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On some mean value results for the zeta-function in short intervals

by

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1. Introduction. As usual, let

$$\Delta(x) := \sum_{n\le x} d(n) - x(\log x + 2\gamma - 1) \tag{1.1}$$

denote the error term in the classical Dirichlet divisor problem. Also let

$$E(T) := \int_0^T |\zeta(1/2+it)|^2\,dt - T\left(\log\left(\frac{T}{2\pi}\right) + 2\gamma - 1\right) \tag{1.2}$$

denote the error term in the mean square formula for $|\zeta(1/2+it)|$. Here $d(n)$ is the number of divisors of n, $\zeta(s)$ is the Riemann zeta-function, and $\gamma = -\Gamma'(1) = 0.577215\ldots$ is Euler's constant. In view of F. V. Atkinson's classical explicit formula for $E(T)$ (see [1], [4, Chapter 15] and [5, Chapter 2]) it was known long ago that there are analogies between $\Delta(x)$ and $E(T)$. However, in this context it seems that instead of the error-term function $\Delta(x)$ it is more exact to work with the modified function $\Delta^*(x)$ (see M. Jutila [14], [15] and T. Meurman [17]), where

$$\begin{aligned}\Delta^*(x) &:= -\Delta(x) + 2\Delta(2x) - \tfrac{1}{2}\Delta(4x)\\ &= \tfrac{1}{2}\sum_{n\le 4x}(-1)^n d(n) - x(\log x + 2\gamma - 1),\end{aligned} \tag{1.3}$$

since it turns out that $\Delta^*(x)$ is a better analogue of $E(T)$ than $\Delta(x)$. Namely, M. Jutila (op. cit.) investigated both the local and global behaviour of the difference

$$E^*(t) := E(t) - 2\pi\Delta^*\left(\frac{t}{2\pi}\right),$$

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and in particular in [14] he proved that

$$\int_T^{T+H} (E^*(t))^2\,dt \ll_\varepsilon HT^{1/3}\log^3 T + T^{1+\varepsilon} \qquad (1 \le H \le T). \tag{1.4}$$

Here and later ε denotes positive constants which are arbitrarily small, but are not necessarily the same ones at each occurrence, while $a(x) \ll_\varepsilon b(x)$ (same as $a(x)=O_\varepsilon(b(x))$) means that $|a(x)| \le Cb(x)$ for some $C=C(\varepsilon)>0$, $x \ge x_0$. The significance of (1.4) is that, in view of (see e.g., [4])

$$\int_0^T (\Delta^*(t))^2\,dt \sim AT^{3/2}, \quad \int_0^T E^2(t)\,dt \sim BT^{3/2} \qquad (A, B > 0,\ T \to \infty),$$

it transpires that $E^*(t)$ is in the mean square sense of a lower order of magnitude than either $\Delta^*(t)$ or $E(t)$.

Later works provided more results on the mean values of $E^*(T)$. Thus in [9] the author sharpened (1.4) (in the case when $H = T$) to the asymptotic formula

$$\int_0^T (E^*(t))^2\,dt = T^{4/3}P_3(\log T) + O_\varepsilon(T^{7/6+\varepsilon}), \tag{1.5}$$

where $P_3(y)$ is a polynomial of degree three in y with positive leading coefficient, and all its coefficients may be evaluated explicitly. This, in particular, shows that (1.4) may be complemented with the lower bound

$$\int_T^{T+H} (E^*(t))^2\,dt \gg HT^{1/3}\log^3 T \qquad (T^{5/6+\varepsilon} \le H \le T), \tag{1.6}$$

which is implied by (1.5). It is likely that the error term in (1.5) is $O_\varepsilon(T^{1+\varepsilon})$, but this seems difficult to prove. In [12] the author showed that (1.6) remains true for $T^{2/3+\varepsilon} \le H \le T$.

For higher moments of $E^*(T)$ the author proved ([6, Part IV])

$$\int_0^T |E^*(t)|^3\,dt \ll_\varepsilon T^{3/2+\varepsilon}, \tag{1.7}$$

and in [6, Part II] that

$$\int_0^T |E^*(t)|^5\,dt \ll_\varepsilon T^{2+\varepsilon}, \tag{1.8}$$

so that by the Cauchy–Schwarz inequality for integrals, (1.7) and (1.8) yield

$$\int_0^T (E^*(t))^4\,dt \ll_\varepsilon T^{7/4+\varepsilon}. \tag{1.9}$$

In [6, Part III] the error-term function $R(T)$ was introduced by

$$\int_0^T E^*(t)\,dt = \frac{3\pi}{4}T + R(T). \tag{1.10}$$

It was shown, by using an estimate for two-dimensional exponential sums, that

$$R(T) = O_\varepsilon(T^{599/912+\varepsilon}), \qquad \frac{599}{912} = 0.65679\ldots. \tag{1.11}$$

