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A ternary Diophantine inequality over primes

by

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1. Introduction. Piatetski-Shapiro [14] initiated the problem of finding, for a given natural number s, a range of values of $c > 1$ ($c \notin \mathbb{N}$) such that the Diophantine inequality

$$|p_1^c + \cdots + p_s^c - R| < R^{-\eta}$$

has many solutions in primes $p_1, \ldots, p_s$, for all sufficiently large positive real numbers R. Here and below, η is a sufficiently small positive constant depending only on c. For $s = 3$ (the smallest s that can be attacked at present), we find papers by Tolev [16], Cai [4], Kumchev and Nedeva [12] and most recently Kumchev [11], where it is shown that the range

$$1 < c < \frac{61}{55} = 1.10909\ldots$$

is permissible. In the present paper we sharpen Kumchev's approach to obtain the following result.

THEOREM 1. *Let $1 < c < 10/9 = 1.11111\ldots$. The number of prime triples satisfying*

$$|p_1^c + p_2^c + p_3^c - R| < R^{-\eta} \tag{1.1}$$

is $\gg R^{3/c-1-\eta}(\log R)^{-3}$ for $R > C_1(c)$.

We elaborate Kumchev's use of Harman's 'alternative sieve' by using two decompositions of $\sum_{X<p\le 2X} e(xp^c)$ in a similar way to Baker and Weingartner [3]. To get satisfactory numerical results, we use five Buchstab iterations in both decompositions: see Sections 4 and 5 for details.

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The quality of the result in Theorem 1 depends on being able to make a satisfactory power saving for exponential sums (I_m denoting a subinterval of $(N, 2N]$)

$$S_{\mathrm{I}} = \sum_{m \le M} a_m \sum_{n \in I_m} e(x m^c n^c) \quad (MN \asymp X,\ X^{-1+8\eta} < x < X^{3\eta})$$

with arbitrary a_m, $|a_m| \le 1$, for as long a range of M as possible (we obtain this for $M < X^{1/2}$); and a similar saving for sums

$$S_{\mathrm{II}} = \sum_{\substack{M < m \le 2M \\ X < mn \le 2X}} \sum_{N < n \le 2N} a_m b_n e(x m^c n^c) \quad (X^{-1+8\eta} < x < X^{3\eta})$$

with arbitrary $a_m, b_n, |a_m| \le 1, |b_n| \le 1$, for sufficiently generous ranges $X^\alpha \le N \le X^\beta$; our ranges for S_{II} are $[\alpha, \beta] = \left[\frac{2}{9}, \frac{127}{470}\right]$ and $[\alpha, \beta] = \left[\frac{10}{27}, \frac{19}{45}\right]$. The latter range would vanish if the constant $10/9$ in Theorem 1 were to be increased. To get our results for S_{I}, we follow Kumchev [11, Lemma 7], but fill in a great many details and aim for maximum generality, with a view to further applications to be considered elsewhere. The first S_{II} range above depends on work of Huxley [10]. The second (as in [11]) depends on work of Sargos and Wu [15]; we take the opportunity to fill in details not given in [15].

We abbreviate '$M < m \le 2M$' to '$m \sim M$' and '$U \ll u \ll U$' to '$u \asymp U$'. We write $f^{(j)}$ for the jth derivative of a real function f on an interval or a holomorphic function f on an open set V in $\mathbb{C}$, and $g^{(i,j)}$ for the partial derivatives of a function g of two real variables. For $0 < \rho_1 < \rho_2$, $0 < \alpha < \pi/4$, we write

$$S(\rho_1, \rho_2, \alpha) = \{re^{it} \in \mathbb{C} : \rho_1 < r < \rho_2,\ |t| < \alpha\}.$$

We reserve the symbol X for a large positive number and write $\mathcal{L} = \log X$.

Constants implied by 'O' or written as $C, C_1, C_2, \ldots$ depend at most on $c, \lambda, \theta, \alpha, \beta$. The numbering of the C_j begins anew in each section. The constant C *need not* be the same in different occurrences in the same section. Constants implied by '$\ll$' are permitted also to depend on η.

2. Type I exponential sums. We shall prove the following result about 'Type I monomial exponential sums' S_{I}.

THEOREM 2. *Let θ, λ be constants, $\theta(\theta-1)(\theta-2)\lambda(\lambda-1)(\theta+\lambda-2)$ $\times (\theta+\lambda-3)(\theta+2\lambda-3)(2\theta+\lambda-4) \ne 0$. Let*

$$S_{\mathrm{I}} = \sum_{m \sim M} \sum_{n \in I_m} a_m e(B m^\lambda n^\theta)$$

where $B > 0$, $M \geq 1$, $N \geq 1$, $|a_m| \leq 1$ *and* I_m *is a subinterval of* $(N, 2N]$. *Let* $F = BM^{\lambda}N^{\theta}$. *Then*

$$(2.1)\qquad S \ll (MN)^{\eta}(F^{3/14}M^{41/56}N^{29/56} + F^{1/5}M^{3/4}N^{11/20} + F^{1/8}M^{13/16}N^{11/16} + M^{3/4}N + MN^{3/4} + MNF^{-1}).$$

We require a number of preliminary lemmas.

LEMMA 1. *Let* $L(Q) = \sum_{j=1}^{J} A_j Q^{a_j} + \sum_{k=1}^{K} B_k Q^{-b_k}$, *where* A_j, a_j, B_k, b_k *are positive. For any* $H > 0$, *there exists* $Q \in (0, H]$ *such that*

$$L(Q) \ll \sum_{j=1}^{J}\sum_{k=1}^{K}(A_j^{b_k}B_k^{a_j})^{1/(a_j+b_k)} + \sum_{k=1}^{K} B_k H^{-b_k}.$$

The implied constant depends only on J, K.

Proof. This is a slight variation of Graham and Kolesnik [7, Lemma 2.4]. ∎

LEMMA 2. *Suppose that* f *has four continuous derivatives on* $I = [a, b]$ *and that* $f'' < 0$ *on* I. *Suppose further that* $I \subseteq [N, 2N]$ *and that* $\alpha = f'(b)$, $\beta = f'(a)$. *Assume that, for some* $F > 0$,

$$f^{(2)}(x) \asymp FN^{-2}, \qquad f^{(j)}(x) \ll FN^{-j} \quad (j = 3, 4)$$

on I. *Let* x_ν *be defined by* $f'(x_\nu) = \nu$ *and let* $\phi(\nu) = \nu x_\nu - f(x_\nu)$. *Then*

$$(2.2)\qquad \sum_{n\in I} e(f(n)) = \sum_{\alpha\leq\nu\leq\beta} \frac{e(-\phi(\nu) - 1/8)}{|f''(x_\nu)|^{1/2}} + O(\log(FN^{-1}+2) + F^{-1/2}N).$$

Proof. This version of van der Corput's B-process is Lemma 3.6 of [7]. ∎

It is helpful to note that if $f^{(j)}(x) = (Kx^{\lambda})^{(j)}(1 + O(\rho))$ $(0 \leq j \leq 1)$ for constants $K > 0$, $\lambda > 0$, $\lambda \neq 1$ with sufficiently small ρ, then

$$(2.3)\qquad \phi(\nu) = C_1 K^{-\frac{1}{\lambda-1}}\nu^{\frac{\lambda}{\lambda-1}}(1 + O(\rho))$$

where $C_1 = \lambda^{-\frac{1}{\lambda-1}} - \lambda^{-\frac{\lambda}{\lambda-1}}$. This formula needs a little modification if $\lambda < 0$, $K < 0$, or both; we disregard this for simplicity of exposition.

LEMMA 3. *Let* $F > 0$. *Let* $\mathcal{A}$ *be a subset of* $\mathcal{R} = [C_2H, C_3H] \times [C_4N, C_5N]$ *and*

$$S = \sum_{(h,n)\in\mathcal{A}} f(h, n)\, e(g(h, n))$$

where f *and* g *are real functions on* $\mathcal{R}$ *with*

$$(2.4)\qquad |f^{(i,j)}(u, v)| < C_6 K H^{-i} N^{-j} \qquad ((u, v) \in R,\ 0 \leq i, j \leq 1).$$

Then for some subrectangle $\mathcal{R}'$ of $\mathcal{R}$,

$$S \ll K \Big| \sum_{(h,n)\in\mathcal{A}\cap\mathcal{R}'} e(g(h,n)) \Big|. \tag{2.5}$$

The implied constant depends on C_2, C_3, C_4, C_5, C_6.

Proof. We apply the identity [9, p. 90]

$$\begin{aligned}\sum_{h=H_1}^{H_2}\sum_{n=N_1}^{N_2} f(h,n)\,G(h,n) = f(H_1,N_1)\sum_{h=H_1}^{H_2}\sum_{n=N_1}^{N_2} G(h,n) \\ + \int_{H_1}^{H_2} f^{(1,0)}(x,N_1)\sum_{h=x}^{H_2}\sum_{n=N_1}^{N_2} G(h,n)\,\mathrm{d}x \\ + \int_{N_1}^{N_2} f^{(0,1)}(H_1,y)\sum_{h=H_1}^{H_2}\sum_{n=y}^{N_2} G(h,n)\,\mathrm{d}y \\ + \int_{N_1}^{N_2}\int_{H_1}^{H_2} f^{(1,1)}(x,y)\sum_{h=x}^{H_2}\sum_{n=y}^{N_2} G(h,n)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Our choices of H_1, H_2 are the smallest and largest integers in $[C_2H, C_3H]$ and similarly for N_1, N_2. Our choice of $G(h,n)$ is $\chi_{\mathcal{A}}(h,n)\,e(g(h,n))$, where $\chi_{\mathcal{A}}$ is the indicator function of $\mathcal{A}$. Each of the four summands on the right side satisfies a bound of the form (2.5), and the lemma follows. ∎

LEMMA 4 (Rouché). *Let γ be a piecewise smooth simple closed curve in a convex domain Ω in $\mathbb{C}$. Suppose that f, g are holomorphic in Ω and*

$$|f(z) - g(z)| < |f(z)| \quad \text{on } \gamma.$$

Then f and g have the same number of zeros (counted with multiplicity) enclosed by γ.

Proof. See [1, p. 153]. ∎

LEMMA 5. *Let θ, σ be constants, $\theta(\theta-1)\sigma(\sigma-1)((\theta-1)\sigma-1) \neq 0$. Let $B \neq 0$, $N \leq X$, $1 \leq q \leq N/\mathcal{L}$, and suppose that the function*

$$f(x) = \big(((x+q)^\theta - x^\theta)^\sigma\big)^{(1)}$$

is positive on $[N, 2N]$. Let $S_t = S(t\eta N, N/(t\eta), \eta/t)$. Then f has a holomorphic extension to S_2 with a holomorphic inverse Φ on $f(S_2)$. Moreover, for $w \in f(S_3)$, $j \geq 0$, we have

$$\Phi^{(j)}(w) = (Lw^{1/\tau})^{(j)}(1 + O(q/N)) \tag{2.6}$$

where $\tau = (\theta-1)\sigma-1$ and $L = (C_7|B|q^\sigma)^{-1/\tau}$ with the constant C_7 depending on θ, σ. The implied constant depends only on θ, σ, η, j.

Proof. In the region $\operatorname{Re} z > 0$, we write $\log z$ for the branch of the logarithm that is real on $(0, \infty)$, and $z^\beta = \exp(\beta \log z)$ ($\beta \in \mathbb{C}$). We suppose for definiteness that $B > 0$, $\theta > 0$, and approximate f (defined in this way on S_1) by g, itself defined by

$$g(z) = B((\theta q z^{\theta-1})^\sigma)^{(1)} = K z^\tau \qquad (\operatorname{Re} z > 0)$$

with $K = C_7 B q^\sigma$.

Applying the binomial expansion to $(1 + q/z)^\theta$, we find that

$$f(z) = g(z)(1 + O(q/N)) \tag{2.7}$$

for $\operatorname{Re} z \geq \eta N$. Now g maps S_1 bijectively onto

$$T_1 := S(K(\eta N)^\tau, K(N/\eta)^\tau, \eta|\tau|).$$

Let $w \in f(S_2)$. We claim that there is exactly one z in S_2 such that $f(z) = w$. This would certainly hold with g in place of f. Now when z is on the boundary of S_1,

$$|(f(z) - w) - (g(z) - w)| < |g(z) - w|.$$

(The left side is $O(\mathcal{L}^{-1}|g(z)|)$ and the right side is $\gg |g(z)|$.) Hence $f(z)-w$, like $g(z) - w$, has exactly one zero in S_2. It is easy to see that $f' \neq 0$ in S_2, so there is a holomorphic inverse Φ of f, $\Phi : f(S_2) \to S_2$.

Let $z = \Phi(w)$, $w \in f(S_2)$. From (2.7),

$$w = C_7 B q^\sigma z^\tau (1 + O(q/N)).$$

An easy calculation gives in turn

$$z^\tau - \frac{w}{C_7 B q^\sigma} \ll \frac{q}{N} N^\tau, \qquad z - \left(\frac{w}{C_7 B q^\sigma}\right)^{1/\tau} \ll \frac{q}{N} N.$$

This gives the case $j = 0$ of the lemma for all w in $f(S_2)$. If w is in the smaller set $f(S_3)$, we apply the Cauchy formula

$$\Phi^{(j)}(w) = \frac{j!}{2\pi i} \int_C \frac{\Phi(\zeta)\, d\zeta}{(\zeta - w)^{j+1}},$$

where the circle C has center w and radius $\gg B q^\sigma N^\tau$, with C and its interior contained in $f(S_2)$. This immediately yields (2.6). ■

Proof of Theorem 2. Suppose first that

$$F \geq MN.$$

We begin the proof like [2, proof of Theorem 4]. With $Q \in [1, \mathcal{L}^{-1} N]$ at our disposal, this yields

$$\frac{S_{\mathrm{I}}^2}{\mathcal{L}^2} \ll \frac{M^2 N^2}{Q} + \frac{MN}{Q} \sum_{q \leq Q} \Big| \sum_{N < n \leq 2N-q} \sum_{m \sim M} e(f(m, n)) \Big| \tag{2.8}$$

with

$$f(m,n) = Bm^{\lambda}((n+q)^{\theta} - n^{\theta}).$$

After conjugating the sum over m, n in (2.8) if necessary (the same device occurs implicitly below), we apply Lemma 2 to the summation over m. This gives rise to functions $x_\nu = x_\nu(n)$ and $\phi(\nu) = \phi(\nu, n)$, say. Explicitly,

$$\phi(\nu,n) = C_8 A^{\sigma}((n+q)^{\theta} - n^{\theta})^{\sigma}\nu^{\lambda/(\lambda-1)},$$

where $C_8 = C_8(\lambda,\theta) \neq 0$ and $\sigma = \frac{-1}{\lambda-1}$, so that

$$\sigma(\sigma-1)(\sigma(\theta-1)-1) \neq 0.$$

As pointed out in the last paragraph of [7, p. 35], we have

$$\frac{1}{|f^{(2,0)}(x_\nu(n))|^{1/2}} = |\phi^{(2,0)}(\nu,n)|^{1/2}.$$

Thus

$$\text{(2.9)}\qquad \frac{S_{\mathrm{I}}^2}{\mathcal{L}^2} \ll \frac{M^2N^2}{Q} + \frac{MN}{Q}\sum_{q\le Q}\Big|\sum_{N<n\le 2N-q}\ \sum_{\nu\in I_1(n)} |\phi^{(2,0)}(\nu,n)|^{\frac12} e(kA^{\sigma}((n+q)^{\theta}-n^{\theta})^{\sigma}\nu^{\frac{\lambda}{\lambda-1}})\Big| + E_1.$$

Here the interval $I_1(n)$ has endpoints $f^{(1,0)}(jM)$ $(j=1,2)$, and E_1 denotes the total error arising from the error terms in (2.2). Clearly

$$\text{(2.10)}\qquad E_1 \ll \mathcal{L}MN^2\bigg(1 + \Big(\frac{FQ}{N}\Big)^{-1/2} M\bigg).$$

Let h_1^{-1} denote the inverse function of $h_1(n) := (n+q)^{\theta} - n^{\theta}$ on $[\eta N, \infty)$. Applying Lemma 3, and rewriting the summation over n, ν,

$$\text{(2.11)}\qquad \frac{S_{\mathrm{I}}^2}{\mathcal{L}^2} \ll \frac{M^2N^2}{Q} + \frac{MN}{Q}\sum_{q\le Q}(FqN^{-1}M^{-2})^{-1/2}\Big|\sum_{\substack{(n,\nu)\in I_2\times J_2\\ n\in I_3(\nu)}} e\big(kA^{\sigma}((n+q)^{\theta}-n^{\theta})^{\sigma}\nu^{\lambda/(\lambda-1)}\big)\Big|,$$

where $I_2 \times J_2$ is a rectangle of the form

$$[N, 2N-q] \times [C_9 Fq(NM)^{-1}, C_{10}Fq(MN)^{-1}],$$

and the interval $I_3(\nu)$ has endpoints $h_1^{-1}\big(\frac{\nu}{\lambda A(jM)^{\lambda-1}}\big)$ $(j=1,2)$.

