ACTA ARITHMETICA
155.4 (2012)

The Bounded Gap Conjecture and bounds between consecutive Goldbach numbers

by

János Pintz (Budapest)

Dedicated to Professor Andrzej Schinzel on his 75th birthday

1. Introduction. It is no surprise that there are connections between the distribution of primes and the Goldbach conjecture. In most cases approximations to the Goldbach conjecture are shown by the circle method using properties of the distribution of primes proved by analytic methods, in many cases in combination with sieve methods (cf. [Pan], [Pin1], [PR]). This is the case, for example, if we look for intervals of type

$$[x, x + f(x)], \quad x > x_0, \tag{1.1}$$

where one can show that almost all even numbers are Goldbach numbers, that is, sums of two primes (cf. [BHP1], [Jia1], [KPP], [Mik], [PP]). We will denote their sequence by $\mathcal{G} = \{g_i\}_{i=1}^{\infty} = \{4, 6, 8, \dots\}$. The Goldbach conjecture can also be formulated as a result about consecutive Goldbach numbers, namely, $g_{n+1} - g_n = 2$ for all $n \geq 1$.

It is interesting to note that it is exactly the problem of estimating the bounds between consecutive Goldbach numbers, that is, estimates of type

$$g_{n+1} - g_n < h(g_n), \tag{1.2}$$

where the best (actually almost all) results follow in a simple way directly from (usually deep) results about the distribution of primes. As was observed first by Montgomery–Vaughan [MV] and Ramachandra [Ram], the two results

$$\pi(x + x^{\vartheta_1}) - \pi(x) \geq \frac{cx^{\vartheta_1}}{\log x}, \tag{1.3}$$

$$\pi(n + n^{\vartheta_2}) - \pi(n) > 0 \quad \text{for almost all } n \in [x, 2x) \text{ for } x \to \infty, \tag{1.4}$$

2010 *Mathematics Subject Classification*: Primary 11N05; Secondary 11N35, 11P32.
Key words and phrases: gaps between primes, Goldbach conjecture.

DOI: 10.4064/aa155-4-4

together imply

$$g_{n+1} - g_n \ll h(g_n) = g_n^{\vartheta_3} = g_n^{\vartheta_1\vartheta_2}. \tag{1.5}$$

Accordingly the best exponent $\vartheta_3 = 21/800$ is a consequence of the results $\vartheta_2 = 1/20$ of Jia [Jia2] and $\vartheta_1 = 21/40$ of R. C. Baker, G. Harman and the author [BHP2].

As a trivial special case of (1.3)–(1.5), we can see that Conjecture 1.1 follows from Conjecture 1.2, where

Conjecture 1.1. *$g_{n+1} - g_n \ll g_n^{\varepsilon}$ for any $\varepsilon > 0$.*

Conjecture 1.2. (1.4) *is true for any exponent $\varepsilon > 0$.*

Thus, some expected regular distribution of primes, the rare occurrence of large gaps between consecutive primes (Conjecture 1.2) implies Conjecture 1.1. We will show in this note that also some unexpected distribution of primes, namely, the non-existence of infinitely many bounded gaps between primes,

$$\liminf_{n\to\infty} (p_{n+1} - p_n) = \infty, \tag{1.6}$$

implies Conjecture 1.1, where $\mathcal{P} = \{p_i\}_{i=1}^{\infty}$, $p_i < p_{i+1}$, denotes the set of all primes; p, p_i will always denote primes.

To reformulate our result we introduce

Conjecture 1.3 (Bounded Gap Conjecture). *We have*

$$\liminf_{n\to\infty} (p_{n+1} - p_n) < \infty. \tag{1.7}$$

Our main result can be formulated as

Theorem 1.4. *At least one of Conjectures* 1.1 *and* 1.3 *is true.*

Remark. Naturally, probably both are true.