In the same paper it was also proved that

$$\int_0^T R^2(t)\,dt = T^2 p_3(\log T) + O_\varepsilon(T^{11/6+\varepsilon}), \tag{1.12}$$

where $p_3(y)$ is a cubic polynomial in y with positive leading coefficient, all coefficients of which may be explicitly evaluated, and that

$$\int_0^T R^4(t)\,dt \ll_\varepsilon T^{3+\varepsilon}. \tag{1.13}$$

The asymptotic formula (1.12) bears resemblance to (1.5), and it is proved by a similar technique. The exponents in the error terms are, in both cases, less than the exponent of T in the main term by $1/6$. From (1.5) one obtains $E^*(T) = \Omega(T^{1/6}(\log T)^{3/2})$, which shows that $E^*(T)$ cannot be too small ($f(x) = \Omega(g(x))$ means that $f(x) = o(g(x))$ does not hold as $x \to \infty$). Likewise, (1.12) yields

$$R(T) = \Omega(T^{1/2}(\log T)^{3/2}). \tag{1.14}$$

It seems plausible that the error term in (1.12) should be $O_\varepsilon(T^{5/3+\varepsilon})$, and one may conjecture that

$$R(T) = O_\varepsilon(T^{1/2+\varepsilon}), \tag{1.15}$$

which is supported by (1.12). In [12] it was proved that, in the range $T^{2/3+\varepsilon} \le H \le T$, we have

$$\int_T^{T+H} R^2(t)\,dt \gg HT\log^3 T, \tag{1.16}$$

and, for $T^\varepsilon \le H \le T$,

$$\int_T^{T+H} R^2(t)\,dt \ll_\varepsilon HT\log^3 T + T^{5/3+\varepsilon}. \tag{1.17}$$

2. Statement of results. Mean values (or moments) of $|\zeta(1/2+it)|$ represent one of the central themes in the theory of $\zeta(s)$. There are two monographs dedicated solely to it: the author's [5], and that of K. Ramachandra [18]. Our results connect bounds for the moments of $|\zeta(1/2+it)|$, $E^*(t)$ and $R(t)$ in short intervals. The meaning of "short interval" is that $[T, T+H]$ is an interval where one can have H much smaller than T, namely $H = o(T)$ as $T \to \infty$. The results are contained in:

THEOREM 1. *For $k \in \mathbb{N}$ fixed, $T^{1/3} \le H = H(T) \le T$, we have*

$$\int_T^{T+H} |\zeta(1/2+it)|^{2k+2}\,dt \ll_k (\log T)^{2k+2} \int_{T-H}^{T+2H} |E^*(t)|^k\,dt + HT^\varepsilon \tag{2.1}$$

and

$$\int_T^{T+H} |E^*(t)|^{2k}\,dt \ll_k (\log T)^{k+2} \int_{T-H}^{T+2H} |R(t)|^k\,dt + HT^\varepsilon. \tag{2.2}$$

THEOREM 2. *Let $k \in \mathbb{N}$ be fixed and $T^\varepsilon \le H = H(T) \le T$. If*

$$\int_0^T |E^*(t)|^k\,dt \ll_{\varepsilon,k} T^{A(k)+\varepsilon} \tag{2.3}$$

for some constant $A(k)$, then $A(k) \ge 1 + k/6$, and

$$\int_T^{2T} \Big(\int_{t-H}^{t+H} |\zeta(1/2+iu)|^2\,du\Big)^k dt \ll_{\varepsilon,k} T^{A(k)+\varepsilon} + TH^k(\log T)^k. \tag{2.4}$$

The term HT^ε in (2.1) can be omitted if $k > 1$, in view of $A(k) \ge 1 + k/6$ in (2.3). When $k = 1$ or $k = 2$, a much more precise result can be obtained for the integral in (2.4). This is contained in

THEOREM 3. *For $T^\varepsilon \le H = H(T) \le T$ we have*

$$\int_T^{2T} \Big(\int_{t-H}^{t+H} |\zeta(1/2+iu)|^2\,du\Big) dt = 2H\Big(T\log\Big(\frac{2T}{e\pi}\Big)\Big) + 4H\gamma T + O(H^2) + O(T^{3/4}) \tag{2.5}$$

and

$$\int_T^{2T} \Big(\int_{t-H}^{t+H} |\zeta(1/2+iu)|^2\,du\Big)^2 dt \ll H^2T(\log T)^2. \tag{2.6}$$

For $T^\varepsilon \le H = H(T) \le T^{1/2-\varepsilon}$ we have the asymptotic formula

$$\int_T^{2T}\Big(\int_{t-H}^{t+H}|\zeta(1/2+iu)|^2\,du\Big)^2\,dt = H^2T(4\log^2 T + e_1\log T + e_0) + HT\sum_{j=0}^{3} d_j\log^j\Big(\frac{\sqrt{T}}{2H}\Big) + O_\varepsilon(T^{1/2+\varepsilon}H^2) + O_\varepsilon(T^{1+\varepsilon}H^{1/2}), \tag{2.7}$$

where the d_j's and e_1, e_0 are suitable constants ($d_3 > 0$).

The proofs of Theorems 1–3 will be given in Section 3. In Section 4 we shall provide some corollaries and remarks to these theorems.

3. Proofs of the theorems. In (2.1) of Theorem 1 we have an estimate for the moments of $|\zeta(1/2+it)|$. In order to deal with these moments we shall use the standard large values technique (see, e.g., [4, Chapter 8]). To transform discrete sums into sums of integrals one uses the bound

$$|\zeta(1/2+it)|^k \ll \log t\int_{t-1}^{t+1}|\zeta(1/2+ix)|^k\,dx + 1 \qquad (k\in\mathbb{N}\text{ fixed}), \tag{3.1}$$

which is Theorem 1.2 of [5] (see also Lemma 7.1 of [4]).