We now apply Lemma 2 for a second time, to the sum over $n \in I_2 \cap I_3(\nu)$ in (2.11). Let us denote the new variable introduced by μ (instead of ν). Rather than x_μ and $\phi(\mu)$, we write $z(\mu,\nu)$ and $f_0(\mu,\nu)$. Thus

$$f_0(\mu,\nu) = \mu\, z(\mu,\nu) - \phi(\nu, z(\mu,\nu)).$$

Let $G = FqN^{-1}$. Using Lemma 5 and the remark after Lemma 2, we obtain the approximation

$$f_0^{(a,b)}(\mu,\nu) = A(\mu^{-\alpha}\nu^{-\beta})^{(a,b)}(1+O(q/N)) \quad (0 \le a,b \le 4) \tag{2.12}$$

for $\nu \asymp G/M$, $\mu \asymp G/N$, where the constant A satisfies

$$A(G/M)^{-\alpha}(G/N)^{-\beta} \asymp G.$$

Here

$$\alpha = \frac{\theta-1}{2-(\theta+\lambda)}, \qquad \beta = \frac{\lambda}{2-(\theta+\lambda)}.$$

Writing $(\alpha)_0 = 1$, $(\alpha)_s = (\alpha)_{s-1}(\alpha+s-1)$ for $s = 1,2,\ldots$, we may verify that

$$(\alpha)_3(\beta)_3(\alpha+\beta+1)_2 \neq 0. \tag{2.13}$$

With a little thought, we see that the range $I_4(\nu)$ of the variable μ when we apply Lemma 2 the second time is a (possibly empty) interval whose endpoints, written as a function of the real variable ν, are continuous piecewise monotonic functions of ν. We obtain, after a second application of Lemma 3,

$$\frac{S_{\mathrm{I}}^2}{\mathcal{L}^2} \ll \frac{M^2N^2}{Q} + \frac{MN}{Q}\sum_{q\le Q}\left(\frac{M^2N}{Fq}\right)^{1/2}\left(\frac{N^3}{Fq}\right)^{1/2}|S_1| + E_1 + E_2, \tag{2.14}$$

where E_2 is the total error arising from the error terms in (2.2) for the second application of Lemma 2, and

$$S_1 = S_1(q) = \sum_{\nu\in I_2}\sum_{\mu\in I_4(\nu)} e(f_0(\mu,\nu)). \tag{2.15}$$

It is easy to see that

$$\begin{aligned} E_2 &\ll \frac{\mathcal{L}MN}{Q}\sum_{q\le Q}\left(\frac{M^2N}{Fq}\right)^{1/2}\frac{Fq}{MN}\left(1+\left(\frac{Fq}{N}\right)^{-1/2}N\right) \\ &\ll \mathcal{L}(MN^{1/2}Q^{1/2} + MN^2). \end{aligned} \tag{2.16}$$

Let us write $X = \max(G/M, G/N)$, $Y = \min(G/M, G/N)$, $W = XY$, $\delta = q/N$. Recalling the condition (2.13) on α,β, a variant of [2, Theorem 7] enables us to give the upper bound

$$\begin{aligned} S_1 \ll \mathcal{L}(&G^{1/3}W^{1/2} + W^{5/6} + G^{-1/8}W^{15/16} + G^{1/2}W^{1/2}Y^{-1/2} \\ &+ \delta^{2/5}W^{1/2}G^{1/5}Y^{2/5} + \delta^{1/4}W^{3/4}Y^{1/4}). \end{aligned} \tag{2.17}$$

The variant is fairly straightforward if the following remarks are noted.

(a) There are two further terms on the right side of the bound in [2] corresponding to (2.17). These can be omitted since

$$\delta^{1/4}G^{1/4}W^{1/2}Y^{1/4} \ll (\delta^{2/5}W^{1/2}G^{1/5}Y^{2/5})^{5/8}(G^{1/3}W^{1/2})^{3/8},$$
$$G^{1/2}W^{1/2}Y^{5/12} \ll (G^{1/2}W^{1/2}Y^{-1/2})^{1/6}(W^{5/6})^{5/6}.$$

(b) Let us write (m,n) instead of (μ,ν) (if $X = G/N$) or instead of (ν,μ) (if $X = G/M$), in order to make comparison with [2] easier. Let

$$f_1(m,n) = f(m+s,n+r) - f(m,n)$$

for a given $(s,r) \in \mathbb{Z}^2 \setminus \{(0,0)\}$. Following the argument in [2], we must estimate averages of $|S(s,r)|$ over a rectangle $\mathcal{R}$,

$$(s,r) \in \mathcal{R} \setminus \{(0,0)\}.$$

Here

$$S(s,r) = \sum_{(m,n)\in\mathcal{D}\cap(\mathcal{D}-(s,r))} e(f_1(m,n))$$

where $\mathcal{D}$ is the set of pairs (m,n) given in the summation (2.15). Let us focus on pairs (s,r) with

$$\rho := \left|\frac{r}{Y}\right| \geq \left|\frac{s}{X}\right|.$$

In [8], f_0 is restricted to the form $Ah_1(u)h_2(v)$ where h_1, h_2 are 'close to' monomials. This does not matter, since for the estimation of $S(s,r)$ we still have the easily verified approximation

$$f_1^{(a,b)}(m,n) = (-1)^{a+b+1}Am^{-\alpha-a}n^{-\beta-b}\frac{r}{n}\left\{T_{a,b}\left(\frac{sn}{rm}\right) + O(\rho+\delta)\right\} \tag{2.18}$$

with

$$T_{a,b}(z) = (\alpha)_{a+1}(\beta)_b z + (\alpha)_a(\beta)_{b+1}.$$

(c) In [2, Theorem 7], only the case of a summation over a rectangle $\mathcal{R}$, rather than the more complicated domain $\mathcal{D}$, is considered. This causes no difficulty when, at certain points of the argument, we sum over subsets $\mathcal{E}$ of $\mathcal{R} \cap (\mathcal{R} - (s,r))$ with the property that vertical and horizontal lines intersect $\mathcal{E}$ in $O(1)$ intervals. (This property holds good if we replace $\mathcal{E}$ by $\mathcal{E} \cap \mathcal{D} \cap (\mathcal{D} - (s,r))$.)

(d) Polynomial approximations arising from (2.18), together with Lemma 4, are used in [2] to prove that certain functions $h(v)$ have a bounded number of zeros. The most complicated example is

$$h(v) := -H^{(1,0)}(k,v)f_1^{(1,1)}(k,v) + H^{(0,1)}(k,v)f_1^{(2,0)}(k,v) \tag{2.19}$$

where k is fixed, v is restricted by

$$(k,v) \in \mathcal{R} \cap (\mathcal{R} - (s,r)),$$

and H denotes the Hessian $f_1^{(1,1)} f_1^{(2,2)} - (f_1^{(1,2)})^2$. Once the polynomial approximation is given, and v is allowed to vary over a suitable open set in $\mathbb{C}$, it suffices to show that a pair of polynomials with coefficients depending on α, β (two cubics in the case (2.19)) are not proportional. For full details, see the argument following (3.10) in Baker and Weingartner [3]. No change is needed in the present discussion because (2.18) remains valid.

We have thus established (2.17). We consider first the case $M \geq N$. We rewrite (2.17) in the form

$$\begin{aligned}\frac{S_1}{\mathcal{L}} \ll \frac{F^{7/4} q^{7/4}}{M^{15/16} N^{43/16}} + \frac{F^{5/3} q^{5/3}}{M^{5/6} N^{5/2}} + \frac{F^{8/5} q^2}{M^{9/10} N^{5/2}} + \frac{F^{4/3} q^{4/3}}{M^{1/2} N^{11/6}} \\ + \frac{F^{7/4} q^2}{M N^{11/4}} + \frac{Fq}{N^{3/2}}.\end{aligned}$$

We use this in conjunction with (2.14), (2.10), (2.16) to obtain

$$\begin{aligned}\frac{S^2}{\mathcal{L}^3} &\ll \frac{M^2 N^2}{Q} + MN^2 + \frac{M^2 N^{5/2}}{F^{1/2} Q^{1/2}} + F^{1/2} M N^{1/2} Q^{1/2} \\ &\quad + F^{3/4} M^{17/16} N^{5/16} Q^{3/4} + F^{2/3} M^{7/6} N^{1/2} Q^{2/3} + F^{3/5} M^{11/10} N^{1/2} Q \\ &\quad + F^{1/3} M^{3/2} N^{7/6} Q^{1/3} + F^{3/4} M N^{1/4} Q + M^2 N^{3/2} \\ &= T_1 + \cdots + T_{10},\end{aligned}$$

say. Since $F \geq MN$, $M \geq N$ and $Q \geq 1$, we have

$$T_2, T_3 \leq T_{10}, \qquad T_4 \leq T_6.$$

Thus (arguing trivially for $Q < 1$) we deduce

$$\begin{aligned}\frac{S^2}{\mathcal{L}^4} &\ll \frac{M^2 N^2}{Q} + F^{3/4} M^{17/16} N^{5/16} Q^{3/4} + F^{2/3} M^{7/6} N^{1/2} Q^{2/3} \\ &\quad + F^{3/5} M^{11/10} N^{1/2} Q + F^{1/3} M^{3/2} N^{7/6} Q^{1/3} + F^{3/4} M N^{1/4} Q + M^2 N^{3/2},\end{aligned}$$

for all Q, $0 < Q \leq \mathcal{L}^{-1} N$. Applying Lemma 1, we find that

$$\begin{aligned}\frac{S^2}{\mathcal{L}^4} &\ll F^{3/7} M^{41/28} N^{29/28} + F^{2/5} M^{3/2} N^{11/10} + F^{3/8} M^{3/2} N^{9/8} \\ &\quad + F^{3/10} M^{31/20} N^{5/4} + F^{1/4} M^{13/8} N^{11/8} + M^2 N + M^2 N^{3/2} \\ &= U_1 + \cdots + U_7,\end{aligned}$$

say. Since $F \geq MN$ and $M \geq N$, we have

$$U_3, U_4 \leq U_2, \qquad U_6 \leq U_7,$$

and Theorem 2 follows in the case $F \geq MN$, $M \geq N$.

Now suppose that $N > M$. Lemma 4 gives

$$\frac{S_1}{\mathcal{L}} \ll \frac{F^{7/4}q^{7/4}}{M^{15/16}N^{43/16}} + \frac{F^{5/3}q^{5/3}}{M^{5/6}N^{5/2}} + \frac{F^{7/4}q^2}{M^{3/4}N^3} + \frac{F^{4/3}q^{4/3}}{M^{1/2}N^{11/6}} + \frac{F^{8/5}q^2}{M^{1/2}N^{29/10}} + \frac{Fq}{M^{1/2}N}.$$

Proceeding as in the case $M \le N$, we see that

$$\begin{aligned}\frac{S^2}{\mathcal{L}^3} &\ll \frac{M^2N^2}{Q} + MN^2 + \frac{M^2N^{5/2}}{F^{1/2}Q^{1/2}} + F^{1/2}MN^{1/2}Q^{1/2} \\ &\quad + F^{3/4}M^{17/16}N^{5/16}Q^{3/4} + F^{2/3}M^{7/6}N^{1/2}Q^{2/3} + F^{3/5}M^{3/2}N^{1/10}Q \\ &\quad + F^{3/4}M^{5/4}Q + F^{1/3}M^{3/2}N^{7/6}Q^{1/3} + M^{3/2}N^2 \\ &= V_1 + \cdots + V_{10},\end{aligned}$$

say. Since $F \ge MN$, $N \ge M$ and $Q \ge 1$, we have

$$V_2, V_3 \le V_{10}, \qquad V_4 \le V_6,$$

and, for $0 < Q \le N$,

$$\begin{aligned}\frac{S^2}{\mathcal{L}^3} &\ll \frac{M^2N^2}{Q} + F^{3/4}M^{17/16}N^{5/16}Q^{3/4} + F^{2/3}M^{7/6}N^{1/2}Q^{2/3} \\ &\quad + F^{3/5}M^{3/2}N^{1/10}Q + F^{3/4}M^{5/4}Q + F^{1/3}M^{3/2}N^{7/6}Q^{1/3}5 + M^{3/2}N^2.\end{aligned}$$

Applying Lemma 1, we find that

$$\begin{aligned}\frac{S^2}{\mathcal{L}^4} &\ll F^{3/7}M^{41/28}N^{29/28} + F^{2/5}M^{3/2}N^{11/10} + F^{3/8}M^{13/8}N \\ &\quad + F^{3/10}M^{7/4}N^{21/20} + F^{1/4}M^{13/8}N^{11/8} + M^2N + M^{3/2}N^2 \\ &= R_1 + \cdots + R_7,\end{aligned}$$

say. Since $F \ge MN$ and $M \le N$, we have

$$R_3, R_4 \le R_2, \qquad R_6 \le R_7,$$

and

$$\frac{S}{\mathcal{L}^2} \ll F^{3/14}M^{41/56}N^{29/56} + F^{1/5}M^{3/4}N^{11/20} + F^{1/8}M^{13/16}N^{11/16} + M^{3/4}N.$$

This completes the proof of Theorem 2 in the case $F \ge MN$.