We will prove the result in a stronger form, which needs the notion of *admissible k-tuples*

$$\mathcal{H} = \{h_i\}_{i=1}^{k}, \qquad h_1 < \cdots < h_k, \qquad \mathcal{H} \subset \mathbb{Z}, \tag{1.8}$$

which are defined by the property that the number $\nu_p = \nu_p(\mathcal{H})$ of residue classes occupied by $\mathcal{H} \bmod p$ satisfies $\nu_p < p$ for all $p \in \mathcal{P}$. This is equivalent to the fact that the associated singular series satisfies

$$\mathfrak{S}(\mathcal{H}) := \prod_p \left(1 - \frac{\nu_p(\mathcal{H})}{p}\right)\left(1 - \frac{1}{p}\right)^{-k} \neq 0. \tag{1.9}$$

We can also introduce one more conjecture, which trivially implies the Bounded Gap Conjecture.

Conjecture 1.5. *If k is sufficiently large ($k > k_0$, an absolute constant) then for an arbitrary admissible k-tuple $\mathcal{H}$, the set $n+\mathcal{H} := \{n+h_i\}_{i=1}^k$ contains at least two primes for infinitely many values of n.*

We will actually prove the following

Theorem 1.6. *At least one of Conjectures* 1.1 *and* 1.5 *is true.*

Remark. Theorem 1.4 trivially follows from this.

2. The basic idea of the proof. The idea of the (non-trivial) proof of Theorem 1.6 slightly resembles the completely trivial implication Conjecture 1.2 ⇒ Conjecture 1.1. Namely, if we want to write an even integer near M as the sum of two primes then setting $\lfloor M/3 \rfloor = N$ we can consider for an $\varepsilon = 1/r$, $r \in \mathbb{Z}^+$ (suppose for simplicity $N^\varepsilon \in \mathbb{Z}$)

$$\mathcal{N} = \{\mathcal{N}_i = [N + (i-1)N^\varepsilon, N + iN^\varepsilon]\}_{i=1}^T, \quad T = (N^\varepsilon)^{r-1} = N^{1-\varepsilon}. \tag{2.1}$$

If at least half of the intervals in $\mathcal{N}$ contain at least one prime (we do not even need almost all in the case of $\vartheta_2 = \varepsilon$ being arbitrarily small!), then we have clearly an integer $i \in [1, T]$ such that the intervals $\mathcal{N}_i$ and $\mathcal{N}_{T-i}$ each contain at least one prime, whose sum satisfies

$$p + p' \in [3N - 2N^\varepsilon, 3N] \subseteq [M - 2N^\varepsilon - 3, M]. \tag{2.2}$$

In the case of Theorem 1.6, we will suppose that Conjecture 1.5 is false for a single $\mathcal{H} = \mathcal{H}_k$, where $k > k_0$ (k_0 will be determined later during the proof). Afterwards we will use a somewhat similar (but fairly non-trivial) argument, where numbers will be weighted by Selberg type weights, used in the proof of

$$\liminf_{n\to\infty} (p_{n+1} - p_n)/\log p_n = 0, \tag{2.3}$$

given recently by D. Goldston, C. Yıldırım and the author [GPY]. The weights depend on the given k-tuple $\mathcal{H} = \mathcal{H}_k$ in the following way:

$$a_n := \Lambda_R(n; \mathcal{H}, l)^2 := \left(\frac{1}{(k+l)!} \sum_{\substack{d \mid \mathcal{P}_\mathcal{H}(n) \\ d \le R}} \mu(d) \left(\log \frac{R}{d} \right)^{k+l} \right)^2, \tag{2.4}$$

$$\mathcal{P}_\mathcal{H}(n) = \prod_{i=1}^k (n + h_i),$$

and n will run in $[N, 2N)$, to be abbreviated by $n \sim N$, with $N = \lfloor M/3 \rfloor$, $N > N_0(k, \mathcal{H}_k)$.