We begin (henceforth let $L = \log T$ for brevity) by noting that, for $T^\varepsilon \ll G \le T$,

$$\begin{aligned}\int_{T-G}^{T+G}|\zeta(1/2+it)|^2\,dt &= \int_{-G}^{G}|\zeta(1/2+iT+iu)|^2\,du\\ &\le e\int_{-\infty}^{\infty}|\zeta(1/2+iT+iu)|^2e^{-(u/G)^2}\,du\\ &= e\int_{-GL}^{GL}|\zeta(1/2+iT+iu)|^2e^{-(u/G)^2}\,du + O(e^{-L^2/2}).\end{aligned}$$

In view of (1.2) we further have, on integrating by parts,

$$\begin{aligned}\int_{-GL}^{GL}|\zeta(1/2+iT+iu)|^2e^{-(u/G)^2}\,du &= \int_{-GL}^{GL}e^{-(u/G)^2}\,dE(T+u)+O(GL)\\ &= 2\int_{-GL}^{GL}uG^{-2}e^{-(u/G)^2}E(T+u)\,du+O(GL).\end{aligned}$$

By the definition of $E^*(T)$ the last integral becomes

$$\frac{1}{G^2}\int_{-GL}^{GL}xE^*(T+x)e^{-(x/G)^2}\,dx + \frac{2\pi}{G^2}\int_{-GL}^{GL}x\Delta^*\Big(\frac{T+x}{2\pi}\Big)e^{-(x/G)^2}\,dx.$$

To bound the integral containing the Δ^* function, we shall use the estimate

$$\sum_{x<n\le x+h} d(n) \ll h\log x \qquad (x^\varepsilon \le h \le x), \tag{3.2}$$

which follows from a general result of P. Shiu [19] on multiplicative functions. Write

$$\int_{-GL}^{GL} x\Delta^*\Big(\frac{T+x}{2\pi}\Big)e^{-(x/G)^2}\,dx = \int_{-GL}^{0}\cdots dx + \int_{0}^{GL}\ldots dx, \tag{3.3}$$

and make the change of variable $y=-x$ in the first integral on the right-hand side. Then (3.3) becomes

$$-\int_0^{GL} y\Delta^*\Big(\frac{T-y}{2\pi}\Big)e^{-(y/G)^2}\,dy + \int_0^{GL} x\Delta^*\Big(\frac{T+x}{2\pi}\Big)e^{-(x/G)^2}\,dx$$
$$= \int_0^{GL} x\Big\{\Delta^*\Big(\frac{T+x}{2\pi}\Big) - \Delta^*\Big(\frac{T-x}{2\pi}\Big)\Big\}e^{-(x/G)^2}\,dx.$$

For $|x|\le T^{\varepsilon/3}$ we use the trivial bound (coming from $d(n)\ll_\varepsilon n^{\varepsilon/3}$)

$$\Delta^*\Big(\frac{T+x}{2\pi}\Big) - \Delta^*\Big(\frac{T-x}{2\pi}\Big) \ll_\varepsilon T^{2\varepsilon/3},$$

while for $T^{\varepsilon/3} < |x| \le GL$ we use (3.2). This yields

$$\int_0^{GL} x\Big\{\Delta^*\Big(\frac{T+x}{2\pi}\Big) - \Delta^*\Big(\frac{T-x}{2\pi}\Big)\Big\}e^{-(x/G)^2}\,dx \tag{3.4}$$
$$\ll_\varepsilon T^{2\varepsilon/3}G^2 + G^3L \ll G^3L,$$

since $G \gg T^\varepsilon$. Therefore (3.4) furnishes the bound

$$\frac{2\pi}{G^2}\int_{-GL}^{GL} x\Delta^*\Big(\frac{T+x}{2\pi}\Big)e^{-(x/G)^2}\,dx \ll GL,$$

and we obtain the starting point for the proof of (2.1), which we formulate as

Lemma 1. *For $T^\varepsilon \le G = G(T) \le T/L$, where $L = \log T$, we have*

$$\int_{T-G}^{T+G} |\zeta(1/2+it)|^2\,dt \le \frac{2e}{G^2}\int_{-GL}^{GL} xE^*(T+x)e^{-(x/G)^2}\,dx + O(GL). \tag{3.5}$$

We return to the proof of (2.1) and suppose now that $\{t_r\}_{r=1}^R$ is a set of points satisfying

$$(3.6)\qquad T < t_1 < \cdots < t_R \le T+H, \quad |\zeta(1/2+it_r)| \ge V \ge T^{\varepsilon}, \quad |t_r - t_s| > 2 \quad (r, s = 1, \ldots, R,\ r \ne s).$$

Since the intervals $[t_r - 1, t_r + 1]$ are disjoint and $V \ge T^{\varepsilon}$, from (3.1) we obtain

$$RV^2 \ll L \sum_{r=1}^{R} \int_{t_r-1}^{t_r+1} |\zeta(1/2+it)|^2\,dt.$$

We now cover $[T, T+H]$ by disjoint subintervals of length G, which is a parameter to be chosen later (see (3.8)) and which satisfies $T^{\varepsilon} \le G \ll H$, starting with $[T, T+G), [T+G, T+2G), \ldots$. The last of these intervals may partially fall out of $[T, T+H]$, which does not matter. Let S $(\le R)$ of these intervals contain some of the points $\{t_r\}_{r=1}^{R}$, and denote their midpoints by $\tau_1, \ldots, \tau_S$. Then each of the intervals $[t_r - 1, t_r + 1]$ is contained in one of the intervals of the form $[\tau_s - 2G/3, \tau_s + 2G/3]$, and if we consider separately the τ_j's with even and odd indices j, these intervals are disjoint. Thus, with a slight abuse of notation, by Lemma 1 we have

$$(3.7)\qquad RV^2 \ll L \sum_{s=1}^{S} \int_{\tau_s-2G/3}^{\tau_s+2G/3} |\zeta(1/2+it)|^2\,dt \le 2eL \sum_{s=1}^{S} \frac{1}{G^2} \int_{-GL}^{GL} xE^*(\tau_s + x)e^{-(x/G)^2}\,dx,$$