Consider finally the case of $F < MN$. By Theorem 1 of [13], which has no restrictions on F,

$$S \ll (MN)^{1+\eta}\left(\left(\frac{F}{MN^2}\right)^{1/4} + \frac{1}{N^{1/2}} + \frac{1}{F}\right). \tag{2.20}$$

The third summand appears in (2.1), and the second summand is acceptable. Finally,

$$(MN)^{\eta}F^{1/4}M^{3/4}N^{1/2} < (MN)^{\eta}MN^{3/4}$$

and the theorem follows in the case $F < MN$. ∎

COROLLARY 1. *Let $M \ge 1$, $N \ge 1$, $MN \asymp X$, $X^{-c+6\eta} < x < X^{3\eta}$, $|a_m| \le 1$, and let I_m be a subinterval of $(N, 2N]$. Then for $1 < c < 10/9$, $M \ll X^{1/2}$, we have*

$$\sum_{m\sim M}\sum_{n\in I_m} a_m\, e(xm^c n^c) \ll \min(X^{1-4\eta}, x^{-1}X^{8/9}).$$

Proof. For $X^{4/9+20\eta} < M \ll X^{1/2}$, we apply Theorem 2. The term MNF^{-1} has a satisfactory bound since $x > X^{-c+6\eta}$. All other terms have satisfactory bounds since $x < X^{3\eta}$, the restriction $M \ll X^{1/2}$ coming from $F^{1/8}M^{13/16}N^{11/16}$ and the restriction $M > X^{4/9+20\eta}$ coming from $M^{3/4}N$.

Now suppose that $M < X^{4/9+20\eta}$. We apply (2.20). We have already discussed the term $(MN)^{1+\eta}F^{-1}$, and the term $(MN)^{1+\eta}N^{-1/2}$ gives no difficulty. For the remaining term,

$$x^{1/4}X^{c/4+\eta}M^{3/4}N^{1/2} < X^{1/2+c/4+2\eta}M^{1/4} < X^{8/9-3\eta},$$

since $c < 10/9$, $M < X^{4/9+20\eta}$, and η is sufficiently small. ∎

3. Type II exponential sums. We begin with a bound for $S_{\mathrm{II}}(x)$ that holds over a wide range of N.

LEMMA 6. *Let*

$$S_{\mathrm{II}}(x) = \sum_{\substack{m\sim M\quad n\sim N\\ X<mn\le 2X}}\sum a_m b_n\, e(xm^c n^c)$$

where $1 < c \le 6/5$, $M \ge 1$, $MN \asymp X$, $|a_m| \le 1$, $|b_n| \le 1$, $X^{-c+8\eta} < x < X^{3\eta}$. Then

$$S_{\mathrm{II}}(x) \ll X^{1-3\eta} \quad \textit{whenever} \quad X^{8\eta} \ll N \ll X^{1/2}.$$

Proof. Let $Q = \eta N$. Arguing as in [2, proof of Theorem 5],

$$S_{\mathrm{II}}(x)^2 \ll \frac{X^2}{Q} + \frac{X}{Q}\sum_{q=1}^{Q}\sum_{n\sim N}\Bigl|\sum_{\substack{m\sim M\\ X<mn\le 2X}} e\bigl(xm^c((n+q)^c - n^c)\bigr)\Bigr|. \tag{3.1}$$

We apply the exponent pair $(1/6, 2/3)$ (see [7]) to the sum over m in (3.1):

$$\sum_{\substack{m\sim M\\ X<mn\le 2X}} e\bigl(xm^c((n+q)^c - n^c)\bigr) \ll (xqN^{-1}X^c)^{1/6}M^{1/2} + \frac{M}{xqN^{-1}X^c}. \tag{3.2}$$

Inserting this into (3.1) shows that the first term on the right in (3.2) produces $\ll X^{2-8\eta}$ since $X^{c/6} \ll X^{2/5}$ and $M \gg X^{1/2}$. The second term on the right in (3.2) produces $\ll X^{2-6\eta}$ since $xX^c > X^{8\eta}$. ∎

We need another four lemmas, the first two due to Bombieri and Iwaniec (see e.g. [7, Lemmas 7.3, 7.5]) and the others respectively to Fouvry and Iwaniec [6, Lemma 2] and Sargos and Wu [15, Theorem 3].

LEMMA 7. *Let* $1 \le M \le N < N_1 \le M_1$. *Let*

$$K(t) = \min\{M_1 - M + 1, (\pi|t|)^{-1}, (\pi|t|)^{-2}\}.$$

Then

$$\Big|\sum_{N<n\le N_1} a_n\Big| \le \int_{-\infty}^{\infty} K(t)\Big|\sum_{M<m\le M_1} a_m e(mt)\Big|\,\mathrm{d}t.$$

Moreover,

$$\int_{-\infty}^{\infty} K(t)\,\mathrm{d}t \ll \log(M_1 - M + 2).$$

LEMMA 8. *Let* $\{x_r\}_{r\sim R}, \{y_s\}_{s\sim S}$ *be two sequences in* $[-1,1]$, *and let* $\varphi_r, \psi_s \in \mathbb{C}$. *Let* $T > 0$,

$$\mathcal{B}_{\varphi,\psi} = \sum_{r\sim R}\sum_{s\sim S} \varphi_r\psi_s\, e(Tx_r y_s),$$

$$\mathcal{B}_{\varphi}(1/T) = \sum\sum_{|x_{r'}-x_{r''}|\le 1/T} |\varphi_{r'}\varphi_{r''}|,$$

$$\mathcal{B}_{\psi}(1/T) = \sum\sum_{|y_{s'}-y_{s''}|\le 1/T} |\psi_{s'}\psi_{s''}|.$$

Then

$$|\mathcal{B}_{\varphi,\psi}|^2 \le 20(1+T)\,\mathcal{B}_{\varphi}(1/T)\mathcal{B}_{\psi}(1/T).$$

LEMMA 9. *Let* $N, Q \ge 1$ *and* $z_n \in \mathbb{C}$. *Then*

$$\Big|\sum_{N<n\le 2N} z_n\Big|^2 \le (2 + N/Q)\sum_{|q|<Q}(1 - |q|/Q)\sum_{N<n-q,n+q\le 2N} z_{n+q}\overline{z}_{n-q}.$$

LEMMA 10. *Let* $1 \le Q \le M^{1-\eta} \le X$, $\varDelta > 0$, $\delta > 0$, $\alpha \in \mathbb{R}$, $\alpha \ne 0, 1, 2$,

$$t(m,q) = (m+q)^\alpha - (m-q)^\alpha.$$

Let $\mathcal{E}(M, Q, \varDelta, \delta)$ *denote the number of quadruples* $(m, \tilde{m}, q, \tilde{q})$ *with* $m, \tilde{m} \sim M$ *and* $Q \le q, \tilde{q} \le (1+\delta)Q$ *satisfying*

$$|t(m,q) - t(\tilde{m},\tilde{q})| \le \varDelta M^{\alpha-1}Q.$$

Then there is a $\delta \in [1/Q, 1]$ *such that*

$$\delta^{-1}\sum_{0\le k\le K} \mathcal{E}(M, Q_k, \varDelta, \delta) \ll \mathcal{L}^4(MQ + \varDelta(MQ)^2 + (MQ^9)^{1/4})$$

where $Q_k = (1+\delta)^k Q$, $K = [(\log 2)/\delta]$.

We are now ready to prove the following result, which is essentially [15, Theorem 9].

THEOREM 3. *Let*

$$S=\sum_{\substack{m\sim M\\ X<mn\le 2X}}\sum_{n\sim N} a_m b_n e(Bm^{\beta}n^{\alpha})$$

where $M\ge 1$, $N\ge 1$, $\alpha(\alpha-1)(\alpha-2)\beta(\beta-1)(\beta-2)\neq 0$, $|a_m|\le 1$, $|b_n|\le 1$. *Suppose that*

$$F:=BM^{\beta}N^{\alpha}\gg MN.$$

Then

$$\begin{aligned}SX^{-\eta}\ll{}& F^{1/20}N^{19/20}M^{29/40}+F^{3/46}N^{43/46}M^{16/23}+F^{1/10}N^{9/10}M^{3/5}\\&+F^{3/28}N^{23/28}M^{41/56}+F^{1/11}N^{53/66}M^{17/22}+F^{2/21}N^{31/42}M^{17/21}\\&+F^{1/5}N^{7/10}M^{3/5}+N^{1/2}M+F^{1/8}N^{3/4}M^{3/4}.\end{aligned}$$

Proof. Obviously we may suppose that $F\le (MN)^2$.

Let $1\le Q\le N^{1-\eta}$. It follows from Lemma 9 together with Cauchy's inequality that

$$S^2\ll\frac{M^2N^2}{Q}+\frac{MN}{Q}\sum_{q\le Q}\sum_{n\sim N}\Big|\sum_{\substack{m\sim M\\ X<mn\le 2X}} e(Bm^{\beta}t(n,q))\Big|.$$

We apply Lemma 2 to the sum over m. After a simple splitting-up argument and a partial summation, we obtain

$$\begin{aligned}\frac{S^2}{\mathcal{L}}\ll{}&\frac{M^2N^2}{Q}\\&+\frac{MN}{Q}\sum_{q\sim Q_1}\sum_{n\sim N}(FqN^{-1}M^{-2})^{-1/2}\Big|\sum_{n_1\in I(n,q)} e\big(C_1(B\,t(n,q))^{\frac{1}{1-\beta}}n_1^{\beta_1}\big)\Big|+E_1\end{aligned}$$

where $\beta_1=\frac{\beta}{\beta-1}$ and $I(n,q)$ is a subinterval of $[N_1,2N_1]$, with $N_1\asymp FQ_1/(MN)$, and

$$E_1=N^2M((FQN^{-1}M^{-2})^{-1/2}+1).$$

Using Lemma 7, we replace the condition $n_1\in I(n,q)$ by $n_1\sim N_1$ at the cost of a factor $\mathcal{L}$. Then we apply Lemma 9 again. Write

$$t_1(n_1,r)=(n_1+r)^{\beta_1}-(n_1-r)^{\beta_1}.$$

We find that for any R, $1\le R\le N_1^{1-\eta}$, there is an R_1, $1\le R_1\le R$, such that

$$\frac{S^4}{\mathcal{L}^4} \ll \frac{M^4N^4}{Q^2} + \frac{N^5M^4}{FQ} + N^4M^2 + \frac{M^4N^4}{FQ^2}\sum_{n\sim N}\sum_{q\sim Q_1}\left\{\frac{N_1^2}{R} + \frac{N_1}{R}\sum_{r\sim R_1}\left(1-\frac{|r|}{R}\right) \times \sum_{n_1\in I(r)} e\big(C_1(B\,t(n,q))^{\frac{1}{1-\beta}}t_1(n_1,r)\big)\right\}. \tag{3.3}$$

Here $I(r)$ is a subinterval of $(N_1, 2N_1]$ depending on r. Let U denote the quadruple exponential sum over n, q, n_1, r on the right side of (3.3). We split up the range of q, r into $(K_1+1)(K_2+1)$ parts as in Lemma 10, so that $\delta_1 = \delta(M, Q_1)$, $K_1 = [(\log 2)/\delta_1]$, $\delta_2 = \delta(N_1, R_1)$, $K_2 = [(\log 2)/\delta_2]$ and

$$U = \sum_{k_1=0}^{K_1}\sum_{k_2=0}^{K_2} U(k_1,k_2), \qquad U^2 \ll (\delta_1\delta_2)^{-1}\sum_{k_1=0}^{K_1}\sum_{k_2=0}^{K_2}|U(k_1,k_2)|^2.$$

Applying Lemma 8 to each subsum $U(k_1,k_2)$ we deduce

$$U^2 \ll (\delta_1\delta_2)^{-1}F_1\sum_{k_1=0}^{K_1}\sum_{k_2=0}^{K_2}\mathcal{E}\left(M, Q(k_1), \frac{1}{F_1}, \delta_1\right)\mathcal{E}\left(N_1, R(k_2), \frac{1}{F_1}, \delta_2\right)$$

where $F_1 \asymp FQ_1R_1N^{-1}N_1^{-1} \asymp MR_1$, $Q(k_1) = (1+\delta_1)^{k_1}Q_1$ and $R(k_2) = (1+\delta_2)^{k_2}R_1$. The bounds arising from Lemma 10 show that

$$\frac{U^2}{\mathcal{L}^8} \ll MR_1\left\{NQ_1 + \frac{(NQ_1)^2}{MR_1} + N^{1/4}Q_1^{9/4}\right\} \times\left\{\frac{FQ_1R_1}{MN} + \frac{F^2Q_1^2R_1}{M^3N^2} + \frac{F^{1/4}Q_1^{1/4}R_1^{9/4}}{M^{1/4}N^{1/4}}\right\}.$$

We insert this bound into (3.3) and multiply out. Noting that all powers of Q_1 and R_1 obtained are positive, we may replace Q_1, R_1 by Q, R in all but three terms:

$$\begin{aligned}\frac{S^8}{\mathcal{L}^{16}} &\ll \frac{N^8M^8}{Q^4} + N^8M^4 + \frac{N^{10}M^8}{F^2Q^2} + \frac{N^6M^4Q_1^6F^2}{Q^4R^2}\\ &\quad + N^{27/4}M^{27/4}Q^{-3/4}R^{5/4}F^{1/4} + N^{31/4}M^{23/4}Q^{1/4}R^{1/4}F^{1/4}\\ &\quad + N^{21/4}M^6Q^{5/4}F + N^{17/4}M^4Q^{9/4}F^2 + \frac{N^7M^5FQ_1^5}{Q^4R}\\ &\quad + N^6M^{27/4}Q^{1/2}R^{5/4}F^{1/4} + N^6M^6F + \frac{N^6M^3F^2Q_1^6}{Q^4R} + N^5M^4QF^2\\ &= T_1 + \cdots + T_{13},\end{aligned}$$

say. From the condition $R \le \frac{Q_1F}{MN}$ we have $N \le \frac{Q_1F}{MR}$. Thus

$$T_2 = N^8M^4 \le N^6\left(\frac{Q_1F}{MR}\right)^2 M^4 \le N^6M^3Q_1^2R^{-1}F^2 = T'_{12},$$

say;

$$T_3 = N^{10}M^8Q^{-2}F^{-2} \le N^6\left(\frac{Q_1F}{MR}\right)^4 M^8Q^{-2}F^{-2} \le \frac{N^6M^4Q_1^4F^2}{Q^2R^2} = T'_4,$$

say. Finally,

$$T_9 = \frac{N^7M^5FQ_1^5}{Q^4R} \le N^6\left(\frac{Q_1F}{MR}\right)\frac{M^5Q_1^5F}{Q^4R} = \frac{N^6M^4Q_1^6F^2}{Q^4R^2} \le T'_4.$$