It is crucial for our proof that in [GPY] we almost proved Conjecture 1.5 (and consequently the Bounded Gap Conjecture) in the sense that we showed that denoting the characteristic function of the primes by $\chi_{\mathcal{P}}$ we have

$$S := \sum_{n\sim N} a_n \sum_{i=1}^{k} \chi_{\mathcal{P}}(n+h_i) \geq \left(1 - \frac{c}{\sqrt{k}}\right) \sum_{n\sim N} a_n =: \left(1 - \frac{c}{\sqrt{k}}\right) A. \tag{2.5}$$

This means that if $\{n+h_i\}_{i=1}^{k}$ contains at most one prime for each $n \sim N$, then

$$A_0 = \sum_{\substack{n\sim N \\ n+h_i \notin \mathcal{P},\, i=1,\dots,k}} a_n \leq \frac{c}{\sqrt{k}} \sum_{n\sim N} a_n = \frac{c}{\sqrt{k}} A, \tag{2.6}$$

which means that we must have *in the weighted sense* for most n at least one (by our assumption exactly one) prime among $n+h_i$, and consequently a bounded distance from n. Therefore (*in the weighted sense*) for most n we can have no primes at a distance at most N^{ε} from $M-n$. This means that all primes in $[N, 2N)$ have to be accumulated near the numbers $M-n$ in (2.6), having relative measure

$$A_0/A \leq c/\sqrt{k}. \tag{2.7}$$

On the other hand, sieve estimates (see Theorem 4.4 of [Mon]) tell us that for $y \in [x^{\varepsilon}, x]$ primes might have just the density at most $2/\varepsilon$ of the expected size:

$$\pi(x+y) - \pi(x) \leq \frac{2y}{\log y} \leq \frac{2y}{\varepsilon \log x}. \tag{2.8}$$

The choice $k > k_0 = C\varepsilon^{-2}$ will lead to a contradiction.

3. The proof of Theorem 1.6. In order to make the sketch of the argument of the previous section precise we will quote special cases of Propositions (2.14)–(2.15) of [GPY] as Lemmas 3.1 and 3.2 using notation (2.4)–(2.5) and

$$B := B_R(N,k,l) := \binom{2l}{l} \frac{N \log^{k+2\ell} R}{(k+2l)!} \qquad (0 \leq l \leq k). \tag{3.1}$$

Lemma 3.1. *If* $R \ll N^{1/2}(\log N)^{-2(k+l)}$ *as* $R, N \to \infty$ *then*

$$A = \sum_{n\sim N} a_n = (\mathfrak{S}(\mathcal{H}) + o(1))B. \tag{3.2}$$

Lemma 3.2. *If* $R \ll N^{1/4}(\log N)^{-C(k)}$ *as* $R, N \to \infty$ *then for* $h_i \in \mathcal{H}$ *we have*

$$S_i := \sum_{n\sim N} a_n \chi_{\mathcal{P}}(n+h_i) = \frac{2(2l+1)}{l+1} \frac{(\mathfrak{S}(\mathcal{H}) + o(1))B}{k+2l+1} \frac{\log R}{\log N}. \tag{3.3}$$

Observing that for any admissible $\mathcal{H}$ with $|\mathcal{H}| = k$ we have

$$\mathfrak{S}(\mathcal{H}) \geq \prod_{p \leq 2k} \frac{1}{p} \prod_{p > 2k} \left(1 - \frac{k}{p}\right)\left(1 - \frac{1}{p}\right)^{-k} = C_0(k), \tag{3.4}$$

we obtain with the optimal choice

$$R = N^{1/4}(\log N)^{-C(k)}, \qquad l = \lfloor \sqrt{k}/2 \rfloor, \tag{3.5}$$

from (3.1)–(3.5), by summing over $i = 1, \ldots, k$, the crucial relation (2.5), and consequently (2.6).