provided that, for some sufficiently small constant $c > 0$, we choose

$$(3.8)\qquad G = cV^2/L.$$

By bounds for $|\zeta(1/2+it)|$ we obtain $G \ll T^{1/3} \ll H$, and we choose a representative set of points τ_ℓ, $\ell = 1, \ldots, S'$ $(\le S)$, from the set $\{\tau_s\}_{s=1}^{S}$ such that the intervals $(\tau_\ell - GL,\ \tau_\ell + GL)$ are disjoint for $\ell = 1, \ldots, S'$. Therefore it follows by Hölder's inequality for integrals that (here we assume $k > 1$, the case $k = 1$ follows directly from (3.7))

$$RV^2 \ll L^2 \sum_{\ell=1}^{S'} G^{-2} \int_{-GL}^{GL} |xE^*(\tau_\ell + x)|e^{-(x/G)^2}\,dx$$
$$\ll L^2 G^{-2} \sum_{\ell=1}^{S'} \Big(\int_{-GL}^{GL} |E^*(\tau_\ell + x)|^k e^{-(x/G)^2}\,dx \Big)^{1/k} \times \Big(\int_{-GL}^{GL} |x|^{k/(k-1)} e^{-(x/G)^2}\,dx \Big)^{1-1/k}$$

$$\ll L^2G^{-2}\Big(\sum_{\ell=1}^{S'}\int\limits_{-GL}^{GL}|E^*(\tau_\ell+x)|^k e^{-(x/G)^2}\,dx\Big)^{1/k}G^{2-1/k}(S')^{1-1/k}$$
$$\ll G^{-1/k}L^2S^{1-1/k}\Big(\int\limits_{T-H}^{T+2H}|E^*(x)|^k\,dx\Big)^{1/k}.$$

Since $S\le R$, in view of (3.8) this gives

$$R\ll V^{-2k}G^{-1}L^{2k}\int\limits_{T-H}^{T+2H}|E^*(x)|^k\,dx \tag{3.9}$$
$$\ll V^{-2k-2}L^{2k+1}\int\limits_{T-H}^{T+2H}|E^*(x)|^k\,dx.$$

The case $H=T/4$ of (3.9) is somewhat sharper than the result for the "long" interval $[T,2T]$ proved by the author in [6, Part II], since that result contained a factor T^ε instead of a log-power. The bound in (2.1) follows if the integral on the left-hand side is split into $O(\log T)$ subintegrals where $T^\varepsilon\le V\le|\zeta(1/2+it)|\le 2V$. Denoting each such integral by I_V, we estimate it as

$$I_V\ll\sum_{r=1}^{R_V}|\zeta(1/2+it_r)|^{2k+2}\ll R_VV^{2k+2}\ll L^{2k+1}\int\limits_{T-H}^{T+2H}|E^*(x)|^k\,dx,$$

where the points t_r are chosen in such a way that $|t_r-t_s|\ge 1$ for $r\ne s$. Finally by estimating the contribution of $V\le T^\varepsilon$ trivially, we obtain (2.1).

Let henceforth C denote generic positive constants, although in each instance we could evaluate C explicitly. To prove (2.2) we need

Lemma 2. *For $T^\varepsilon\le G=G(T)\ll T$, $t\asymp T$, $L=\log T$ we have*

$$E^*(t)\le\frac{C}{G}\int\limits_t^{t+G}\varphi_+(u)E^*(u)\,du+CGL \tag{3.10}$$

and

$$E^*(t)\ge\frac{C}{G}\int\limits_{t-G}^{t}\varphi_-(u)E^*(u)\,du-CGL. \tag{3.11}$$

Here φ_+ is a non-negative, smooth function supported in $[t,t+G]$ such that $\varphi_+(u)=1$ for $t+G/4\le u\le t+3G/4$. Similarly, φ_- is a non-negative, smooth function supported in $[t-G,t+G]$ such that $\varphi_-(u)=1$ for $t-3G/4\le u\le t-G/4$. We have $\varphi'_\pm(u)\ll 1/G$.

The proofs of these inequalities are similar, so we only treat (3.10). From (1.2) we have, for $0\le u\ll T$,

$$0 \le \int_T^{T+u} |\zeta(1/2+it)|^2\,dt = (T+u)\Big(\log\Big(\frac{T+u}{2\pi}\Big)+2\gamma-1\Big)$$
$$- T\Big(\log\Big(\frac{T}{2\pi}\Big)+2\gamma-1\Big) + E(T+u) - E(T).$$

By the mean-value theorem this implies

$$E(T) \le E(T+u) + O(u\log T),$$

giving by integration and change of notation

$$E(t) \le \frac{C}{G}\int_t^{t+G} \varphi_+(u)E(u)\,du + CG\log T \qquad (1 \ll G \ll T,\ C>0,\ t \asymp T). \tag{3.12}$$

By using (3.2) again it is established that, for $T^\varepsilon \le G \le T$, $t \asymp T$,

$$\Delta^*\Big(\frac{t}{2\pi}\Big) = \frac{C}{G}\int_t^{t+G} \varphi_+(u)\Delta^*\Big(\frac{u}{2\pi}\Big)\,du + O(G\log T). \tag{3.13}$$

