], [16], [17], [20]).

**4. Proof of Theorem 1.4.** Write $\psi(x) = a_1x^k + a_2x^{k-1} + \cdots + a_{k-t+1}x^t$ where $a_{k-t+1} \ne 0$. Let $\delta = \bar{d}_{\mathcal{P}}(P)$. Since $\bar{d}_{\mathcal{P}}(P) > 0$, there exist infinitely many $n$ such that

$$|P \cap [1, n]| \ge \frac{4\delta}{5} \frac{n}{\log n}.$$

Define

$$w(n) = \max\{w \le \log\log\log n : n \ge 16\mathcal{W}(w)N(\delta, \mathcal{W}(w), \psi_{\mathcal{W}(w)})\},$$

where $N(\delta, W, \psi)$ is as defined in Theorem 3.1 and $\mathcal{W}(w) = \prod_{p \le w,\, p \,\text{prime}} p$. Clearly $\lim_{n\to\infty} w(n) = \infty$. Let $w = w(n)$ and $\mathcal{W} = \mathcal{W}(w)$. Then

$$\sum_{\substack{x \in P\cap[1,n] \\ (x,\mathcal{W})=1}} \log x \ge \sum_{x \in P\cap[n^{2/3},n]} \log x \ge \frac{2\log n}{3} (|P \cap [1, n]| - n^{2/3}) \ge \frac{\delta}{2} n.$$

Hence there exists $1 \le b \le \mathcal{W}^t$ with $(b, \mathcal{W}) = 1$ such that

$$\sum_{\substack{x \in P\cap[1,n] \\ x \equiv b \,(\mathrm{mod}\, \mathcal{W}^t)}} \log x \ge \frac{\delta}{2\phi(\mathcal{W}^t)} n.$$

Let

$$A = \{(x - b)/\mathcal{W}^t : x \in P \cap [1, n],\, x \equiv b \pmod{\mathcal{W}^t}\}.$$

Let $N$ be a prime in the interval $(2n/\mathcal{W}^t, 4n/\mathcal{W}^t]$. Define $\lambda_{b,\mathcal{W}^t,N} = \lambda_{b,\mathcal{W}^t}/N$ and $a = \mathbf{1}_A \lambda_{b,\mathcal{W}^t,N}$. Then

$$\sum_x a(x) \ge \frac{\phi(\mathcal{W}^t)}{\mathcal{W}^t N} \frac{\delta n}{2\phi(\mathcal{W}^t)} \ge \frac{\delta}{8}.$$

Let

$$\psi_{\mathcal{W}}(x) = \psi(\mathcal{W}x)/\mathcal{W}^t = a_1\mathcal{W}^{k-t}x^k + \cdots + a_{k-t+1}x^t.$$

Clearly $\psi_{\mathcal{W}}(z)$ is positive and strictly increasing for $z \geq 1$, whenever $\mathcal{W}$ is sufficiently large.

Below we consider $A$ as a subset of $\mathbb{Z}_N$. Let

$$M = \max\{z \in \mathbb{N} : \psi_{\mathcal{W}}(z) < N/2\}.$$

If $x, y \in A$ and $1 \leq z \leq M$ satisfy $x - y = \psi_{\mathcal{W}}(z)$ in $\mathbb{Z}_N$, then we also have $x - y = \psi_{\mathcal{W}}(z)$ in $\mathbb{Z}$. In fact, since $1 \leq x, y < N/2$ and $1 \leq z \leq M$, it is impossible that $x - y = \psi_{\mathcal{W}}(z) - N$ in $\mathbb{Z}$. For a function $f : \mathbb{Z}_N \to \mathbb{C}$, define

$$\tilde{f}(r) = \sum_{x \in \mathbb{Z}_N} f(x) e(-xr/N).$$

LEMMA 4.1 (Bourgain [4], [5] and Green [12]). *Suppose that $\rho > 2$. Then*

$$\sum_r |\tilde{a}(r)|^\rho \leq C(\rho),$$

*where $C(\rho)$ is a constant only depending on $\rho$.*

*Proof.* See [12, Lemma 6.6]. ∎

LEMMA 4.2.