In order to make the second part (2.7)–(2.8) of our heuristic argument precise we have to calculate the *weighted sum* of primes of the form $M+h-n$ for $h \leq H \asymp x^{\varepsilon}$ (at least on average over $h \in [0, H]$); in order to do this we need first a variant of Proposition 2 of [GPY], which we state as

LEMMA 3.3. *If $\mathcal{H}$ is admissible, $R \ll N^{1/4}(\log N)^{-C(k)}$ as $R, N \to \infty$, $2N < m \ll N$, $m \equiv h_1 \pmod 2$ then*

$$S^*_m := \sum_{n \sim N} a_n \chi_{\mathcal{P}}(m-n) = \frac{(\mathfrak{S}(\mathcal{H} \cup \{-m\}) + o(1))B}{\log N}. \tag{3.6}$$

The proof is nearly the same as that of the first case ($h_0 \notin \mathcal{H}$) of Proposition 2 of [GPY] or that of Lemma 2 in [GMPY]; the only change being that the role of

$$\varDelta := \prod_{1 \leq i < j \leq k} (h_j - h_i) \tag{3.7}$$

is now played by

$$\varDelta^*_m := \varDelta \prod_{1 \leq i \leq k} (m + h_i). \tag{3.8}$$

Accordingly the estimate $\varDelta \ll (\log N)^{k^2}$ of [GPY] has to be replaced by the weaker

$$\varDelta^*_m \ll_k N^k \varDelta \ll N^{2k}. \tag{3.9}$$

Consequently, the parameter $U := Ck^2(\log 2h)$ in (6.14) of [GPY] has to be replaced by

$$\log \varDelta^*_m \leq U^* := 2k \log N. \tag{3.10}$$

Hence the estimate (7.12) of [GPY] takes now for $\operatorname{Re} s_i = \sigma_i$ the form

$$\begin{aligned} &|G_{\mathcal{H}}(s_1, s_2)| \\ &\qquad \ll \exp\bigl(Ck(\log N)^{\delta_1+\delta_2} \log\log\log N\bigr), \text{ where } \delta_i = \max(-\sigma_i, 0). \end{aligned} \tag{3.11}$$

This estimate is exactly the same as in (1.6) of [GMPY], which uses the above weaker but still sufficient upper bound of $G_{\mathcal{H}}(s_1, s_2)$. The rest of the proof is the same as in [GPY] or [GMPY]. □

However, in this case we cannot use the argument of [GPY] or [GMPY] which makes use of Gallagher's theorem [Gal] about the mean value of $\mathfrak{S}(\mathcal{H})$ for $\mathcal{H} \subset [1, H]$, the relation ($k$ fixed)

$$\sum_{\mathcal{H}\subset[1,H],\,|\mathcal{H}|=k} \mathfrak{S}(\mathcal{H}) \sim H^k \quad \text{as } H \to \infty. \tag{3.12}$$

The argument that we now use instead is contained in [Pin2], which also gives a very quick alternative proof of Gallagher's theorem.

The lemma below is a small variation of Theorem 2 of [Pin2].

LEMMA 3.4. *Let $\mathcal{H}_k \subset [0, H]$ be a fixed k-element admissible set, C a sufficiently large constant, $k > k_0$, $M \in \mathbb{Z}$, and $\delta > 0$. Then for $H > \exp\bigl(\frac{Ck}{\delta \log k}\bigr)$ we have*

$$S_{\mathcal{H}}(M, H) = \frac{1}{H} \sum_{m\in[M,M+H)} \frac{\mathfrak{S}(\mathcal{H} \cup \{m\})}{\mathfrak{S}(\mathcal{H})} \geq 1 - \delta. \tag{3.13}$$

REMARK. As we see we allow here M and thereby m to be negative as well.

REMARK. A slightly more elaborate proof (see Theorem 1 of [Pin2]) would give $S_{\mathcal{H}}(M, H) \sim 1$ if $H > \exp(k^{C/\varepsilon})$, but the above weaker estimate (3.13) is sufficient for our purposes too.