Therefore by combining (3.12) and (3.13) one obtains (3.10), since

$$E^*(t) = E(t) - 2\pi\Delta^*\Big(\frac{t}{2\pi}\Big).$$

In proving (2.2) we use (3.10) if $E^*(t) > 0$, and (3.11) otherwise. Suppose $E^*(t) > 0$. Then by integrations by parts from (3.10) we obtain

$$\begin{aligned}
E^*(t) &\le \frac{C}{G}\int_t^{t+G} \varphi_+(u)E^*(u)\,du + CGL \qquad (3.14)\\
&= \frac{C}{G}\int_0^u E^*(v)\,dv \cdot \varphi_+(u)\Big|_{u=t}^{t+G} + CGL\\
&\quad - \frac{C}{G}\int_t^{t+G} \varphi_+'(u)\int_0^u E^*(v)\,dv\,du\\
&= -\frac{C}{G}\int_t^{t+G}\Big(\frac{3\pi}{4}u + R(u)\Big)\varphi_+'(u)\,du + CGL\\
&= -\frac{C3\pi}{4G}\Big(u\varphi_+(u)\Big|_{u=t}^{t+G} - \int_t^{t+G}\varphi_+(u)\,du\Big)\\
&\quad - \frac{C}{G}\int_t^{t+G} R(u)\varphi_+'(u)\,du + CGL\\
&= O(GL) - \frac{C}{G}\int_t^{t+G} R(u)\varphi_+'(u)\,du.
\end{aligned}$$

Combining (3.14) with the corresponding lower bound and using the fact that $\varphi'_\pm(u) \ll 1/G$, we have proved

LEMMA 3. *For $T^\varepsilon \le G = G(T) \ll T$, $t \asymp T$, we have*

$$|E^*(t)| \ll \frac{1}{G^2}\int_{t-G}^{t+G} |R(u)|\,du + CGL. \tag{3.15}$$

If we suppose that $R(T) \ll_\varepsilon T^{\alpha+\varepsilon}$ then from (3.15), (3.5) of Lemma 1 and (3.1) we obtain

$$\zeta(1/2+it) \ll_\varepsilon t^{\alpha/4+\varepsilon}, \qquad E^*(T) \ll_\varepsilon T^{\alpha/2+\varepsilon}, \tag{3.16}$$

so that with the value $\alpha = 593/912 = 0.6502129\ldots$ (see (1.11)) we have the bounds

$$\begin{aligned} \zeta(1/2+it) &\ll_\varepsilon |t|^{593/3648+\varepsilon}, \quad 593/3648 = 0.164199\ldots, \\ E^*(T) &\ll_\varepsilon T^{593/1824+\varepsilon}, \quad 593/1824 = 0.32510\ldots. \end{aligned} \tag{3.17}$$

If the conjectural $\alpha = 1/2$ held ($\alpha < 1/2$ is impossible by (1.14)), then from (3.15) we would obtain

$$\zeta(1/2+it) \ll_\varepsilon |t|^{1/8+\varepsilon}, \qquad E^*(T) \ll_\varepsilon T^{1/4+\varepsilon},$$

which is out of reach by present day methods. See (4.5) for the best known bound for $\zeta(1/2+it)$; the best known exponent for $E^*(T)$ is $131/416 = 0.31490\ldots$. This exponent was proved for $E(T)$ by N. Watt [20], but since the same exponent holds for $\Delta(x)$ and $\Delta^*(x)$, it holds for $E^*(T)$ as well. Thus, although the bounds in (3.17) are non-trivial, they are not the best ones known at present.

We return now to our proof of (2.2). The method is similar to the proof of (2.1), so we shall be relatively brief. Suppose now that $|E^*(t)| \ge V \ge T^\varepsilon$ on a set of points $\{t_r\}_{r=1}^{\mathcal{R}}$ lying in $[T, T+H]$ and spaced at least CG apart. We take $G = \delta V/L$ $(< H)$ for sufficiently small $\delta > 0$. Then from (3.15) we have, for a representative set of the t_r's such that the intervals $(t_r - G,\, t_r + G)$ are disjoint,

$$\begin{aligned} \mathcal{R}V^3L^{-2} &\ll \sum_{r=1}^{\mathcal{R}} \int_{t_r-G}^{t_r+G} |R(u)|\,du \ll \sum_{r=1}^{\mathcal{R}} \Big(\int_{t_r-G}^{t_r+G} |R(u)|^k\,du\Big)^{1/k} G^{1-1/k} \\ &\ll \Big(\sum_{r=1}^{\mathcal{R}} \int_{t_r-G}^{t_r+G} |R(u)|^k\,du\Big)^{1/k} (\mathcal{R}G)^{1-1/k}, \end{aligned}$$

on applying Hölder's inequality for integrals. Since the intervals $(t_r - G,$ $t_r + G)$ are disjoint, and their union is contained in $[T-H,\, T+2H]$, the preceding bound gives

$$\mathcal{R} \ll \int_{T-H}^{T+2H} |R(u)|^k \, du \cdot V^{-3k} L^{2k} G^{k-1},$$

which simplifies to

$$\mathcal{R} \ll \int_{T-H}^{T+2H} |R(u)|^k \, du \cdot V^{-1-2k} L^{k+1}. \tag{3.18}$$

Splitting $\int_T^{T+H} |E^*(t)|^{2k} \, dt$ into $O(\log T)$ integrals I_V, in each of which

$$V \le |E^*(t)| \le 2V,$$

we estimate each of these integrals by (3.18), keeping in mind that $T^\varepsilon \le V \le T^{1/3} \ll H$. On estimating trivially the contribution of $V \le T^\varepsilon$, the bound in (2.2) follows at once.

An obvious corollary of Theorem 1 is that

$$\int_T^{T+H} |\zeta(1/2+it)|^{4k+2} \, dt \ll (\log T)^{5k+4} \int_{T-2H}^{T+4H} |R(t)|^k \, dt \qquad (T^{1/3} \ll H \ll T). \tag{3.19}$$

From (1.17) and (3.19) with $k = 2$ we obtain

$$\int_{T-H}^{T+H} |\zeta(1/2+it)|^{10} \, dt \ll_\varepsilon T^\varepsilon (HT + T^{5/3}) \qquad (T^{1/3} \ll H \ll T). \tag{3.20}$$

It seems that this bound is new when H is close to $T^{2/3}$. It gives, by (3.1), the classical bound $\zeta(1/2+it) \ll_\varepsilon |t|^{1/6+\varepsilon}$.