Hence

$$\begin{aligned}\frac{S^8}{\mathcal{L}^{16}} \ll N^6M^6F &+ N^8M^8Q^{-4} + N^{21/4}M^6Q^{5/4}F + N^{17/4}M^4Q^{9/4}F^2 \\ &+ N^5M^4QF^2 + \frac{N^6M^4Q_1^4F^2}{Q^2R^2} + N^6M^3Q_1^2R^{-1}F^2 \\ &+ N^{31/4}M^{23/4}Q^{1/4}R^{1/4}F^{1/4} + N^{27/4}M^{27/4}Q^{-3/4}R^{5/4}F^{1/4} \\ &+ N^6M^{27/4}Q^{1/2}R^{5/4}F^{1/4}.\end{aligned}$$

If we recall the first appearance of R in (3.3), this estimate holds trivially for $0 < R < 1$. Optimizing via Lemma 1 over $0 < R \le \left(\frac{Q_1F}{MN}\right)^{1-\eta}$, we obtain

$$\begin{aligned}\frac{S^8}{(MN)^{4\eta}} \ll{}& N^6M^6F + N^8M^8Q^{-4} + N^{21/4}M^6Q^{5/4}F \\ &+ N^{17/4}M^4Q^{9/4}F^2 + N^5M^4QF^2 + N^{68/9}M^{50/9}Q^{4/9}F^{4/9} \\ &+ N^{37/5}M^{26/5}Q^{3/5}F^{3/5} + N^{84/13}M^{74/13}Q^{4/13}F^{12/13} \\ &+ N^{19/3}M^{14/3}Q^{7/9}F^{11/9} + N^6M^{14/3}Q^{4/3}F^{11/9} \\ &+ N^6M^{74/13}Q^{14/13}F^{12/13} + N^8M^6 + N^7M^4QF \\ ={}& V_1 + \cdots + V_{13},\end{aligned} \tag{3.4}$$

say. We can discard V_9 and V_{10}, because

$$V_9 = N^{19/3}M^{14/3}Q^{7/9}F^{11/9} = V_5^{4/9}V_7^{5/9},$$
$$V_{10} = N^6M^{14/3}Q^{4/3}F^{11/9} = V_4^{4/9}V_7^{5/9}.$$

Since (3.4) is trivial for $Q < 1$, we can optimize the remaining expression on the right side of (3.4) over $0 < Q < N^{1-\eta}$ to obtain

$$\begin{aligned}\frac{S^8}{(MN)^{8\eta}} &\ll N^{38/5}M^{29/5}F^{2/5} + N^{172/23}M^{128/23}F^{12/23}\\ &\quad + N^{36/5}M^{24/5}F^{4/5} + N^{46/7}M^{41/7}F^{6/7}\\ &\quad + N^{212/33}M^{68/11}F^{8/11} + N^{124/21}M^{136/21}F^{16/21}\\ &\quad + N^{28/5}M^{24/5}F^{8/5} + N^{28/5}M^{136/25}F^{32/25}\\ &\quad + N^4M^8 + N^8M^6 + N^6M^6F\\ &= U_1 + \cdots + U_{11},\end{aligned} \tag{3.5}$$

say. Since $F \ge MN$,

$$U_{10} = N^8M^6 \le N^8M^6\left(\frac{F}{MN}\right)^{2/5} \le N^{38/5}M^{29/5}F^{2/5} = U_1.$$

Also

$$U_8 = N^{28/5}M^{136/25}F^{32/25} = U_6^{21/75}U_7^{50/75}U_9^{4/75}.$$

Removing these two terms from (3.5), we obtain the theorem. ∎

COROLLARY 2. *Let $S_{\mathrm{II}}(x)$ be as in Lemma 6. Suppose now that $1 < c < 10/9$,*

$$X^{1-c} < x < X^{\eta}.$$

Then

$$S_{\mathrm{II}}(x) \ll x^{-1}X^{1-c-4\eta} \tag{3.6}$$

whenever

$$X^{10/27} \ll N \ll X^{19/45}. \tag{3.7}$$

Proof. We apply Theorem 3 with

$$F \asymp xX^c \gg MN.$$

The term $F^{3/28}N^{23/28}M^{41/56}$ gives rise to the condition $N \ll X^{19/45}$; the term $F^{2/21}N^{31/42}M^{17/21}$ gives rise to the condition $N \gg X^{10/27}$; and the term $F^{1/8}N^{3/4}M^{3/4}$ gives rise to the condition $c < 10/9$. The other terms are easily dealt with, and the corollary follows. ∎

By using two results of Huxley [9], we can alter the endgame in the proof of Theorem 3 to obtain (3.6) for a different range of N.

LEMMA 11. *Let $G(w)$ be four times continuously differentiable on $[1,2]$. Suppose that*

$$G^{(r)}(w) \asymp 1 \tag{3.8}$$

for $r = 2, 3, 4$ *and*

$$|G^{(2)}(w)\, G^{(4)}(w) - 3\, G^{(3)}(w)| \gg 1. \tag{3.9}$$

Let

$$S = \sum_{m=M}^{M_1} e(TG(m/M)),$$

where $1 \le M \le M_1 \le 2M$ *and*

$$T^{141/328+\eta} \le M \le T^{187/328-\eta}. \tag{3.10}$$

Then

$$S \ll M^{1/2} T^{32/205+\eta}.$$

Proof. This is a consequence of [9, Theorem 1]. ∎

Note that (3.8), (3.9) are satisfied if

$$(G(w))^{(j+2)} = (w^\beta)^{(j)}(1 + O(\eta)) \quad (1 \le j \le 2),$$

for a real β with $\beta(\beta + 1/2) \ne 0$.

LEMMA 12. *Let* $G(w, y)$ *be a function on* $\mathcal{R} = [1, 2] \times [0, 1]$ *having partial derivatives* $G^{(i,j)}$ $(i, j \le 5)$. *Suppose that on* $\mathcal{R}$,

$$G^{(r,0)}(w, y) \asymp 1 \tag{3.11}$$

for $r = 2, 3, 4$ *and*

$$|G^{(r+1,0)}(w, y)\, G^{(r+1,1)}(w, y) - G^{(r,1)}(w, y)\, G^{(r+2,0)}(w, y)| \asymp 1 \tag{3.12}$$

for $r = 2, 3$. *Let* $y_1, \ldots, y_J$ *satisfy*

$$0 \le y_1 < \cdots < y_J \le 1, \qquad y_{j+1} - y_j \gg J^{-1}. \tag{3.13}$$

Let

$$S(y) = \sum_{m=M_1(y)}^{M_2(y)} e(TG(m/M, y)),$$

where $1 \le M \le M_1(y) \le M_2(y) \le 2M$ *and*

$$T^{1/3+\eta} \le M \le T^{1/2}. \tag{3.14}$$

Then

$$\begin{aligned} \sum_{j=1}^{J} |S(y_j)|^5 \tag{3.15} \\ \ll M^\eta (J^{43/69} M^{449/138} T^{63/138} + J M^{59/34} T^{37/54} + J M^{5/2} T^{141/190}). \end{aligned}$$

Proof. This is a consequence of [9, Theorem 2]. ∎

Let $G(w, y) = g_1(w) g_2(y)$ where

$$(g_1(w))^{(j+2)} = (w^\beta)^{(j)}(1 + O(\eta)), \qquad (g_2(y))^{(j)} = ((y+1)^\gamma)^{(j)}(1 + O(\eta))$$

for $j \le 5$. It may be verified that (3.11), (3.12) hold provided that we have $\gamma\beta(\beta-1)(\beta-2) \ne 0$.

Lemma 13. *The conclusion of Corollary* 2 *remains valid if* (3.7) *is replaced by*

$$X^{2/9} \ll N \ll X^{127/470}.$$

Proof. We begin with (3.3), where now $\alpha = \beta = c$, $B = x$, $F \asymp xX^c$. We choose $Q = X^{2c-2+9\eta}$ and $R = Q_1^3 X^{c-2} N x^5$. It is easy to check that $Q \le N^{1-\eta}$ since $N \gg X^{2/9}$. With $N_1 \asymp FQ_1/(MN)$, we may verify that $R \le N_1^{1-\eta}$. Because of the choice of Q and R, the terms M^4N^4/Q^2 and $M^4N^5N_1^2Q_1/(FQ^2R)$ on the right side of (3.3) are acceptable. The terms $N^5M^4/(FQ)$ and N^4M^2 are also acceptable:

$$\frac{N^5M^4}{FQ} \ll \frac{N^5M^4}{xX^cQ} \ll \frac{N^4M^4}{Q^2x^2} \quad \text{since } QN \ll N^2 \ll X^{c-3\eta},$$

$$N^4M^2 \ll \frac{N^4M^4}{Q^2x^2} \quad \text{since } Q \ll N \ll MX^{-1/3}.$$

We now choose $q \sim Q_1$ and $r \sim R_1$ so that the remaining term in (3.3) is bounded by

$$\ll \frac{M^4N^4N_1qr}{FQ^2R} \sum_{n\sim N} \Big| \sum_{n_1 \in I(r)} e\big(C_1(B\,t(n,q))^{\frac{1}{1-\beta}} t_1(n_1,r)\big)\Big|$$

$$= \frac{M^4N^4N_1qr}{FQ^2R} \sum_{n\sim N} |V_n|,$$

say. Thus we need to show that

$$\sum_{n\sim N} |V_n| \ll \frac{FR}{qrN_1x^4}. \tag{3.16}$$

We shall show that one of Lemmas 11, 12 is applicable to $\sum_{n\sim N}|V_n|$ (with n_1, N_1 in the roles of m, M). Let us write

$$T = C_2\frac{Fq}{N}\cdot\frac{r}{N_1} \asymp \frac{Xr}{N}.$$

We show to begin with that

$$N_1 \ll T^{187/328}X^{-\eta},$$

that is,

$$X^{328c-328}(qx)^{328} \ll X^{187-C\eta}r^{187}N^{-187}.$$

It suffices to show that

$$N^{187} < X^{187-984(c-1)-C\eta} = X^{1171-984c-C\eta}.$$

For this, $N < X^{3/10}$ suffices.

We show next that

$$N_1 \gg T^{1/3} X^{\eta},$$

that is,

$$(X^{c-1} qx)^3 \gg X^{1+\eta} r/N. \tag{3.17}$$

The right side of (3.17) cannot exceed $q^3 X^{c-1+2\eta} x^5$, and (3.17) follows at once.

We now divide the argument into two cases.

CASE 1: $N_1 > T^{141/328} X^{\eta}$. We shall obtain (3.16) by showing for each $n \sim N$ that

$$V_n \ll \frac{FR}{qrN_1 x^4 N}.$$

It is clear that Lemma 11 is applicable, since the exponent in the approximating monomial for $f(n_1) := C_1(B\,t(n,q))^{1/(1-\beta)} t_1(n_1, r)$ is

$$\frac{c}{c-1} - 1 = \frac{1}{c-1} > 9,$$

and

$$f^{(j)} \asymp T N_1^{-j}.$$

Thus it remains to verify that

$$(X^{c-1} qx)^{1/2} \left(\frac{Xr}{N}\right)^{32/205} X^{\eta} \ll \frac{FR}{qrN_1 x^4 N} \asymp \frac{xX^{c-1} q}{r}.$$

We require

$$r^{237/205} \ll q^{1/2} x^{1/2} X^{(c-1)/2 - 32/205 - \eta} N^{32/205}.$$

We recall that

$$X^{328(c-1)} (qx)^{328} \asymp N_1^{328} > T^{141} X^{\eta} > X^{141} r^{141} N^{-141}$$

or

$$r^{141} \ll X^{328c-469} N^{141} (qx)^{328}.$$

Hence it suffices to show that

$$(X^{328c-469} N^{141} q^{328} x^{328})^{237/(205\cdot 141)} \ll q^{1/2} x^{1/2} X^{(c-1)/2 - 32/205 - \eta} N^{32/205}.$$

In verifying this, the worst case is $qx = X^{2c-2+12\eta}$. After a short calculation, the condition on N reduces to

$$N^{57810} \ll X^{437511 - 379701c - C\eta},$$

which is a consequence of $c < 10/9$ and $N \ll X^{127/470}$.

CASE 2: $N_1 \le T^{141/328}X^\eta$. We apply Lemma 12 with

$$G(w,y) = \frac{NN_1}{qr}\left\{\left(w+\frac{r}{N_1}\right)^{\frac{c}{c-1}} - w^{\frac{c}{c-1}}\right\}\left\{\left(y+1+\frac{q}{N}\right)^{c} - (y+1)^c\right\}^{\frac{-1}{c-1}}$$
$$(1 \le w \le 2,\ 0 \le y \le 1),$$

taking (N, N_1) in the role of (J, M) and $y_{n-N} = (n-N)/N$ $(N < n \le 2N)$. If C_2 is suitably chosen, then

$$TG(n_1/N_1, y_{n-N}) = C_1(B\,t(n,q))^{\frac{-1}{c-1}}t_1(n_1,r).$$

Lemma 12 gives the estimate

$$\sum_{n\sim N}|V_n|^5 \ll X^\eta(N^{43/69}N_1^{449/138}T^{63/138} + NN_1^{59/34}T^{37/34} + NN_1^{5/2}T^{141/190}).$$

By Hölder's inequality

$$(3.18)\qquad \sum_{n\sim N}|V_n| \le N^{4/5}\Big(\sum_{n\sim N}|V_n|^5\Big)^{1/5}$$
$$\ll X^\eta(N^{319/345}N_1^{449/690}T^{63/690} + NN_1^{59/170}T^{37/170} + NN_1^{1/2}T^{141/950}).$$

There is in fact something to spare in bounding the right side of (3.18) by the expression on the right of (3.16). The worst case is $x = X^{3\eta}$, $q \asymp X^{2c-2+9\eta}$, $r \asymp X^{7c-8+C\eta}N$, so that $N_1 \asymp X^{3c-3+C\eta}$, $T \asymp X^{7c-7+C\eta}$. We require

$$N^{319/345}(X^{3c-3})^{449/690}(X^{7c-7})^{63/690} \ll X^{5-4c-C\eta},$$
$$N(X^{3c-3})^{59/170}(X^{7c-7})^{37/170} \ll X^{5-4c-C\eta},$$
$$N(X^{3c-3})^{1/2}(X^{7c-7})^{141/950} \ll X^{5-4c-C\eta}.$$

Each of these bounds follows from $N \ll X^{127/470}$, $c < 10/9$. This concludes the proof in Case 2, and the proof of Lemma 13 is complete. ∎

4. The alternative sieve. We require a variant of Theorem 3.1 of Harman [8]. The details are intricate and deserve a full discussion.