Proof of Lemma 3.4. Let us study $\mathfrak{S}(\mathcal{H}')/\mathfrak{S}(\mathcal{H})$ with $\mathcal{H}' = \mathcal{H} \cup \{m\}$ with a fixed $m \in [M, M + H]$, $m \notin \mathcal{H}$ for $k > k_0$ with the notation

$$\nu'_p = \nu_p(\mathcal{H}'), \quad y := \frac{5}{6} \log H, \quad P := \prod_{p\leq y} p. \tag{3.14}$$

Then

$$\frac{\mathfrak{S}(\mathcal{H}')}{\mathfrak{S}(\mathcal{H})} = \prod_p \frac{1 - \frac{\nu'_p}{p}}{\bigl(1 - \frac{\nu_p}{p}\bigr)\bigl(1 - \frac{1}{p}\bigr)} = \prod_{p\leq y}{}_1 \cdot \prod_{p>y}{}_2, \tag{3.15}$$

where

$$\prod{}_2 \geq \prod_{p>y} \frac{1 - \frac{\nu_p+1}{p}}{1 - \frac{\nu_p+1}{p} + \frac{k}{p^2}} = \prod\left(1 - O\left(\frac{k}{p^2}\right)\right) = 1 + O\left(\frac{k}{y \log y}\right). \tag{3.16}$$

Since $\prod_1(m)$ is periodic with period P, it is sufficient to study it for $m \in [1, P]$. Since for any prime p we have exactly ν_p possibilities for m mod p with $\nu'_p = \nu_p$ and $p - \nu_p$ possibilities with $\nu'_p = \nu_p + 1$, we obtain

$$\frac{1}{P} \sum_{m=1}^{P} \prod{}_1(m) = \prod_{p|P} \frac{\bigl\{\frac{\nu_p}{p}\bigl(1 - \frac{\nu_p}{p}\bigr) + \bigl(1 - \frac{\nu_p}{p}\bigr)\bigl(1 - \frac{\nu_p+1}{p}\bigr)\bigr\}}{\bigl(1 - \frac{\nu_p}{p}\bigr)\bigl(1 - \frac{1}{p}\bigr)} = 1. \tag{3.17}$$

Since the possible non-complete period from $M+rP$ to $M+H$ (if $\lfloor H/P\rfloor = r$) adds to $S_{\mathcal{H}}(M,H)$ just $O(P/H)=o(1)$ (as $H\to\infty$), (3.16)–(3.17) together prove Lemma 3.4. ■

Lemmas 3.3 and 3.4 together imply for $H > H_0(\varepsilon,k)$ a lower bound for the weighted sum below. Namely, interchanging the order of summations we obtain

$$\begin{aligned}(3.18)\qquad \sum_{m\in(M,\,M+H]} S_m^* &= \sum_{n\sim N} a_n \sum_{h=1}^{H} \chi_{\mathcal{P}}(M+h-n)\\ &\geq (1-\varepsilon)\frac{B\mathfrak{S}(\mathcal{H})}{\log N}\sum_{m=-M-H}^{-M-1}\frac{\mathfrak{S}(\mathcal{H}\cup\{m\})}{\mathfrak{S}(\mathcal{H})}\\ &\geq (1-2\varepsilon)\frac{B\mathfrak{S}(\mathcal{H})}{\log N}H.\end{aligned}$$

Hence there exist $m\in(M,M+H]$ and $n\in[N,2N)$ such that $m-n$ is prime. If for the n found here there exists an i, $1\leq i\leq k$, such that $n+h_i\in\mathcal{P}$ then $m+h_i=(m-n)+(n+h_i)$ is a sum of two primes and $M<m+h_i\leq M+2H$, which on taking $H=N^{\epsilon}/2$ proves the theorem. This leaves for consideration only n in the set

$$(3.19)\qquad \mathcal{N}_0 := \bigl\{n : n\sim N,\ n+h_i\notin\mathcal{P}\ (1\leq i\leq k)\bigr\}$$