We shall now pass to the proof of Theorem 2. To obtain (2.4) we use (3.5) of Lemma 1 with $G \equiv H$. This gives, for fixed $k \in \mathbb{N}$, $T^\varepsilon \le H = H(T) \le T$,

$$\int_T^{2T} \Big(\int_{t-H}^{t+H} |\zeta(1/2+iu)|^2 \, du \Big)^k dt \ll H^{-k} \int_T^{2T} \Big(\int_{-HL}^{HL} |E^*(t+x)| e^{-(x/H)^2} \, dx \Big)^k dt + TH^k L^k. \tag{3.21}$$

Hölder's inequality for integrals shows that the integral on the right-hand side of (3.21) is

$$(3.22)\qquad \le \int_T^{2T}\int_{-HL}^{HL} |E^*(t+x)|^k e^{-(x/H)^2}\,dx\cdot\Bigl(\int_{-HL}^{HL} e^{-(x/H)^2}\,dx\Bigr)^{k-1}\,dt$$
$$\ll H^{k-1}\int_{-HL}^{HL} e^{-(x/H)^2}\Bigl(\int_{T-HL}^{2T+HL}|E^*(t)|^k\,dt\Bigr)\,dx.$$

From (3.21) and (3.22) we obtain (2.4) if we take into account (2.3). Note that the constant $A(k)$ in (2.3) must actually satisfy $A(k) \ge 1 + k/6$ for any $k \ge 1$, and not only when k is an integer. If $k \ge 2$, then by Hölder's inequality for integrals

$$\int_T^{2T} |E^*(t)|^2\,dt \le \Bigl(\int_T^{2T}|E^*(t)|^k\,dt\Bigr)^{2/k} T^{1-2/k},$$

and the desired bound for $A(k)$ follows from the mean square formula (1.5). If $1 \le k \le 2$ then it follows in a similar fashion from (1.5) and (1.7). We remark that if $A(k) = 1 + k/6$ for some k, then (2.1) and (3.1) yield the bound

$$\zeta(1/2+it) \ll_\varepsilon |t|^{(k+6)/(12(k+1))+\varepsilon},$$

and this improves the exponent $32/205 = 0.15609\ldots$ (see (4.5)) for $k \ge 5$, since for $k = 5$ it gives $11/72 = 0.152777\ldots$.

It remains to prove Theorem 3. We begin by noting that the author [11] proved the following result, which improves on an earlier result of M. Jutila [16]: If $1 \ll U = U(T) \le \frac{1}{2}\sqrt{T}$, then we have ($c_3 = 8\pi^{-2}$)

$$(3.23)\qquad \int_T^{2T} (\Delta(x+U)-\Delta(x))^2\,dx = TU\sum_{j=0}^{3} c_j \log^j\Bigl(\frac{\sqrt{T}}{U}\Bigr)$$
$$+ O_\varepsilon(T^{1/2+\varepsilon}U^2) + O_\varepsilon(T^{1+\varepsilon}U^{1/2}),$$

a similar result being true if $\Delta(x+U)-\Delta(x)$ is replaced by $E(x+U)-E(x)$, with different constants c_j ($c_3 > 0$). This is in tune with the analogy (see the Introduction) between $E(T)$ and $\Delta(x)$, as indicated first by F. V. Atkinson [1], who proved an explicit formula for $E(T)$ with error term which is only $O(\log^2 T)$ (see also [4, Chapter 15] and [5, Chapter 2]). But the integral in (2.7) can be reduced to the evaluation of the mean square of $E(t+h) - E(t-h)$, since by (1.2) one has

$$(3.24)\qquad \int_{t-H}^{t+H} |\zeta(1/2+it)|^2\,dt$$
$$= E(t+H) - E(t-H) + 2H\Bigl(\log\Bigl(\frac{t}{2\pi}\Bigr) + 2\gamma\Bigr) + O\Bigl(\frac{H^2}{T}\Bigr).$$

Therefore

$$\int_T^{2T}\Big(\int_{t-H}^{t+H}|\zeta(1/2+iu)|^2\,du\Big)^2\,dt = I_1+2I_2+I_3,$$

say, where

(3.25)

$$\begin{aligned} I_1 &:= \int_T^{2T}(E(t+H)-E(t-H))^2\,dt\\ &= \int_{T-H}^{2T-H}(E(x+2H)-E(x))^2\,dx,\\ I_2 &:= \int_T^{2T}2H\bigg(\log\Big(\frac{t}{2\pi}\Big)+2\gamma+O\Big(\frac{H}{T}\Big)\bigg)(E(t+H)-E(t-H))\,dt,\\ I_3 &:= \int_T^{2T}4H^2\bigg(\log\Big(\frac{t}{2\pi}\Big)+2\gamma+O\Big(\frac{H}{T}\Big)\bigg)^2\,dt. \end{aligned}$$

To evaluate I_1 we write

$$I_1=\int_{T-H}^{2T-H} = \int_T^{2T}+\int_{T-H}^{T}-\int_{2T-H}^{2T} = J_1+J_2-J_3,$$

say. By trivial estimation, in view of $E(t)\ll t^{1/3}$ (see, e.g., [4, Ch. 15]), it follows that

$$J_2-J_3\ll HT^{2/3}.$$

To evaluate J_1 we use the analogue of (3.23) (with $U=2H$) for $E(x+U)-E(x)$. This gives, with suitable constants d_j ($d_3>0$) and $1\ll H\ll\sqrt{T}$,

$$J_1 = TH\sum_{j=0}^{3}d_j\log^j\bigg(\frac{\sqrt{T}}{2H}\bigg)+O_\varepsilon(T^{1/2+\varepsilon}H^2)+O_\varepsilon(T^{1+\varepsilon}H^{1/2}).$$

One can evaluate I_3 in a straightforward way to obtain

$$\begin{aligned} I_3 &= 4H^2\int_T^{2T}\bigg(\log^2\Big(\frac{t}{2\pi}\Big)+4\gamma^2+4\gamma\log\Big(\frac{t}{2\pi}\Big)+O\Big(\frac{H\log T}{T}\Big)\bigg)\,dt\\ &= H^2T(4\log^2T+e_1\log T+e_0)+O(H^3\log T) \end{aligned}$$

with suitable constants e_0 and e_1.