$$\sum_{r \in \mathbb{Z}_N} \Big| \sum_{z=1}^{M} \psi_{\mathcal{W}}^{\Delta}(z-1) \lambda_{1,\mathcal{W}W}(z) e(-\psi_{\mathcal{W}}(z) r/N) \Big|^\rho \leq C'(\rho) |a_{k-t+1}| N^\rho$$

*provided that $\rho \geq k\, 2^{k+3}$, where $C'(\rho)$ is a constant only depending on $\rho$.*

*Proof.* This is an immediate consequence of Lemma 2.10 since $\gcd(\psi_{\mathcal{W}}) \leq |a_{k-t+1}|$. ∎

Let $\eta$ and $\varepsilon$ be two positive real numbers to be chosen later. Let

$$R = \{r \in \mathbb{Z}_N : |\tilde{a}(r)| \geq \eta\}, \quad B = \{x \in \mathbb{Z}_N : \|xr/N\| \leq \varepsilon \text{ for all } r \in R\},$$

where $\|x\| = \min\{|x - z| : z \in \mathbb{Z}\}$. Define $\beta = \mathbf{1}_B/|B|$ and $a' = a * \beta * \beta$, where

$$f * g(x) = \sum_{y \in \mathbb{Z}_N} f(y) g(x - y).$$

Let $\varrho = k\, 2^{k+3}$.

LEMMA 4.3.

$$\sum_{\substack{x,y \in \mathbb{Z}_N \\ 1 \leq z \leq M \\ x - y = \psi_{\mathcal{W}}(z)}} (a'(x)a'(y) - a(x)a(y)) \psi_{\mathcal{W}}^{\Delta}(z-1) \lambda_{1,\mathcal{W}W}(z) \leq C(\varepsilon^2 \eta^{-5/2} + \eta^{1/\varrho}).$$

*Proof.* It is not difficult to check that

$$\sum_{\substack{x,y\in\mathbb{Z}_N\\1\le z\le M\\x-y=\psi_{\mathcal{W}}(z)}} a(x)a(y)\psi_{\mathcal{W}}^{\Delta}(z-1)\lambda_{1,\mathcal{W}W}(z)$$
$$=\frac{1}{N}\sum_{r\in\mathbb{Z}_N}\tilde a(r)\tilde a(-r)\Big(\sum_{z=1}^{M}\psi_{\mathcal{W}}^{\Delta}(z-1)\lambda_{1,\mathcal{W}W}(z)e(-\psi_{\mathcal{W}}(z)r/N)\Big).$$

Also, it is easy to see that $(f*g)\tilde{}=\tilde f\tilde g$. Then

$$\sum_{\substack{x,y\in\mathbb{Z}_N\\1\le z\le M\\x-y=\psi_{\mathcal{W}}(z)}} a'(x)a'(y)\psi_{\mathcal{W}}^{\Delta}(z-1)\lambda_{1,\mathcal{W}W}(z)-\sum_{\substack{x,y\in\mathbb{Z}_N\\1\le z\le M\\x-y=\psi_{\mathcal{W}}(z)}} a(x)a(y)\psi_{\mathcal{W}}^{\Delta}(z-1)\lambda_{1,\mathcal{W}W}(z)$$
$$=\frac{1}{N}\sum_{r\in\mathbb{Z}_N}\tilde a(r)\tilde a(-r)(\tilde\beta(r)^2\tilde\beta(-r)^2-1)$$
$$\times\Big(\sum_{z=1}^{M}\psi_{\mathcal{W}}^{\Delta}(z-1)\lambda_{1,\mathcal{W}W}(z)e(-\psi_{\mathcal{W}}(z)r/N)\Big).$$