LEMMA 14. *Let $w(n)$ be a complex function with support in $(X, 2X] \cap \mathbb{Z}$, $|w(n)| \le X^{1/\eta}$ $(n \sim X)$. For $r \in \mathbb{N}$, $z \ge 2$, let $P(z) = \prod_{p<z} p$ and*

$$S(r,z) = \sum_{(n,P(z))=1} w(rn).$$

Suppose that, for some $\alpha > 0$, $\beta \le 1/2$, $M \ge 1$, $Y > 0$, we have (for any coefficients a_m, $|a_m| \le 1$, and b_n, $|b_n| \le \tau(n)$, the number of positive *divisors of* n)

$$\sum_{m\le M} a_m \sum_n w(mn) \ll Y, \tag{4.1}$$

$$\sum_{X^\alpha\le m\le X^{\alpha+\beta}} a_m \sum_n b_n w(mn) \ll Y. \tag{4.2}$$

Let u_r $(r \le R)$, v_s $(s \le S)$ *be complex numbers with* $|u_r| \le 1$, $|v_s| \le 1$, $u_r = 0$ *for* $(r, P(X^\eta)) > 1$, $v_s = 0$ *for* $(s, P(X^\eta)) > 1$,

$$R < X^\alpha, \qquad S < MX^{-\alpha}. \tag{4.3}$$

Then

$$\sum_{r\le R}\sum_{s\le S} u_r v_s S(rs, X^\beta) \ll Y\mathcal{L}^3.$$

Proof. We write $z = X^\beta$ and define

$$\psi(m) = \sum_n w(mn).$$

We have

$$\begin{aligned} S(rs,z) &= \sum_n \Big(\sum_{\substack{d\mid P(z)\\ d\mid n}} \mu(d)\Big) w(rsn) \\ &= \sum_{d\mid P(z)} \mu(d)\psi(rsd) = \sum\nolimits_1(r,s) + \sum\nolimits_2(r,s), \end{aligned}$$

where

$$\sum\nolimits_1(r,s) = \sum_{\substack{d\mid P(z)\\ rsd\le M}} \mu(d)\,\psi(rsd), \qquad \sum\nolimits_2(r,s) = \sum_{\substack{d\mid P(z)\\ rsd> M}} \mu(d)\,\psi(rsd).$$

Now

$$\begin{aligned} \sum_{r\le R} u_r \sum_{s\le S} v_s \sum\nolimits_1(r,s) &= \sum_{r\le R}\sum_{s\le S} u_r v_s \sum_{\substack{d\mid P(z)\\ rsd\le M}} \mu(d)\,\psi(rsd) \\ &= \sum_{m\le M} a_m \sum_n w(mn), \end{aligned}$$

where

$$a_m = \sum_{d\mid P(z)} \sum_{\substack{r\le R, s\le S\\ rsd=m}} u_r v_s \mu(d).$$

Because m has at most η^{-1} prime factors $\ge X^\eta$, we get $|a_m| \le (2^{1/\eta})^2$. Thus

$$\sum_{r\le R}\sum_{s\le S} u_r v_s \sum\nolimits_1(r,s) \ll Y,$$

and it remains to show that

$$\sum_{r\le R}\sum_{s\le S} u_r v_s {\sum}_2(r,s) \ll Y\mathcal{L}^3. \tag{4.4}$$

We make repeated use of the identity

$$\sum_{d\mid P(z)} \mu(d)g(d) = g(1) - \sum_{p<z}\sum_{d\mid P(p)} \mu(d)g(dp) \tag{4.5}$$

(see [8, (3.1.2)]). Fix $r \le R$, $s \le S$ and take

$$g(d) = \begin{cases} \psi(drs) & \text{if } drs > M,\\ 0 & \text{otherwise.}\end{cases}$$

Then $g(1) = 0$ from (4.3). Hence

$${\sum}_2(r,s) = -\sum_{\substack{p<z\\ pdrs>M}}\ \sum_{d\mid P(p)} \mu(d)\psi(pdrs) = -\Big({\sum}_3(r,s) + {\sum}_4(r,s)\Big),$$

where $pr < X^\alpha$ in $\sum_3(r,s)$ and $pr \ge X^\alpha$ in $\sum_4(r,s)$.

We repeat this splitting procedure for $\sum_3(r,s)$. Let us give the general form of the recursive step. For $t \ge 1$, let $\pi_t = p_1\cdots p_t$ and

$${\sum}_3(r,s,t) = \sum_{\substack{p_t<\cdots<p_1<z\\ \pi_t drs>M\\ \pi_t r<X^\alpha}}\ \sum_{d\mid P(p_t)} \mu(d)\psi(drs\pi_t)$$

so that $\sum_3(r,s,1) = \sum_3(r,s)$. We apply (4.5) for given r, s, $p_1,\ldots,p_t$, with

$$g(d) = \begin{cases} \psi(drs\pi_t) & \text{if } drs\pi_t > M,\\ 0 & \text{otherwise.}\end{cases}$$

For $r \le R$, $s \le S$, $\pi_t r < X^\alpha$, we have

$$(r\pi_t)s < X^\alpha(MX^{-\alpha}) = M.$$

Hence $g(1) = 0$,

$$\begin{aligned}{\sum}_3(r,s,t) &= -\sum_{\substack{p_t<\cdots<p_1<z\\ \pi_t drs>M\\ \pi_t r<X^\alpha}}\ \sum_{\substack{p_{t+1}<p_t\\ drs\pi_t p_{t+1}>M}}\ \sum_{d\mid P(p_{t+1})} \mu(d)\psi(drs\pi_t p_{t+1})\\ &= -\Big({\sum}_3(r,s,t+1) + {\sum}_4(r,s,t+1)\Big),\end{aligned}$$

where $\pi_{t+1}r < X^\alpha$ in $\sum_3(r,s,t+1)$ (in accordance with our notation for $\sum_3(r,s,\ldots)$) and $\pi_t r < X^\alpha$, $\pi_{t+1}r \ge X^\alpha$ in $\sum_4(r,s,t+1)$.

We shall show that

$$\sum_{r\le R}\sum_{s\le S} u_r v_s {\sum}_4(r,s) \ll Y\mathcal{L}^2, \tag{4.6}$$

$$\sum_{r\le R}\sum_{s\le S} u_r v_s {\sum}_4(r,s,t+1) \ll Y\mathcal{L}^2 \quad (t\ge 1). \tag{4.7}$$

Since $\sum_3(r,s,t)$ is clearly empty for $t > C_1\mathcal{L}/\log\mathcal{L}$, (4.4) follows from (4.6) and (4.7).

The key to proving (4.7) is that

$$r\pi_t p_{t+1} \ge X^\alpha, \qquad r\pi_t p_{t+1} < X^\alpha p_{t+1} < X^{\alpha+\beta}$$

in the sum. But we need a little more work before we use (4.2), because the groups of variables $r\pi_t p_{t+1}$, ds are 'linked' by the condition $d\,|\,P(p_{t+1})$.

Let $\sigma(u)$ be the indicator function of (M,∞). By (4.5),

$$\begin{aligned}\sum_{r\le R}\sum_{s\le S} u_r v_s {\sum}_4(r,s,t+1) &= \sum_{r\le R}\sum_{s\le S} u_r v_s \sum_{\substack{p_{t+1}<\cdots<p_1<z\\ \pi_t r<X^\alpha,\, \pi_{t+1}r\ge X^\alpha}} \Big(\sigma(rs\pi_{t+1})\psi(rs\pi_{t+1})\\ &\qquad - \sum_{d|P(p_{t+2})}\sum_{\substack{p_{t+2}<p_{t+1}\\ drs\pi_{t+1}p_{t+2}>M}} \mu(d)\psi(drs\pi_{t+1}p_{t+2})\Big).\end{aligned} \tag{4.8}$$

We rewrite the subtracted part as

$$-\sum_{r\le R}\sum_{s\le S} u_r v_s \sum_{\substack{p_{t+2}<p_{t+1}<\cdots<p_1<z\\ \pi_t r<X^\alpha,\, \pi_{t+1}r\ge X^\alpha}} \sum_{\substack{d|P(p_{t+2})\\ drs\pi_{t+1}p_{t+2}>M}} \mu(d)\sum_k w(rsd\pi_{t+1}p_{t+2}k).$$

Grouping the variables as $m = p_1\cdots p_{t+1}r$, $n = p_{t+2}sdk$, there are just two joint conditions of summation $p_{t+2} < p_{t+1}$, $drs\pi_{t+1}p_{t+2} > M$. These can be removed at a cost of a factor $\mathcal{L}^2$; see [8, Section 3.2] for the discussion of this standard 'cosmetic surgery', which we shall use again later in Section 4. Moreover, for given m, the coefficient of m is $\ll 1$ because $p_1,\dots,p_t$ are determined by r. The coefficient of n is $\ll \tau(n)$ because in the equation

$$n = p_{t+2}sdk,$$

once k is specified, there are $O(1)$ possibilities for s, and p_{t+2} is the largest prime factor of the remaining factor $p_{t+2}d$. We conclude that the subtracted portion in (4.8) is

$$\ll Y\mathcal{L}^2.$$

The residual part of the right side of (4.8) can be bounded similarly. The treatment of the sum in (4.6) is similar but simpler. This establishes (4.6), (4.7), and the proof of Lemma 14 is complete. ∎

In the remainder of the paper, let $1 < c < 10/9$. Let R be a large positive number, $X = (R/4)^{1/c}$. As in [3], we employ a continuous function $\varphi : \mathbb{R} \to [0,1]$ such that

$$\varphi(y) = 0 \quad (|y| \ge R^{-\eta}), \qquad \varphi(y) = 1 \quad (|y| \le 4R^{-\eta}/5) \tag{4.9}$$

with Fourier transform $\Phi(x) = \int_{-\infty}^{\infty} e(-xy)\varphi(y)\,\mathrm{d}y$ satisfying

$$\int\limits_{|x|>X^{3\eta}} |\Phi(x)|\,\mathrm{d}x \ll X^{-3}. \tag{4.10}$$

We write briefly $\mathrm{d}\mu = e(-Rx)\Phi(x)\mathrm{d}x$ and define

$$\tau = X^{8\eta - c}.$$

We also write

$$T(x) = \sum_{p\sim X} e(p^c x), \qquad I_0(x) = \int\limits_X^{2X} e(t^c x)\,\mathrm{d}t, \qquad I(x) = \int\limits_X^{2X} \frac{e(t^c x)}{\log t}\,\mathrm{d}t.$$

We let $U(x)$ denote an arbitrary sum of the form $U(x) = \sum_{n\sim X} u_n e(n^c x)$ with real $u_n \ll 1$ $(n \sim X)$, and $U^+(x)$ denote a sum with the further property $u_n \ge 0$ $(n \sim X)$. It is convenient to write

$$a = \frac{2}{9}, \qquad b = \frac{127}{470}, \qquad d = \frac{10}{27}, \qquad f = \frac{19}{45}$$

and

$$g = f - d = \frac{7}{135}, \qquad h = \frac{1}{2} - d = \frac{7}{54}, \qquad l = 0.291954.$$

In writing our exponential sums containing variables $p_1, \ldots, p_j$, we set

$$\boldsymbol{\alpha}_j = (\alpha_1, \ldots, \alpha_j) = ((\log p_1)/\mathcal{L}, \ldots, (\log p_j)/\mathcal{L}),\ s_i = \alpha_1 + \cdots + \alpha_i \qquad (1 \le i \le j),$$

$$F(m) = \begin{cases} e(m^c x) & (m \sim X), \\ 0 & \text{otherwise}. \end{cases}$$

Let P_j be the region of $\mathbb{R}^j$ given by

$$P_j = \{(y_1, \ldots, y_j) : g \le y_j < y_{j-1} < \cdots < y_1,\ y_1 + \cdots + y_{j-1} + 2y_j \le 1 + (\log 3)/\mathcal{L}\}.$$

Let $G = [a,b] \cup [d,f] \cup [1-f, 1-d]$ and

$$G_j = \Big\{\mathbf{y}_j = (y_1, \ldots, y_j) \in P_j : \sum_{i\in\sigma} y_i \in G \text{ for some set } \sigma \subseteq \{1, \ldots, j\}\Big\},$$

$$B_j = P_j \setminus G_j.$$

For $\mathbf{y}_{j+1}=(y_1,\ldots,y_{j+1})\in P_{j+1}$, write $\mathbf{y}^*_{j+1}=(y_1,\ldots,y_j)$. For $E\subseteq P_j$, let

$$E'=\{\mathbf{y}_{j+1}\in P_{j+1}:\mathbf{y}^*_{j+1}\in E\}.$$

Let $\{1,\ldots,j\}$ have a partition into two (disjoint) sets σ_1,σ_2. We say that a point $\mathbf{y}_j\in P_j$ *splits* using σ_1,σ_2 if

$$\sum_{j\in\sigma_1}y_j<d,\qquad \sum_{j\in\sigma_2}y_j\le h.$$

Lemma 15. *Let $K(x)$ be either of the following:*

(i) *for Q_j a polytope (i.e. a finite intersection of half-spaces) with $Q_j\subseteq G_j$,*

$$K(x)=\sum_{\substack{(\alpha_1,\ldots,\alpha_j)\in Q_j\\ (n,P(p_j))=1}}F(\pi_j n);$$

(ii) *for some partition σ_1,σ_2 of $\{1,\ldots,j\}$,*

$$K(x)=\sum_{\substack{(\alpha_1,\ldots,\alpha_j)\in L_j\\ (n,P(X^g))=1}}F(\pi_j n)$$

where L_j is a polytope, $L_j\subseteq P_j$ and every point of L_j splits using σ_1,σ_2.

Then for any $U(x)$,

$$\int_\tau^\infty T(x)U(x)K(x)\,\mathrm{d}\mu\ll X^{3-c-2\eta}. \tag{4.11}$$

Proof. Recalling (4.10), it suffices to show that

$$\int_y^{y'} T(x)U(x)K(x)\,\mathrm{d}\mu\ll X^{3-c-2\eta}\mathcal{L}^{-1}$$

whenever $\tau\le y<X^{3\eta}$, $y<y'\le 2y$. Now

$$\begin{aligned}\int_y^{y'}|U(x)|^2\,\mathrm{d}x&=\int_y^{y'}\Big\{\sum_{n\sim X}u_n^2+2\sum_{X<n<n+j\le 2X}u_nu_{n+j}\,e\big((n^c-(n+j)^c)x\big)\Big\}\,\mathrm{d}x\\&\ll Xy+\sum_{n\sim X}\sum_{j\le X}\frac{1}{(n+j)^c-n^c}\\&\ll Xy+\sum_{n\sim X}\frac{1}{n^{c-1}}\sum_{j\le X}j^{-1}\ll Xy+X^{2-c}\mathcal{L}.\end{aligned}$$

The same bound applies to $\int_y^{y'} |TU| \,\mathrm{d}x \le \frac{1}{2}\int_y^{y'} (|T|^2 + |U|^2)\,\mathrm{d}x$. Since $\|\varPhi\|_\infty \le R^{-\eta}$ from (4.9),

$$\int_y^{y'} TUK\,\mathrm{d}\mu \ll X^{-c\eta}\int_y^{y'} |TUK|\,\mathrm{d}x \ll \sup_{[y,2y]} |K(x)|(X^{1-c\eta}y + X^{2-c-c\eta}\mathcal{L}), \tag{4.12}$$

and it suffices to show that

$$K(x) \ll \min(X^{1-\eta}, X^{2-c-\eta}x^{-1}). \tag{4.13}$$

In case (i), we rewrite the sum as

$$\sum_{k<20} \sum_{\substack{(\alpha_1,\ldots,\alpha_j)\in Q_j\\ p_j\le p_{j+1}\le\cdots\le p_k}} F(\pi_j p_{j+1}\cdots p_k).$$

We use cosmetic surgery to remove the conditions $p_s < p_t$, $p_s \le p_t$, and further conditions arising from $(\alpha_1,\ldots,\alpha_j) \in Q_j$, at the cost of a log power. Now we group the variables into products m_1, m_2 with $(\log m_1)/\mathcal{L} \in G$. The desired bound (4.13) follows from Lemma 6, Corollary 2 and Lemma 13.