Finally to bound I_2 we invoke a result of J. L. Hafner and the author [2]

$$\text{(3.26)}\quad E_1(T):=\int_2^T E(u)\,du = \pi T+O(G(T)),\quad G(T)=O(T^{3/4})\quad (T>2).$$

Actually in [2] an explicit expression is given for $G(T)$ (from which one can deduce that $G(T) = \Omega_\pm(T^{3/4})$). Thus from (3.25), (3.26) we obtain, on integrating by parts,

$$I_2 = 2H\left\{(E_1(t+H) - E_1(t-H))\left(\log\frac{t}{2\pi} + 2\gamma\right)\right\}\Big|_{t=T}^{2T}$$
$$- 2H\int_T^{2T}(E_1(t+H) - E_1(t-H))\,\frac{dt}{t} + O(H^2T^{1/3})$$
$$= O(H^2\log T) + O(HT^{3/4}\log T) + O(H^2T^{1/3}) = O(HT^{3/4}\log T)$$

in view of the range for H, namely $T^\varepsilon \le H = H(T) \le T^{1/2-\varepsilon}$.

Combining the expressions for I_1, I_2 and I_3 we obtain (2.7), which in the range $T^\varepsilon \le H = H(T) \le T^{1/2-\varepsilon}$ provides an asymptotic formula for the integral in question. Note that in this range $HT^{3/4}L \ll T^{1+\varepsilon}H^{1/2}$, so only the error terms in (2.7) remain. For $T^{1/2-\varepsilon} \le H \le T$ the upper bound in (2.6) follows easily from (2.4) and $A(2) \le 4/3$; see (4.1).

It remains yet to prove (2.5). Note that, by (3.24), the integral in question is easily seen to be equal to

$$(3.27)\quad 2H\int_T^{2T}\left(\log\frac{t}{2\pi} + 2\gamma + O\left(\frac{H}{T}\right)\right)dt + \int_T^{2T}(E(t+H) - E(t-H))\,dt.$$

But by using (3.26) again it is seen that (3.27) reduces to

$$2H\left(T\log\left(\frac{2T}{e\pi}\right)\right) + 4H\gamma T + O(H^2) + O(T^{3/4}).$$

Hence, for $T^\varepsilon \le H = H(T) \le T$,

$$\int_T^{2T}\left(\int_{t-H}^{t+H}|\zeta(1/2+iu)|^2\,du\right)dt$$
$$= 2H\left(T\log\left(\frac{4T}{e}\right)\right) + 4H\gamma T + O(H^2) + O(T^{3/4}),$$

as asserted in (2.5).

4. Some corollaries and remarks. If $A(k)$ is defined by (2.3), then from (1.7)–(1.9) we have

$$(4.1)\qquad A(2) \le \frac{4}{3},\quad A(3) \le \frac{3}{2},\quad A(4) \le \frac{7}{4},\quad A(5) \le 2.$$

We also have $A(1) \le 7/6$ by $A(2) \le 4/3$ and the Cauchy–Schwarz inequality. Then (with $H = T$) (2.1) of Theorem 1 yields

$$\int_0^T |\zeta(1/2+it)|^8\,dt \ll_\varepsilon T^{3/2+\varepsilon},$$

$$\int_0^T |\zeta(1/2+it)|^{10}\,dt \ll_\varepsilon T^{7/4+\varepsilon}, \tag{4.2}$$

$$\int_0^T |\zeta(1/2+it)|^{12}\,dt \ll_\varepsilon T^{2+\varepsilon},$$

with $k = 3, 4, 5$, respectively. The bounds in (4.2) (up to T^ε, which can be replaced by a log-factor) are the sharpest known bounds for the moments in question (see e.g. [4, Chapter 8]).

On the other hand, by using (1.4), from (2.1) we also have

$$\int_T^{T+H} |\zeta(1/2+it)|^6\,dt \ll_\varepsilon HT^{1/3}\log^9 T + T^{1+\varepsilon} \quad (T^{1/3} \le H \le T). \tag{4.3}$$

Although this is not trivial, it can be improved if one uses the bound of H. Iwaniec [13]

$$\int_T^{T+H} |\zeta(1/2+it)|^4\,dt \ll_\varepsilon T^\varepsilon(H + TH^{-1/2}) \quad (T^\varepsilon \le H \le T). \tag{4.4}$$

The bound in (4.4) was obtained by sophisticated methods from the spectral theory of the non-Euclidean Laplacian, and if coupled with the best known bound of M. N. Huxley [3] for $|\zeta(1/2+it)|$, namely

$$\zeta(1/2+it) \ll_\varepsilon |t|^{32/205+\varepsilon}, \qquad 32/205 = 0.15609\ldots, \tag{4.5}$$

one gets an improvement of (4.3). Note that the famous, yet unsettled Lindelöf conjecture states that, instead of (4.5), one has $\zeta(1/2+it) \ll_\varepsilon |t|^\varepsilon$.