whose "measure" satisfies (2.6). Since (2.8) gives an upper estimate, for each individual n, for the number of primes between $M+1-n$ and $M+H-n$, we deduce from (2.6)–(2.8) and (3.18)–(3.19) that

$$\begin{aligned}(3.20)\qquad \frac{(1-2\varepsilon)BH\mathfrak{S}(\mathcal{H})}{\log N} &\leq \sum_{n\in\mathcal{N}_0} a_n\bigl(\pi(M+H-n)-\pi(M-n)\bigr)\\ &\leq \frac{cB\mathfrak{S}(\mathcal{H})}{\sqrt{k}}\cdot\frac{2H}{\log H},\end{aligned}$$

which is a contradiction if $k\geq c'/\varepsilon^2$.

Remark. It is easy to see from (3.3) that one can take for c any number exceeding 2, which means that one can choose any $c'>4$ in the above lower bound if ε is sufficiently small. This means that a more quantitative form of Theorem 1.6 is also true.

Theorem 1.6′. *Let $c'>4$ be a fixed constant, and $\varepsilon<\varepsilon_0$ sufficiently small. Then one of the following assertions is true:*

(i) $g_{n+1}-g_n \ll g_n^{\varepsilon}$.
(ii) *Let $\mathcal{H}=\{h_i\}_{i=1}^{k}$ be an admissible k-tuple with $k>c'\varepsilon^{-2}$. Then for infinitely many values of n the set $n+\mathcal{H}$ contains at least two primes.*

4. Concluding remarks. After the proof of (2.3) in [GPY], the question arose whether the method was able to show

$$\varDelta_\nu = \liminf_{n\to\infty} (p_{n+\nu} - p_n)/\log p_n = 0 \tag{4.1}$$

for some $\nu \ge 2$. The problem remained entirely open for all $\nu \ge 3$, while (4.1) was shown to be true for $\nu = 2$ under the assumption of the very deep Elliott–Halberstam conjecture [EH], asserting that primes have level of distribution $\vartheta = 1$. We may remark that, on the other hand, the seemingly deeper Bounded Gap Conjecture could be shown under the weaker assumption that the primes have some distribution level exceeding 1/2. It may be interesting to remark that if besides the falsity of the Bounded Gap Conjecture we also assume the falsity of (4.1) for an arbitrarily chosen fixed ν, then we obtain a much smaller upper bound for the gaps between consecutive Goldbach numbers. The result is as follows.

THEOREM 4.1. *Suppose $\varepsilon > 0$, $\varDelta_\nu > 0$ for some $\nu \ge 2$ and there exists an admissible k-element set $\mathcal{H}$ such that for each sufficiently large n the set $n + \mathcal{H}$ contains at most one prime. Then*

$$g_{n+1} - g_n \le \varepsilon \log g_n \quad \textit{for} \ \ n > n_1(k,\nu,\varepsilon) \ \textit{and} \ k > k_0(\nu,\varepsilon). \tag{4.2}$$

The proof is very similar to the proof of Theorem 1.6. However, choosing a fixed $\varepsilon < \varDelta_\nu$ and

$$H = \varepsilon \log N, \quad N > N_0(k, \varDelta_\nu, \varepsilon) \tag{4.3}$$

we can clearly replace (2.8) by

$$\pi(M + H - n) - \pi(M + H) \le \nu + 1. \tag{4.4}$$

Hence we obtain, in place of (3.20),

$$\frac{(1-2\varepsilon)BH\mathfrak{S}(\mathcal{H})}{\log N} \le \frac{cB\mathfrak{S}(\mathcal{H})}{\sqrt{k}} \cdot (\nu + 1), \tag{4.5}$$

which is a contradiction if k is large enough ($k > (2c\nu/\varepsilon)^2$).