In case (ii), we apply Lemma 14 with

$$w(n) = \begin{cases} \int_y^{y'} T(x)U(x)e(n^c x)\,\mathrm{d}\mu & \text{if } n \sim X,\\ 0 & \text{otherwise.}\end{cases}$$

Take $Y = X^{3-c-2\eta}$, $M = X^{1/2}$, $\alpha = d$, $\beta = g$. Then

$$\sum_{m\le M} a_m w(mn) = \int_y^{y'} T(x)U(x) \sum_{\substack{m\le X^{1/2}\\ mn\sim X}} a_m\, e((mn)^c x)\,\mathrm{d}\mu, \tag{4.14}$$

and

$$\sum_{X^\alpha\le m\le X^{\alpha+\beta}} a_m b_n w(mn) = \int_y^{y'} T(x)U(x) \sum_{\substack{X^\alpha\le m\le X^{\alpha+\beta}\\ mn\sim X}} a_m b_n\, e((mn)^c x)\,\mathrm{d}\mu. \tag{4.15}$$

The right-hand side in (4.14), (4.15) is seen to be $\ll Y$, by arguing as in (4.12), (4.13), using Corollary 1, Lemma 6 and Corollary 2. We conclude that

$$\sum_{r<X^d}\sum_{s\le X^h} u_r v_s S(rs, X^g) = \int_y^{y'} T(x)U(x) \sum_{r<X^d}\sum_{s\le X^h} u_r v_s \sum_{\substack{rsn\sim X\\ (n,P(X^g))=1}} e((rsn)^c x)\,\mathrm{d}\mu \ll Y\mathcal{L}^3.$$

We can bring $\int_y^{y'} TUK\,\mathrm{d}\mu$ into the form of the latter integral by writing $r=\prod_{i\in\sigma_1} p_i$, $s=\prod_{i\in\sigma_2} p_i$, and summing over $(\alpha_1,\ldots,\alpha_j)\in L_j$. We have to vary the proof of Lemma 14 to accomodate a bounded number of joint conditions of summation coming from $\boldsymbol{\alpha}_j\in L_j$, but the loss of a power of $\mathcal{L}$ via cosmetic surgery is harmless. This completes the proof of Lemma 15. ∎

Our proof of Theorem 1 requires two decompositions:

$$T(x)=K^{(1)}(x)-D^{(1)}(x)=K^{(2)}(x)-D^{(2)}(x)+D^{(3)}(x) \tag{4.16}$$

where $K^{(j)}$ is of the form $U(x)$, $D^{(j)}$ is of the form $U^{+}(x)$, and (for any U)

$$\int_{\tau}^{\infty} TUK^{(j)}\,\mathrm{d}\mu \ll X^{3-c-2\eta}. \tag{4.17}$$

(K is for 'keep', D for 'discard'!) We obtain the decompositions by using Buchstab's identity. We have

$$\begin{aligned} T(x) &= \sum_{\substack{n\sim X\\ (n,P((3X)^{1/2}))=1}} F(n) \\ &= \sum_{\substack{n\sim X\\ (n,P(X^g))=1}} F(n) - \sum_{\substack{X^g\le p_1<(3X)^{1/2}\\ (n,P(p_1))=1}} F(p_1 n). \end{aligned} \tag{4.18}$$

When we iterate the procedure, our general step has the following shape. Let E_j be a polytope, $E_j\subseteq P_j$, and let H_{j+1},E_{j+1} be a partition into polytopes of $B_{j+1}\cap E'_j$ ($j=1,\ldots,5$), with E_5 empty. We shall choose E_j so that every point of E_j splits using suitable sets of indices. Let

$$S_j=\sum_{\substack{\boldsymbol{\alpha}_j\in E_j\\ (n,P(p_j))=1}} F(\pi_j n),\qquad K_j=\sum_{\substack{\boldsymbol{\alpha}_j\in E_j\\ (n,P(X^g))=1}} F(\pi_j n), \tag{4.19}$$

$$K^*_{j+1}=\sum_{\boldsymbol{\alpha}_{j+1}\in E'_j\cap G_{j+1}} F(\pi_j n), \tag{4.20}$$

$$D_{j+1}=\sum_{\boldsymbol{\alpha}_{j+1}\in H_{j+1}} F(\pi_{j+1} n),\qquad S_{j+1}=\sum_{\substack{\boldsymbol{\alpha}_{j+1}\in E_{j+1}\\ (n,P(p_{j+1}))=1}} F(\pi_{j+1} n). \tag{4.21}$$

Since E'_j partitions into $E'_j\cap G_{j+1}$, H_{j+1}, E_{j+1}, we have

$$S_j=K_j-K^*_{j+1}-D_{j+1}-S_{j+1}. \tag{4.22}$$

Similarly, we partition the domain of α_1 in the subtracted part in (4.18) into G_1,H_1,E_1, where $H_1\cup E_1=B_1$, giving

$$T=K_0-K^*_1-D_1-S_1 \tag{4.23}$$

in a notation analogous to (4.19)–(4.21).

For a small part of H_5, we iterate twice more. Let

$$\widehat{H}_5 = \{\boldsymbol{\alpha}_5 \in H_5 : s_4 > b\}, \qquad L = H_5 \setminus \widehat{H}_5$$

so that

$$D_5(x) = \widehat{D}_5(x) + K_6(x) \tag{4.24}$$

where $\widehat{D}_5$ is a sum over $\widehat{H}_5$. The point is that each of the sums in

$$K_6(x) = \sum_{\substack{\boldsymbol{\alpha}_5 \in L \\ (n,P(X^g))=1}} F(\pi_5 n) - \sum_{\substack{\boldsymbol{\alpha}_6 \in L' \\ (n,P(X^g))=1}} F(\pi_6 n) + \sum_{\substack{\boldsymbol{\alpha}_7 \in (L')' \\ (n,P(p_7))=1}} F(\pi_7 n)$$

can be handled via Lemma 15. We have $s_5 < a$ in the first sum; $s_6 < 3a/2 < d$ in the second sum. In the third sum, $\boldsymbol{\alpha}_7$ could not be in B_7, since this would lead to

$$s_7 < 7a/4 < f, \quad \text{hence} \quad s_7 < d, \qquad s_5 < d - 2g < b,$$

and finally

$$5g \le s_5 < a,$$

which is absurd. So $K(x) = K_6(x)$ has the property (4.11).

We assemble (4.22)–(4.24) to get

$$T(x) = K(x) - D^-(x) + D^+(x)$$

where K is a sum of terms $\pm K_j, \pm K_j^*$,

$$D^+ = D_2 + D_4, \qquad D^- = D_1 + D_3 + \widehat{D}_5.$$

For the splitting property of E_j, it clearly suffices to have

$$\alpha_1 < d \quad (\alpha_1 \in E_1), \tag{4.25}$$

$$\alpha_1 + \alpha_2 < d \quad (\boldsymbol{\alpha}_2 \in E_2), \tag{4.26}$$

$$\alpha_3 \le h \quad (\boldsymbol{\alpha}_3 \in E_3), \tag{4.27}$$

$$\alpha_1 + \alpha_2 + \alpha_4 \le h \quad \text{or} \quad \alpha_3 + \alpha_4 \le h \qquad (\boldsymbol{\alpha}_4 \in E_4). \tag{4.28}$$

We are now ready to write down the decompositions (4.16). We first consider $K^{(2)} - D^{(2)} + D^{(3)}$. Here we let

$$\begin{aligned}
&E_1 = [g, a) \cup (b, l), \\
&E_2 = \{\boldsymbol{\alpha}_2 \in E_1' \cap B_2 : \alpha_1 + \alpha_2 < d\}, \\
&E_3 = \{\boldsymbol{\alpha}_3 \in E_2' \cap B_3 : \alpha_3 \le h\}, \\
&E_4 = \{\boldsymbol{\alpha}_4 \in E_3' \cap B_4 : \alpha_1 + \alpha_2 + \alpha_4 < d \ \textit{or} \ \alpha_3 + \alpha_4 \le h\},
\end{aligned}$$

which fulfils (4.25)–(4.28). Hence

$$\begin{aligned}
H_1 &= [l,d) \cup \left(f, \frac{1}{2} + \frac{\log 3}{2d}\right),\\
H_2 &= \{\boldsymbol{\alpha}_2 \in E_1' \cap B_2 : \alpha_1+\alpha_2 > f\},\\
H_3 &= \{\boldsymbol{\alpha}_3 \in E_2' \cap B_3 : \alpha_3 > h\},\\
H_4 &= \{\boldsymbol{\alpha}_4 \in E_3' \cap B_4 : \alpha_1+\alpha_2+\alpha_4 > f,\ \alpha_3+\alpha_4 > h\},\\
\widehat{H}_5 &= \{\boldsymbol{\alpha}_5 \in E_4' \cap B_5 : \alpha_1+\alpha_2+\alpha_3+\alpha_4 > b\}.
\end{aligned}$$

We shall prove several properties of $H_3, H_4, \widehat{H}_5$.

LEMMA 16. *Let* $\boldsymbol{\alpha}_j \in B_j$, $\sigma \subseteq \{1,\ldots,j\}$, $s = \sum_{t\in\sigma}\alpha_t$.

(i) *Let* $\sigma' = (\sigma \setminus \{i\}) \cup \{k\}$ *where* $i \in \sigma$, $k \notin \sigma$. *If* $|\alpha_i - \alpha_k| < 0.0479$, *then* s *and* $s' = \sum_{t\in\sigma'}\alpha_t$ *are either both to the left, or both to the right, of* $[a,b]$. *If* $|\alpha_i-\alpha_k| < g$, *then* s *and* s' *are either both to the left, or both to the right, of* $[d,f]$.

(ii) *Let* $i \notin \sigma$, $k \notin \sigma \cup \{i\}$. *If* $s+\alpha_i+\alpha_k < d$, *then* $s < a$. *If* $s > b$, *then* $s+\alpha_i+\alpha_k > f$.

(iii) *Let* $\sigma = \{i,i',i''\}$, $i < i' < i''$. *If* $s < d$, *then* $\alpha_{i'}+\alpha_{i''} < a$.

(iv) *If* $j=5$, $s_4 < d$, *then* $\alpha_4 - \alpha_5 < 0.041$.

(v) *If* $j=5$, $s_3 < d$, *then* $s_5 < 1-f$.

(vi) *If* $\boldsymbol{\alpha}_3 \in H_3$, *then* $\alpha_1 < a$.

Proof. The first assertion in (i) follows from $|s-s'| = |\alpha_i-\alpha_k| < b-a$. The second assertion is proved similarly.

For the first assertion in (ii), we need only note that $s < d-2g < b$. The second assertion now follows.

For (iii), we observe that $\alpha_{i'}+\alpha_{i''} < 2d/3 < b$.

For (iv), we use $\alpha_4-\alpha_5 < d/4 - g < 0.041$.

For (v), we have $s_5 < 5s_3/3 < 5d/3 < 1-d$, hence $s_5 < 1-f$.

For (vi), we use $\alpha_1 < d-\alpha_2 < d-\alpha_3 < d-h < b$. ∎

We can 'concatenate' in (i): for instance, if we have $\alpha_1+\alpha_2+\alpha_3 > b$ and $\max(\alpha_3-\alpha_4, \alpha_4-\alpha_5) < 0.0479$, then $\alpha_1+\alpha_2+\alpha_5 > b$.

LEMMA 17. *Let* $\boldsymbol{\alpha}_4 \in H_4$. *Then* $g \le \alpha_4 < \alpha_3 < \alpha_2 < \alpha_1$; *either* $\alpha_1 < a$ *or* $b < \alpha_1 \le l$; $\alpha_1+\alpha_2 < d$; $\alpha_3 \le h$; $\alpha_1+\alpha_2+\alpha_4 > f$; $\alpha_3+\alpha_4 > h$; $\max(a-(\alpha_2+\alpha_3), \alpha_2+\alpha_3-b) > 0$; $\max(a-(\alpha_2+\alpha_4), \alpha_2+\alpha_4-b) > 0$; $\max(d-(\alpha_1+\alpha_3+\alpha_4), \alpha_1+\alpha_3+\alpha_4-f) > 0$; $\max(a-(\alpha_2+\alpha_3+\alpha_4), \alpha_2+\alpha_3+\alpha_4-b) > 0$. *Moreover,*

(i) $\alpha_3+\alpha_4 < a$,

(ii) $s_4 < 1-f$,

(iii) $\alpha_1+\alpha_4 > b$,

(iv) $\alpha_2+\alpha_3+\alpha_4 < d$.

Proof. Everything except (i)–(iv) follows from the definition.