If we combine (1.17) and (2.2) (with $k = 2$), it follows that

$$\int_T^{T+H} |E^*(t)|^4\,dt \ll_\varepsilon HT\log^7 T + T^{5/3+\varepsilon} \qquad (T^\varepsilon \le H \le T). \tag{4.6}$$

The bound in (4.6) does not follow from (1.9), as it is better for $H \le T^{3/4}$.

As a corollary to Theorem 2, with (4.1) we obtain

$$\int_T^{2T} \Big(\int_{t-H}^{t+H} |\zeta(1/2+it)|^2\,du\Big)^3 dt \ll_\varepsilon T^{3/2+\varepsilon} + TH^3L^3,$$

$$\int_T^{2T} \Big(\int_{t-H}^{t+H} |\zeta(1/2+it)|^2\,du\Big)^4 dt \ll_\varepsilon T^{7/4+\varepsilon} + TH^4L^4, \tag{4.7}$$

$$\int_T^{2T} \Big(\int_{t-H}^{t+H} |\zeta(1/2+it)|^2\,du\Big)^5 dt \ll_\varepsilon T^{2+\varepsilon} + TH^5L^5.$$

All the bounds in (4.7) are valid for $T^\varepsilon \le H \le T$, but since (see e.g. K. Ramachandra [18])

$$\int_{t-H}^{t+H} |\zeta(1/2+it)|^{2k}\,dt \gg_k H(\log H)^{k^2} \qquad (\log\log T \ll H \le T,\ k \in \mathbb{N}),$$

we have the expected upper bounds $T(HL)^m$ $(m = 3,4,5)$ for the integrals in (4.7). Indeed, from (4.7) we obtain

$$\begin{aligned}
&\int_T^{2T}\Big(\int_{t-H}^{t+H} |\zeta(1/2+iu)|^2\,du\Big)^3 dt \ll TH^3L^3 \quad (H \ge T^{1/6+\varepsilon}),\\
&\int_T^{2T}\Big(\int_{t-H}^{t+H} |\zeta(1/2+iu)|^2\,du\Big)^4 dt \ll TH^4L^4 \quad (H \ge T^{3/16+\varepsilon}),\\
&\int_T^{2T}\Big(\int_{t-H}^{t+H} |\zeta(1/2+iu)|^2\,du\Big)^5 dt \ll TH^5L^5 \quad (H \ge T^{1/5+\varepsilon}).
\end{aligned}\tag{4.8}$$

The bounds in (4.8) seem to be the best unconditional bounds yet.

For the analogous, but less difficult, problem of moments of

$$J_k(t,G) := \frac{1}{\sqrt{\pi}G}\int_{-\infty}^{\infty} |\zeta(1/2+it+iu)|^{2k} e^{-(u/G)^2}\,du \qquad (t \asymp T, T^\varepsilon \le G \ll T),$$

where k is a natural number, we refer the reader to the author's work [7]. Not only do we have

$$\int_{T-G}^{T+G} |\zeta(1/2+it)|^{2k}\,dt = \int_{-G}^{G} |\zeta(1/2+iT+iu)|^{2k}\,du \le \sqrt{\pi}\,eG\,J_k(T,G),$$

but the presence of the smooth Gaussian exponential factor in $J_k(T,G)$ facilitates the ensuing estimations. We know (this is [7, Theorem 1]) that

$$\int_T^{2T} J_1^m(t,G)\,dt \ll_\varepsilon T^{1+\varepsilon}\tag{4.9}$$

for $T^\varepsilon \le G \le T$ if $m = 1,2$ for $T^{1/7+\varepsilon} \le G \le T$ if $m = 3$, and for $T^{1/5+\varepsilon} \le G \le T$ if $m = 4$; and these bounds were sharpened in [10] to $T^{7/36} \le G \le T$ when $m = 4$, $T^{1/5} \le G \le T$ when $m = 5$, and $T^{2/9} \le G \le T$ when $m = 6$. The bounds in (4.9) can be compared to those in (4.8).

We remark that in [8] the author proved that

$$\int_T^{2T}\Big(\int_{t-G}^{t+G} |\zeta(1/2+iu)|^4\,du\Big)^2 dt \ll_\varepsilon G^2T^{1+\varepsilon}\tag{4.10}$$

for $T^{1/2} \le G = G(T) \ll T$. In fact, (4.10) is connected with the following, more general result (Theorem 1 of [8]): Let $T < t_1 < t_2 < \cdots < t_R < 2T$, $t_{r+1} - t_r \ge G$ for $r = 1, \ldots, R-1$. If, for fixed $m, k \in \mathbb{N}$, we have

$$\int_T^{2T} \Big(\frac{1}{G} \int_{t-G}^{t+G} |\zeta(1/2+iu)|^{2k}\, du \Big)^m dt \ll_\varepsilon T^{1+\varepsilon} \tag{4.11}$$

for $T^{\alpha_{k,m}} \le G = G(T) \ll T$ and $0 \le \alpha_{k,m} \le 1$, then

$$\sum_{r=1}^{R} \int_{t_r-G}^{t_r+G} |\zeta(1/2+it)|^{2k}\, dt \ll_\varepsilon (RG)^{(m-1)/m} T^{1/m+\varepsilon}.$$

In this notation, (4.10) is implied by $\alpha_{2,2} = 1/2$. In fact, if (4.11) holds, then

$$\int_0^T |\zeta(1/2+it)|^{2km}\, dt \ll_\varepsilon T^{1+(m-1)\alpha_{k,m}+\varepsilon}.$$

Non-trivial bounds of the type (4.11) (with $0 \le \alpha_{k,m} \le 1$) are hard to obtain when $m > 2$ or $k > 2$.

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