If $r\in R$, then by the proof of Lemma 6.7 of [12], we know that

$$|\tilde\beta(r)^2\tilde\beta(-r)^2-1|\le 2^{16}\varepsilon^2.$$

And applying Lemma 2.2 with $\alpha=a=q=1$,

$$\sum_{z=1}^{M}\psi_{\mathcal{W}}^{\Delta}(z-1)\lambda_{1,\mathcal{W}W}(z)=\sum_{z=1}^{M}\psi_{\mathcal{W}}^{\Delta}(z-1)+O(\psi_{\mathcal{W}}^{\Delta}(M)Me^{-c\sqrt{\log M}})$$
$$\le 2\psi_{\mathcal{W}}(M).$$

Therefore

$$\Big|\sum_{r\in R}\tilde a(r)\tilde a(-r)(\tilde\beta(r)^2\tilde\beta(-r)^2-1)\Big(\sum_{z=1}^{M}\psi_{\mathcal{W}}^{\Delta}(z-1)\lambda_{1,\mathcal{W}W}(z)e(-\psi_{\mathcal{W}}(z)r/N)\Big)\Big|$$
$$\le 2^{16}\varepsilon^2\sum_{r\in R}|\tilde a(r)|^2\Big|\sum_{z=1}^{M}\psi_{\mathcal{W}}^{\Delta}(z-1)\lambda_{1,\mathcal{W}W}(z)e(-\psi_{\mathcal{W}}(z)r/N)\Big|$$
$$\le 2^{17}\varepsilon^2\psi_{\mathcal{W}}(M)|R|.$$

In view of Lemma 4.1 with $\rho=5/2$, we have $|R|\le C''\eta^{-5/2}$. On the other hand, by the Hölder inequality,

$$\Big|\sum_{r\notin R}\tilde a(r)\tilde a(-r)(\tilde\beta(r)^2\tilde\beta(-r)^2-1)$$
$$\times\Big(\sum_{z=1}^{M}\psi_{\mathcal W}^{\Delta}(z-1)\lambda_{1,\mathcal W W}(z)e(-\psi_{\mathcal W}(z)r/N)\Big)\Big|$$
$$\le 2\sup_{r\notin R}|\tilde a(r)|^{1/\varrho}\Big(\sum_{r\notin R}|\tilde a(r)|^{\frac{2\varrho-1}{\varrho-1}}\Big)^{\frac{\varrho-1}{\varrho}}$$
$$\times\Big(\sum_{r\notin R}\Big|\sum_{z=1}^{M}\psi_{\mathcal W}^{\Delta}(z-1)\lambda_{1,\mathcal W W}(z)e(-\psi_{\mathcal W}(z)r/N)\Big|^{\varrho}\Big)^{1/\varrho}$$
$$\le 2\eta^{1/\varrho}C((2\varrho-1)/(\varrho-1))^{1-1/\varrho}(|a_{k-t+1}|C'(\varrho))^{1/\varrho}N,$$

where in the last step we apply Lemma 4.1 with $\rho=(2\varrho-1)/(\varrho-1)$ and Lemma 4.2 with $\rho=\varrho$. ∎

LEMMA 4.4. *If* $\varepsilon^{|R|}\ge 2\log\log w/w$, *then* $|a'(x)|\le 2/N$ *for any* $x\in\mathbb Z_N$.

*Proof.* See [12, Lemma 6.3]. ∎

Let $A'=\{x\in\mathbb Z_N : a'(x)\ge\frac{1}{16}\delta N^{-1}\}$. Then

$$\frac{2}{N}\,|A'|+\frac{\delta}{16N}\,(N-|A'|)\ge\sum_{x\in\mathbb Z_N}a'(x)=\sum_{x\in\mathbb Z_N}a(x)\ge\frac{\delta}{8},$$

whence $|A'|/N\ge\delta/32$. Let $A_1'=A'\cap[1,(N-1)/2]$ and

$$A_2'=\{x-(N-1)/2 : x\in A'\cap[(N+1)/2,N-1]\}.$$

Clearly there exists $i\in\{1,2\}$ such that $|A_i'|/N\ge\delta/64$, say $|A_1'|/N\ge\delta/64$. Applying Theorem 3.1, we know that

$$|\{(x,y,z): x,y\in A_1',\ z\in\Lambda_{1,\mathcal W W}\cap[1,M],\ x-y=\psi_{\mathcal W}(z)\}|$$
$$\ge c(\delta/64,a_{k-t+1})\,\frac{\mathcal W W(N/2)^{1+1/k}(a_1\mathcal W^{k-t})^{-1/k}}{\phi(\mathcal W W)\log N}.$$