Finally we remark that without any further assumption a refinement of the present method in combination with some deeper results about the distribution of primes can yield a stronger form of Theorem 1.6 (thereby of Theorem 1.4) where the estimate $g_{n+1} - g_n \ll g_n^\varepsilon$ is replaced by $g_{n+1} - g_n \ll (\log g_n)^C$ with some explicitly given value of C.

Acknowledgements. This research was supported by OTKA (Grants nos. K72731, K67676, K100291) and ERC-AdG.228005.

References

[BHP1] R. C. Baker, G. Harman and J. Pintz, *The exceptional set for Goldbach's problem in short intervals*, in: Sieve Methods, Exponential Sums, and Their Applications in Number Theory (Cardiff, 1995), London Math. Soc. Lecture Note Ser. 237, Cambridge Univ. Press, Cambridge, 1997, 1–54.

[BHP2] R. C. Baker, G. Harman and J. Pintz, *The difference between consecutive primes. II*, Proc. London Math. Soc. (3) 83 (2001), 532–562.

[EH] P. D. T. A. Elliott and H. Halberstam, *A conjecture in prime number theory*, in: Symposia Mathematica 4 (INDAM, Rome, 1968/69), Academic Press, London, 1970, 59–72.

[Gal] P. X. Gallagher, *On the distribution of primes in short intervals*, Mathematika 23 (1976), 4–9.

[GMPY] D. A. Goldston, Y. Motohashi, J. Pintz and C. Y. Yıldırım, *Small gaps between primes exist*, Proc. Japan Acad. 82A (2006), 61–65.

[GPY] D. A. Goldston, J. Pintz and C. Y. Yıldırım, *Primes in tuples. I*, Ann. of Math. (2) 170 (2009), 819–862.

[Jia1] C. H. Jia, *On the exceptional set of Goldbach numbers in a short interval*, Acta Arith. 77 (1996), 207–287.

[Jia2] C. H. Jia, *Almost all short intervals containing prime numbers*, Acta Arith. 76 (1996), 21–84.

[KPP] J. Kaczorowski, A. Perelli and J. Pintz, *A note on the exceptional set for Goldbach's problem in short intervals*, Monatsh. Math. 116 (1995), 275–282; Corrigendum, ibid. 119 (1995), 215–216.

[Mik] H. Mikawa, *On the exceptional set in Goldbach's problem*, Tsukuba J. Math. 16 (1992), 513–543.

[Mon] H. L. Montgomery, *Topics in Multiplicative Number Theory*, Lecture Notes in Math. 227, Springer, Berlin, 1971.

[MV] H. L. Montgomery and R. C. Vaughan, *The exceptional set in Goldbach's problem*, Acta Arith. 27 (1975), 353–370.

[Pan] C. D. Pan, *On the representation of even numbers as the sum of a prime and a product of not more than 4 primes*, Sci. Sinica 12 (1963), 455–473.

[Pin1] J. Pintz, *Recent results on the Goldbach conjecture*, in: Elementare und analytische Zahlentheorie (Proc. ELAZ-Conference, 2004), Steiner Verlag, Stuttgart, 2006, 220–254.

[Pin2] J. Pintz, *On the singular series in the prime k-tuple conjecture*, arXiv:1004.10841 [math.NT].

[PP] A. Perelli and J. Pintz, *On the exceptional set for Goldbach's problem in short intervals*, J. London Math. Soc. (2) 47 (1993), 41–49.

[PR] J. Pintz and I. Z. Ruzsa, *On Linnik's approximation to Goldbach's problem. I*, Acta Arith. 109 (2003), 169–194.

[Ram] K. Ramachandra, *On the number of Goldbach numbers in small intervals*, J. Indian Math. Soc. (N.S.) 37 (1973), 157–170.

János Pintz
Rényi Mathematical Institute
of the Hungarian Academy of Sciences
Reáltanoda u. 13–15
H-1053 Budapest, Hungary
E-mail: pintz@renyi.hu

Received on 6.10.2011
and in revised form on 11.8.2012 (6857)