For (i), $\alpha_3+\alpha_4 < 2\alpha_3 \le 2h < b$. For (ii), $(\alpha_1+\alpha_2)+(\alpha_3+\alpha_4) < d+a < 1-d$. For (iii), $\alpha_1+\alpha_4 > f-\alpha_2 > f-d/2 > a$. Finally, $\alpha_2+(\alpha_3+\alpha_4) < d/2+a < f$, yielding (iv). ∎

LEMMA 18. *Let* $\boldsymbol{\alpha}_5 \in \widehat{H}_5$. *Then* $g \le \alpha_5 < \alpha_4 < \alpha_3 < \alpha_2 < \alpha_1$; $\alpha_3 \le h$; $s_2 < d$; *either* $\alpha_1+\alpha_2+\alpha_4 < d$ *or* $\alpha_3+\alpha_4 \le h$; $s_4 > b$; $\alpha_1 < a$. *Moreover, one of the following alternatives holds:*

(i) $s_4 < d$, $s_2 < a$, $\alpha_3+\alpha_4+\alpha_5 > b$, $s_5 > f$;
(ii) $s_4 < d$, $s_2 < a$, $\alpha_2+\alpha_3+\alpha_4 < a$, $\alpha_1+\alpha_4+\alpha_5 > b$, $s_5 > f$;
(iii) $s_4 < d$, $\alpha_1+\alpha_3+\alpha_4+\alpha_5 > b$, $s_3 < a$;
(iv) $s_4 > f$, $s_2 < a$, $\alpha_1+\alpha_2+\alpha_3+\alpha_5 < d$;
(v) $\alpha_1+\alpha_2+\alpha_4+\alpha_5 > f$, $s_3 < d$, $\alpha_2+\alpha_3+\alpha_4+\alpha_5 < d$, $\alpha_2+\alpha_3 < a$, $\alpha_1+\alpha_4+\alpha_5 > b$, $\alpha_1+\alpha_2 > b$, $\alpha_3+\alpha_4+\alpha_5 < a$, $s_5 < 1-f$;
(vi) $\alpha_1+\alpha_3+\alpha_4+\alpha_5 > f$, $s_3 < d$, $\alpha_2+\alpha_3+\alpha_4+\alpha_5 < d$, $\alpha_2+\alpha_3 < a$, $\alpha_3+\alpha_4+\alpha_5 > b$, $\alpha_1+\alpha_5 > b$, $s_5 < 1-f$;
(vii) $\alpha_2+\alpha_3+\alpha_4+\alpha_5 > f$, $s_2 < a$, $s_5 < 1-f$;
(viii) $\alpha_2+\alpha_4 < a$, $\alpha_1+\alpha_3 > b$, $\alpha_1+\alpha_2+\alpha_4 < d$, $s_3 > f$, $\alpha_2+\alpha_3+\alpha_5 > b$, $\alpha_1+\alpha_3+\alpha_4+\alpha_5 > f$, $s_5 < 1-f$.

Proof. The first assertion we need to prove is $\alpha_1 < a$. This follows from Lemma 16 if $\alpha_1+\alpha_2+\alpha_4 < d$. If $\alpha_1+\alpha_2+\alpha_4 > f$, then $\alpha_3+\alpha_4 \le h$. This leads to a contradiction if $\alpha_1 > b$: we would have $\alpha_1+\alpha_3+\alpha_4 > f$ from Lemma 16, hence $\alpha_1 > f-h > l$, which is absurd. So $\alpha_1 < a$.

To show that one of (i)–(viii) holds, we observe that one of the following alternatives is clearly valid:

(A) $s_4 < d$, $\alpha_2+\alpha_3+\alpha_4 > b$;
(B) $s_4 < d$, $\alpha_2+\alpha_3+\alpha_4 < a$, $\alpha_1+\alpha_3+\alpha_4 > b$;
(C) $s_4 < d$, $\alpha_1+\alpha_3+\alpha_4 < a$, $s_3 > b$;
(D) $s_4 < d$, $s_3 < a$;
(E) $\alpha_1+\alpha_2+\alpha_3+\alpha_5 > f$, $s_3 < d$, $\alpha_2+\alpha_3+\alpha_4+\alpha_5 < d$;
(F) $\alpha_2+\alpha_3+\alpha_4+\alpha_5 > f$, $s_3 < d$;
(G) $s_3 > f$, $\alpha_1+\alpha_2+\alpha_4 < d$;
(H) $\alpha_1+\alpha_2+\alpha_4 > f$, $\alpha_3+\alpha_4 \le h$.

Suppose (A) holds. Then $\alpha_1 < d-b < 0.1002$. We cannot have $\alpha_4 < \alpha_2 - 0.04$, since then $\alpha_2+\alpha_3+\alpha_4 < 0.3006-0.04 < b$. Moreover, $\alpha_4-\alpha_5 < 0.041$ by Lemma 16(iv), so we obtain $\alpha_3+\alpha_4+\alpha_5 > b$ by concatenation. Further, $s_2 < a$ and $s_5 > f$ from Lemma 16. So (i) holds.

Suppose (B) holds. Then $s_2 < a$, $s_5 > f$ from Lemma 16. Now $\alpha_3+\alpha_4 < 2a/3$, $\alpha_3 < 2a/3 - g$, $\alpha_3 - \alpha_5 < 2a/3 - 2g < 0.045$. Hence $\alpha_1 + \alpha_4 + \alpha_5 > b$ from Lemma 16, and (ii) holds.

Suppose (C) holds. Then $s_2 < a$, $s_5 > f$ from Lemma 16. Hence $\alpha_2 < a/2$, $\alpha_2 + \alpha_5 > f - (\alpha_1 + \alpha_3 + \alpha_4) > f - a = 1/5$, $\alpha_5 > 1/5 - a/2 > 0.08$, $\alpha_1 + \alpha_3 + \alpha_4 > 0.24$, which is absurd.

Suppose (D) holds. We have $2\alpha_2 < a - g$, $\alpha_2 < 0.09$, $\alpha_2 - \alpha_5 < 0.04$. Hence $\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 > b$ from Lemma 16. So (iii) holds.

Suppose (E) holds. Since $s_2 < a$ from Lemma 16, (iv) holds.

Suppose (F) holds. Then $\alpha_2+\alpha_3 < a$, $\alpha_2 < a-g$, $s_5 < 1-f$ by Lemma 16. Now $\alpha_1 + \alpha_3 > \alpha_2 + \alpha_5$, so $\alpha_1 + \alpha_3 > f/2$, $\alpha_1 + \alpha_3 + \alpha_5 > f/2 + g > a$ and $\alpha_1+\alpha_3+\alpha_5 > b$. Also $\alpha_3+\alpha_5 < d/2$, $\alpha_1+\alpha_2 > f-d/2 > a$, so $\alpha_1+\alpha_2 > b$, giving $\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5 > f$ from Lemma 16. So $\alpha_1 + \alpha_4 + \alpha_5 > f - \alpha_2 > f - a + g > a$ and $\alpha_1 + \alpha_4 + \alpha_5 > b$. If $\alpha_3 + \alpha_4 + \alpha_5 < a$, we get (v). If $\alpha_3 + \alpha_4 + \alpha_5 > b$, we have $\alpha_1 + (\alpha_3 + \alpha_4 + \alpha_5) > b/2 + b > d$, hence $\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 > f$. Also, $\alpha_3 + \alpha_4 + \alpha_5 > b$ implies $\alpha_2 < d - b < 0.1002$. Suppose $\alpha_1 + \alpha_5 < a$. Then, by Lemma 16, $\alpha_2 - \alpha_5 > 0.0479$, $\alpha_5 < 0.0523$, so $\alpha_3 + \alpha_4 + \alpha_5 < 0.2004 + 0.0523 < b$, which is absurd. Hence $\alpha_1 + \alpha_5 > b$ and (vi) holds.

Suppose (G) holds. Lemma 16 yields $s_5 < 1 - f$, $\alpha_2 + \alpha_3 < a$. We claim that $s_2 < a$. For suppose $s_2 > b$. Now $\alpha_4+\alpha_5 > f-a = 1/5$, $\alpha_3+\alpha_4+\alpha_5 > 3/10$, and $\alpha_1 + \alpha_2 < 0.278$, $\alpha_2 < 0.139$. Also $\alpha_3 < d - b < 0.1002$, hence $\alpha_5 > 0.3 - 0.2004 = 0.0996$, and $\alpha_2 - \alpha_5 < 0.04$. We thus get $\alpha_1 + \alpha_5 > b$ from $\alpha_1 + \alpha_2 > b$. Now $s_5 > b + \alpha_2 + \alpha_3 + \alpha_4 > b + 3f/4 > 1 - f$, which is absurd. So $s_2 < a$ and (vii) holds.

Suppose (H) holds. Then $\alpha_2 + \alpha_4 < a$ from Lemma 16. Next, $s_5 = (\alpha_1 + \alpha_3 + \alpha_5) + (\alpha_2 + \alpha_4) < d + a < 1 - d$, so $s_5 < 1 - f$. Now $\alpha_2 < d/2$, $\alpha_1 + \alpha_3 > f - d/2 > a$, so $\alpha_1 + \alpha_3 > b$. Hence $\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 > f$ from Lemma 16. Finally, $\alpha_2 + \alpha_3 > f - a = 1/5$, therefore $\alpha_2 + \alpha_3 + \alpha_5 > a$ and $\alpha_2 + \alpha_3 + \alpha_5 > b$. So (viii) holds.

Suppose (I) holds. Now $\alpha_2 < d/2$, so $\alpha_1 + \alpha_4 > f - d/2 > a$, and $\alpha_1 + \alpha_4 > b$. We obtain $\alpha_1 + \alpha_4 + \alpha_3 + \alpha_5 > f$ from Lemma 16. Hence $\alpha_3 + \alpha_4 + \alpha_5 > f - \alpha_1 > 1/5$, contrary to the bound $\alpha_3 + \alpha_4 + \alpha_5 \leq 3h/2$. This completes the proof of Lemma 18. ∎

We now turn to the decomposition

$$T(x) = K^{(1)}(x) - D^{(1)}(x).$$

We use (4.19)–(4.24) with H_2, H_4 empty, and so $D_2 = D_4 = 0$. We write our choice of E_i as $\mathcal{E}_i$ and $\mathcal{H}_1, \ldots, \mathcal{H}_4, \widehat{\mathcal{H}}_5$ rather than $H_1, \ldots, H_4, \widehat{H}_5$.

Let

$$\begin{aligned}
\mathcal{E}_1 &= [g, 19/90];\\
\mathcal{E}_2 &= \mathcal{E}_1' \cap B_2 \quad \text{(so that } \mathcal{H}_2 = \emptyset);\\
\mathcal{H}_3 &= \{\boldsymbol{\alpha}_3 \in \mathcal{E}_2' \cap B_3 : s_3 > f\}, \quad \text{so that}\\
\mathcal{E}_3 &= \{\boldsymbol{\alpha}_3 \in \mathcal{E}_2' \cap B_3 : s_3 < d\};\\
\mathcal{E}_4 &= \mathcal{E}_3' \cap B_4 \quad \text{(so that } \mathcal{H}_4 = \emptyset).
\end{aligned}$$

Thus

$$\widehat{\mathcal{H}}_5 = \{\boldsymbol{\alpha}_5 \in \mathcal{E}_4' \cap B_5 : s_4 > b\}.$$

Note that in $\mathcal{E}_2$, $s_2 < f$, hence $s_2 < d$; so any $\boldsymbol{\alpha}_i$ in $\mathcal{E}_i$ obviously splits for $i = 1, 2, 3$. Any $\boldsymbol{\alpha}_4$ in $\mathcal{E}_4$ splits using $\{1, 2, 3\}, \{4\}$, since $\alpha_4 < \alpha_3 < d/3 < h$.

It is easy to write down the conditions satisfied by points of $\mathcal{H}_3$, so we simply note the following lemma for $\widehat{\mathcal{H}}_5$.

LEMMA 19. *Let $\boldsymbol{\alpha}_5 \in \widehat{\mathcal{H}}_5$. Then*

$$g \le \alpha_5 < \alpha_4 < \alpha_3 < \alpha_2 < \alpha_1 \le 19/90, \qquad s_3 < d.$$

Moreover, one of the alternatives (i)–(vii) *of Lemma* 18 *holds.*

Proof. The first two assertions follow from the definition. Since $\widehat{\mathcal{H}}_5 \subseteq \mathcal{H}_5$, one of (i)–(viii) of Lemma 18 holds, and (viii) is ruled out since $s_3 < d$. ∎

In Section 5 we shall need bounds for several integrals. Let $f_1(\alpha_1) = \alpha_1^{-2}$, $f_2(\boldsymbol{\alpha}_2) = \alpha_1^{-1}\alpha_2^{-2}$, and generally $f_j(\boldsymbol{\alpha}_j) = (\alpha_1 \cdots \alpha_{j-1})^{-1}\alpha_j^{-2}$ $(j \ge 2)$. Let $\omega(\ldots)$ denote Buchstab's function (see e.g. [8] for more information). Now let

$$J_j = \int\limits_{H_j} f_j(\boldsymbol{\alpha}_j)\, \omega\bigg(\frac{1 - s_j}{\alpha_j}\bigg)\, d\alpha_1 \cdots d\alpha_j,$$

$$J_5 = \int\limits_{\widehat{H}_5} f_5(\boldsymbol{\alpha}_5)\, \omega\bigg(\frac{1 - s_5}{\alpha_5}\bigg)\, d\alpha_1 \cdots d\alpha_5.$$

Define $J_1^\dagger, J_3^\dagger, J_5^\dagger$ in the same way with $\mathcal{H}_1, \mathcal{H}_3, \widehat{\mathcal{H}}_5$ in place of $H_1, H_3, \widehat{H}_5$. Computer calculations yield

$$\begin{aligned}
&J_1^\dagger < 0.992255, \qquad J_1 < 0.704010,\\
&J_2 < 0.126406,\\
&J_3^\dagger < 0.094570, \qquad J_3 < 0.050281,\\
&J_4 < 0.003991,\\
&J_5^\dagger < 0.006422, \qquad J_5 < 0.007383.
\end{aligned}$$

The integrals in j dimensions are bounded for $j \le 3$ using a precise evaluation. For each of the other integrals, we allow a possibly larger region of integration defined via Lemma 17, 18 or 19, and multiply its measure by an upper bound for the integrand to give the upper bound quoted.

5. Proof of Theorem 1. We need just two more lemmas. Write $\mathrm{d}\boldsymbol{\alpha}_j$ for $\mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_j$.