Let $c'=\frac{1}{16k}c(\delta/64,a_{k-t+1})$. Clearly

$$|\{(x,y,z): x,y\in A_1',\ z\in\Lambda_{1,\mathcal W W}\cap[1,c'M],\ x-y=\psi_{\mathcal W}(z)\}|$$
$$\le\frac{\mathcal W W(c'M)}{\phi(\mathcal W W)\log M}\,N.$$

Therefore

$$|\{(x,y,z): x,y\in A_1',\ z\in\Lambda_{1,\mathcal W W}\cap(c'M,M],\ x-y=\psi_{\mathcal W}(z)\}|$$
$$\ge\frac{c(\delta/64,a_{k-t+1})}{8}\,\frac{\mathcal W W N^{1+1/k}(a_1\mathcal W^{k-t})^{-1/k}}{\phi(\mathcal W W)\log N}.$$

It follows that

$$\sum_{\substack{x,y\in A_1'\\ 1\le z\le M\\ x-y=\psi_{\mathcal{W}}(z)}} \psi_{\mathcal{W}}^{\Delta}(z-1)\lambda_{1,\mathcal{W}W}(z)$$

$$\ge \frac{c(\delta/64,a_{k-t+1})}{8}\,\frac{\mathcal{W}WN^{1+1/k}(a_1\mathcal{W}^{k-t})^{-1/k}}{\phi(\mathcal{W}W)\log N}\,\frac{\psi_{\mathcal{W}}^{\Delta}(c'M)\phi(\mathcal{W}W)\log M}{2\mathcal{W}W}$$

$$\ge \frac{c(\delta/64,a_{k-t+1})c'^{k-1}}{64}\,N^2.$$

So

$$\sum_{\substack{x,y\in \mathbb{Z}_N\\ 1\le z\le M\\ x-y=\psi_{\mathcal{W}}(z)}} a(x)a(y)\psi_{\mathcal{W}}^{\Delta}(z-1)\lambda_{1,\mathcal{W}W}(z)$$

$$\ge \sum_{\substack{x,y\in \mathbb{Z}_N\\ 1\le z\le M\\ x-y=\psi_{\mathcal{W}}(z)}} a'(x)a'(y)\psi_{\mathcal{W}}^{\Delta}(z-1)\lambda_{1,\mathcal{W}W}(z) - C(\varepsilon^2\eta^{-5/2}+\eta^{1/\varrho})$$

$$\ge \frac{\delta^2}{2^8N^2}\sum_{\substack{x,y\in A_1'\\ 1\le z\le M\\ x-y=\psi_{\mathcal{W}}(z)}} \psi_{\mathcal{W}}^{\Delta}(z-1)\lambda_{1,\mathcal{W}W}(z) - C(\varepsilon^2\eta^{-5/2}+\eta^{1/\varrho})$$

$$\ge c''(\delta,a_{k-t+1}) - C(\varepsilon^2\eta^{-5/2}+\eta^{1/\varrho}).$$

Finally, we may choose $\eta,\varepsilon>0$ satisfying $\varepsilon^{C''\eta^{-5/2}}\ge 2\log\log w/w$ such that

$$C(\varepsilon^2\eta^{-5/2}+\eta^{1/\varrho}) < c''(\delta,a_{k-t+1})/2$$

whenever $w$ is sufficiently large. Hence

$$\sum_{\substack{x,y\in \mathbb{Z}_N\\ 1\le z\le M\\ x-y=\psi_{\mathcal{W}}(z)}} a(x)a(y)\psi_{\mathcal{W}}^{\Delta}(z-1)\lambda_{1,\mathcal{W}W}(z) \ge \frac{c''(\delta,a_{k-t+1})}{2} > 0$$

for sufficiently large $N$. ■

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