LEMMA 20. *Let E be a polytope, $E \subseteq P_j$. Let*

$$f(E;X) = \sum_{\boldsymbol{\alpha}_j \in E} \sum_{j+1 \le k \le 19} \sum_{\substack{p_j \le p_{j+1} \le \cdots \le p_{k-1} \\ \pi_{k-1} p_{k-1} \le 2X}} \frac{1}{\pi_{k-1} \log(X/\pi_{k-1})}.$$

As $X \to \infty$,

$$f(E;X) = (1+o(1)) \frac{1}{\mathcal{L}} \int_E f_j(\boldsymbol{\alpha}_j)\, \omega\Big(\frac{1-s_j}{\alpha_j}\Big)\, \mathrm{d}\boldsymbol{\alpha}_j.$$

Proof. Fix $p_1, \ldots, p_j$ with $\boldsymbol{\alpha}_j \in E$. Let $\mathcal{N}(\boldsymbol{\alpha}_j)$ be the number of integers n with $\pi_j n \sim X$, $(n, P(p_j)) = 1$. The solution of

$$(X/\pi_j)^{1/u} = p_j$$

is $u = (1-s_j)/\alpha_j$. Using a well-known asymptotic formula (see e.g. [8]), we deduce that

$$\mathcal{N}(\boldsymbol{\alpha}_j) = (1+o(1)) \frac{u\,\omega(u)}{\log(X/\pi_j)} \frac{X}{\pi_j} = (1+o(1)) \frac{X}{\mathcal{L}\pi_j \alpha_j}\, \omega\Big(\frac{1-s_j}{\alpha_j}\Big)$$

as $X \to \infty$, uniformly for $\boldsymbol{\alpha}_j \in E$. Hence

$$\sum_{\boldsymbol{\alpha}_j \in E} \mathcal{N}(\boldsymbol{\alpha}_j) = (1+o(1)) \frac{X}{\mathcal{L}} \sum_{\boldsymbol{\alpha}_j \in E} \frac{1}{\pi_j \alpha_j}\, \omega\Big(\frac{1-s_j}{\alpha_j}\Big).$$

Using the prime number theorem to approximate the sum by an integral in standard fashion,

$$\sum_{\boldsymbol{\alpha}_j \in E} \mathcal{N}(\boldsymbol{\alpha}_j) = (1+o(1)) \frac{X}{\mathcal{L}} \int_E f_j(\boldsymbol{\alpha}_j)\, \omega\Big(\frac{1-s_j}{\alpha_j}\Big)\, \mathrm{d}\boldsymbol{\alpha}_j. \tag{5.1}$$

On the other hand,

$$\sum_{\boldsymbol{\alpha}_j \in E} \mathcal{N}(\boldsymbol{\alpha}_j) = \sum_{\boldsymbol{\alpha}_j \in E} \sum_{j+1 \le k \le 19} \sum_{\substack{p_j \le p_{j+1} \le \cdots \le p_{k-1} \\ \pi_{k-1} p_{k-1} \le 2X}} \sum_{\substack{X < p_k \le 2X \\ p_k \ge p_{k-1}}} 1.$$

The error incurred in removing the condition $p_k \ge p_{k-1}$ from the last summation is 0 unless

$$p_{k-1} \sim Y := X/\pi_{k-1},$$

in which case the error is $O(Y\mathcal{L}^{-1})$. Thus the prime number theorem yields

$$\begin{aligned}(5.2)\qquad &\sum_{\boldsymbol{\alpha}_j\in E}\mathcal{N}(\boldsymbol{\alpha}_j)\\ &= (1+o(1))\sum_{\boldsymbol{\alpha}_j\in E}\ \sum_{j+1\le k\le 19}\ \sum_{\substack{p_j\le p_{j+1}\le\cdots\le p_{k-1}\\ \pi_{k-1}p_{k-1}\le 2X}}\frac{X}{\pi_{k-1}\log(X/\pi_{k-1})}\\ &\quad + O\biggl(\frac{X}{\mathcal{L}}\sum_{\substack{X^g\le p_1\le\cdots\le p_{k-2}\\ \pi_{k-2}\le X}}\frac{1}{\pi_{k-2}}\sum_{p_{k-1}\sim Y}\frac{1}{p_{k-1}}\biggr).\end{aligned}$$

The error term on the right side of (5.2) is readily seen to be $O(X\mathcal{L}^{-2})$, while the main term is clearly $\gg X\mathcal{L}^{-1}$ for nonempty E. Hence the lemma follows on comparing (5.1), (5.2). ∎

Lemma 21. *Let E be a polytope, $E\subseteq P_j$. Let k be fixed, $j+1\le k\le 19$. Then for $0<x\le\tau$,*

$$\begin{aligned}(5.3)\qquad &\sum_{\boldsymbol{\alpha}_j\in E}\ \sum_{p_j\le p_{j+1}\le\cdots\le p_k}F(\pi_j p_{j+1}\cdots p_k)\\ &= \sum_{\boldsymbol{\alpha}_j\in E}\ \sum_{p_j\le p_{j+1}\le\cdots\le p_k}\frac{1}{\pi_{k-1}}\int\limits_{\max(\pi_{k-1}p_{k-1},X)}^{2X}\frac{e(t^c x)}{\log(t/\pi_{k-1})}\,\mathrm{d}t\\ &\quad + O(X\exp(-C_1\mathcal{L}^{1/4})).\end{aligned}$$

Proof. By a slight variant of [3, Lemma 24], we have, for $1<A<A'\le 2A$, $0<y\le A^{-c+1-2\eta}$,

$$(5.4)\qquad \sum_{A\le p_k<A'}e(p_k^c y)=\int\limits_A^{A'}\frac{e(u^c y)}{\log u}\,\mathrm{d}u+O(A\exp(-3(\log A)^{1/4})).$$

Fix $\boldsymbol{\alpha}_j\in E$ and $p_{j+1},\dots,p_{k-1}$ with $p_j\le p_{j+1}\le\cdots\le p_{k-1}$, $\pi_{k-1}p_{k-1}\le 2X$ (other tuples give an empty sum on both sides of (5.3)). Set

$$A=\max(p_{k-1},X/\pi_{k-1}),\qquad A'=2X/\pi_{k-1},\qquad y=\pi_{k-1}^c x$$

so that $\log A\ge g\mathcal{L}$. We verify that

$$y\le A^{-c+1}X^{-2\eta}.$$

Indeed, we have

$$yA^{c-1}X^{2\eta}\le \pi_{k-1}^c X^{-c+10\eta}(X/\pi_{k-1})^{c-1}=\pi_{k-1}X^{-1+10\eta}\le 1,$$

since $\pi_{k-1}X^g \le 2X$. Now (5.4) yields

$$\sum_{\substack{p_k \ge p_{k-1} \\ X < \pi_{k-1}p_k \le 2X}} e((p_1 \cdots p_k)^c x)$$
$$= \int_{\max(p_{k-1}, X/\pi_{k-1})}^{2X/\pi_{k-1}} \frac{e(u^c \pi_{k-1}^c x)}{\log u}\, \mathrm{d}u + O\biggl(\frac{X}{\pi_{k-1}} \exp(-C_1 \mathcal{L}^{1/4})\biggr)$$

with $C_1 = 3g^{1/4}$. A change of variables gives the integral in the form

$$\frac{1}{\pi_{k-1}} \int_{\max(\pi_{k-1}p_{k-1}, X)}^{2X} \frac{e(t^c x)}{\log(t/\pi_{k-1})}\, \mathrm{d}t,$$

and the lemma follows on summing over $p_1, \ldots, p_{k-1}$. ∎

Proof of Theorem 1. Let $\mathcal{N}$ be the number of solutions of (1.1) with $p_i \sim X$ $(1 \le i \le 3)$. Using an initial step that goes back to Davenport and Heilbronn [5], we observe that

$$\mathcal{N} \ge \sum_{\substack{p_i \sim X \\ (1 \le i \le 3)}} \varphi(p_1^c + p_2^c + p_3^c - R) = \int_{-\infty}^{\infty} T^3\, \mathrm{d}\mu \tag{5.5}$$
$$= \int_{-\infty}^{\infty} T^2(K^{(2)} - D^{(2)} + D^{(3)})\, \mathrm{d}\mu$$
$$= \int_{-\infty}^{\infty} T^2(K^{(2)} + D^{(3)})\, \mathrm{d}\mu - \int_{-\infty}^{\infty} T(K^{(1)} - D^{(1)})D^{(2)}\, \mathrm{d}\mu$$
$$\ge \int_{-\infty}^{\infty} (T^2 K^{(2)} - T K^{(1)} D^{(2)})\, \mathrm{d}\mu.$$

(Compare the argument below [3, (5.4)] for the last step.) In view of (4.17), then,

$$\mathcal{N} \ge \int_{-\tau}^{\tau} (T^2 K^{(2)} - T K^{(1)} D^{(2)})\, \mathrm{d}\mu + O(X^{3-c-2\eta}) \tag{5.6}$$
$$= \int_{-\tau}^{\tau} (T^3 - T^2 D^{(3)} - T D^{(1)} D^{(2)})\, \mathrm{d}\mu + O(X^{3-c-2\eta}).$$

We now use approximations to $T, D^{(i)}$ that arise from (5.4) with $(A, A') = (X, 2X)$ and from (5.3). In $[-\tau, \tau]$,

$$D^{(l)}(x) = I_l(x) + O(X \exp(-C_1 \mathcal{L}^{1/4})) \quad (1 \le l \le 3).$$

Here

$$I_1(x) = Q_1(x) + Q_3(x) + Q_5(x),$$
$$I_2(x) = Q_2(x) + Q_4(x),$$
$$I_3(x) = Q_6(x) + Q_7(x) + Q_8(x),$$

with

$$Q_j(x) = \sum_{\boldsymbol{\alpha}_j \in H_j} \sum_{j+1 \le k \le 19} \sum_{\substack{p_j \le p_{j+1} \le \cdots \le p_{k-1} \\ \pi_{k-1} p_{k-1} \le 2X}} \frac{1}{\pi_{k-1}} \int\limits_{\max(\pi_{k-1}p_{k-1}, X)}^{2X} \frac{e(t^c x)}{\log(t/\pi_{k-1})} \, \mathrm{d}t$$

for $1 \le j \le 4$; Q_5, Q_6, Q_7, Q_8 are defined similarly with H_j replaced respectively by $\widehat{H}_5, \mathcal{H}_1, \mathcal{H}_3, \widehat{\mathcal{H}}_5$.

We make the simple observation that for any functions f_1, f_2 chosen from $\{D_1, D_2, D_3, I_1, I_2, I_3\}$ and for $f = D^{(l)}$, $f_0 = I_l$,

$$\int\limits_{-\tau}^{\tau} f f_1 f_2 \, \mathrm{d}\mu = \int\limits_{-\tau}^{\tau} f_0 f_1 f_2 \, \mathrm{d}\mu + O(X^{3-c-c\eta} \exp(-C_1 \mathcal{L}^{1/4})).$$

A similar approximation is discussed below [3, (5.13)]. Replacing $T, D^{(1)}$, $D^{(2)}, D^{(3)}$ by I, I_1, I_2, I_3 one step at a time, we deduce from (5.6) that

$$\mathcal{N} \ge \int\limits_{-\tau}^{\tau} (I^3 - I^2 I_2 - I I_1 I_3) \, \mathrm{d}\mu + O(X^{3-c-c\eta} \exp(-C_1 \mathcal{L}^{1/4})). \tag{5.7}$$

To extend this integral to infinity, we use the same bound as in [3], namely

$$I(x), I_l(x) \ll |x|^{-1} \mathcal{L}^{-1} X^{1-c} \qquad (x \ne 0,\ 1 \le l \le 3).$$

Since

$$\int\limits_{\tau}^{\infty} x^{-3} X^{3(1-c)} \, \mathrm{d}x < \tau^{-2} X^{3(1-c)} = X^{3-c-16\eta},$$

we infer from (5.7) that

$$\begin{aligned} \mathcal{N} &\ge \int\limits_{-\infty}^{\infty} (I^3 - I^2 I_2 - I I_1 I_3) \, \mathrm{d}\mu + O(X^{3-c-c\eta} \exp(-C_1 \mathcal{L}^{1/4})) \\ &= \int\limits_{-\infty}^{\infty} \{I^3 - I^2(Q_2 + Q_4) - I(Q_1 + Q_3 + Q_5)(Q_6 + Q_7 + Q_8)\} \, \mathrm{d}\mu \\ &\quad + O(X^{3-c-c\eta} \exp(-C_1 \mathcal{L}^{1/4})). \end{aligned}$$

We now rewrite the various integrals $\int_{-\infty}^{\infty} I^2 Q_j \, \mathrm{d}\mu$, $\int_{-\infty}^{\infty} I Q_j Q_k \, \mathrm{d}\mu$ in terms of $\int_{-\infty}^{\infty} I_0^3 \, \mathrm{d}\mu$. Consider, for example, the contribution to $\int_{-\infty}^{\infty} I Q_2 Q_7 \, \mathrm{d}\mu$ from $\boldsymbol{\alpha}_2 \in H_2$, $p_2 \le p_3 \le \cdots \le p_{k-1}$, $\pi_k p_{k-1} \le 2X$ and $\boldsymbol{\alpha}_3' \in \mathcal{H}_3$,

$p'_3 \le p'_4 \le \cdots \le p'_{l-1}$, $\pi'_{l-1}p'_{l-1} \le 2X$ (in an obvious notation). We write

$$s(\mathbf{t}) = t_1^c + t_2^c + t_3^c - R, \qquad \mathrm{d}\mathbf{t} = \mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3.$$

This contribution may be brought, using Fubini's theorem, to the form

$$\begin{aligned}
&\sum_{p_1,\ldots,p_{k-1}}\sum_{p'_1,\ldots,p'_{l-1}} \frac{1}{\pi_k\pi'_{l-1}} \int_{X_3}^{2X}\int_{X_2}^{2X}\int_{X}^{2X}\int_{-\infty}^{\infty} \frac{e(xs(\mathbf{t}))\varPhi(x)\,\mathrm{d}x\,\mathrm{d}\mathbf{t}}{(\log t_1)(\log(t_2/\pi_{k-1}))(\log(t_3/\pi'_{l-1}))}\\
&\quad = \sum_{p_1,\ldots,p_{k-1}}\sum_{p'_1,\ldots,p'_{l-1}} \frac{1}{\pi_k\pi'_{l-1}} \int_{X_3}^{2X}\int_{X_2}^{2X}\int_{X}^{2X} \frac{\varphi(s(\mathbf{t}))\,\mathrm{d}\mathbf{t}}{(\log t_1)(\log(t_2/\pi_{k-1}))(\log(t_3/\pi'_{l-1}))}\\
&\quad \le H \sum_{p_1,\ldots,p_{k-1}}\sum_{p'_1,\ldots,p'_{l-1}} \frac{1}{\pi_k\pi'_{l-1}} \frac{1}{(\log X)(\log(X/\pi_{k-1}))(\log(X/\pi'_{l-1}))},
\end{aligned}$$

where

$$H = \int_{-\infty}^{\infty} I_0(x)^3\,\mathrm{d}\mu.$$

For the last step we replace the positive integrand by a larger one using $\log t_j \ge \log X$, and then reverse the order of integration. In this way we see that

$$\mathcal{N} \ge \left(\frac{1}{(\log 2X)^3} - \frac{W_1}{\mathcal{L}^2} - \frac{W_2W_3}{\mathcal{L}}\right)H + O(X^{3-c-c\eta}\exp(-C_1\mathcal{L}^{1/4})),$$

where

$$\begin{aligned}
W_1 &= f(H_2;X) + f(H_4;X),\\
W_2 &= f(H_1;X) + f(H_3;X) + f(H_5;X),\\
W_2 &= f(\mathcal{H}_1;X) + f(\mathcal{H}_3;X) + f(\mathcal{H}_5;X).
\end{aligned}$$

Now we use Lemma 20 to obtain, as $X \to \infty$,

$$\begin{aligned}
\mathcal{N} \ge (1+o(1))\frac{H}{\mathcal{L}^3}&(1 - (J_2+J_4) - (J_1+J_3+J_5)(J_1^\dagger + J_3^\dagger + J_5^\dagger))\\
&+ O(X^{3-c-c\eta}\exp(-C_1\mathcal{L}^{1/4})).
\end{aligned}$$

For large X, the upper bounds given in Section 4 for $J_i, J_i^\dagger$ yield

$$\mathcal{N} \ge \frac{0.0369H}{\mathcal{L}^3}.$$

Since

$$H \gg X^{3-c-c\eta}$$

(Tolev [16]), this completes the proof of Theorem 1. ∎

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