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Egyptian fractions with restrictions

by

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1. Introduction. Egyptian fractions or unit fractions have been extensively studied (see [1], [8], [14, D11], [17]). Some studies concern the question which fractions can be written as a sum of k unit fractions, others restrict the denominators, still others count the number of solutions. In particular, solutions of the diophantine equation $1 = \sum_{i=1}^{k} 1/x_i$ have been extensively studied. Sierpiński [22] noted that there is a solution with distinct odd integers, and Breusch [24] and Stewart [25] independently proved that each fraction a/b with odd denominator can be written as a finite sum of distinct unit fractions with odd denominators. More recently Shiu [20] and Burshtein [5] proved that the equation $\sum_{i=1}^{9} 1/x_i = 1$ has only five solutions in distinct odd numbers that can be easily found with a computer. Motivated by this, let $T_o(k)$ denote the number of solutions of $\sum_{i=1}^{k} 1/x_i = 1$ in odd numbers $1 < x_1 < \cdots < x_k$. It is easy to see that $T_o(k) = 0$ for all even values of k. One natural problem is: how large can $T_o(k)$ be for odd k? In this paper we present a lower bound for $T_o(k)$ which grows faster than exponentially.

The literature contains many results either stating that there are solutions of $\sum_{i=1}^{k} 1/x_i = 1$ of a special type, which is an indication that the equation has many solutions, or stating that certain types of solutions cannot exist, or bounding the number of solutions. For example, Martin [17] showed that $\sum_{i=1}^{k} 1/x_i = 1$ has solutions in which a dense set of possible denominators occur. Croot [8] showed that for any r-colouring of the positive integers there is a monochromatic solution of $\sum_{i=1}^{k} 1/x_i = 1$. This is some measure of saying the equation has many solutions, and these are closely interlinked, as otherwise one could construct a bad colouring.

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In 2007 Z. W. Sun [26] conjectured the following strengthening of this: If $A \subset \mathbb{N}$ is a set of positive upper asymptotic density, then there is a finite subset $\{x_1, \ldots, x_k\}$ of A such that $\sum_{i=1}^{k} 1/x_i = 1$.

In this paper we examine for which set of primes there is a solution of the diophantine equation $\sum_{i=1}^{k} 1/x_i = 1$ for which all denominators have the given prime factors only, and we give upper and lower bounds on the number of these solutions. We introduce the following notation. Let $\mathbb{N}_0$ be the set of all nonnegative integers. For distinct primes $p_1, \ldots, p_t$, let

$$S(p_1, \ldots, p_t) = \{p_1^{\alpha_1} \cdots p_t^{\alpha_t} \mid \alpha_i \in \mathbb{N}_0,\ i = 1, 2, \ldots, t\}$$

and let $T_k(p_1, \ldots, p_t)$ be the number of solutions of $\sum_{i=1}^{k} 1/x_i = 1$ with $1 < x_1 < \cdots < x_k$ and $x_i \in S(p_1, \ldots, p_t)$ $(1 \le i \le k)$.

As a very special case Burshtein [6] proved that the equation $\sum_{i=1}^{11} 1/x_i = 1$ with $1 < x_1 < \cdots < x_{11}$ and $x_i \in \{3^\alpha 5^\beta 7^\gamma : \alpha, \beta, \gamma \in \mathbb{N}_0\}$ $(1 \le i \le 11)$ has exactly 17 solutions, in other words $T_{11}(3, 5, 7) = 17$.

In this paper we establish a necessary and sufficient condition on the set $\{p_1, \ldots, p_t\}$ of primes for a solution to exist, and give upper and lower bounds of exponential type on $T_k(p_1, \ldots, p_t)$. The upper bounds are stronger than those that would follow from Evertse's result [11] on S-unit equations. (For details see the next section.)

There is a closely related problem, where not all denominators are necessarily distinct. Let us review some known results on counting such solutions. Let $U(k)$ denote the number of solutions of $\sum_{i=1}^{k} 1/x_i = 1$ in integers $1 \le x_1 \le \cdots \le x_k$. Erdős, Graham and Straus (unpublished but see [10, p. 32]) proved that

$$e^{k^{2-\varepsilon}} < U(k) < c_0^{2^k},$$

where $c_0 = 1.264085\ldots$. Sándor [19] improved this to

$$e^{ck^3/\log k} \le U(k) \le c_0^{(1+\varepsilon)2^{k-1}}, \quad k \ge k_0.$$

The upper bound was recently improved by Browning and Elsholtz [4] to

$$U(k) \le c_0^{(5/48+\varepsilon)2^k}, \quad k \ge k_0.$$

Finally, let us remark that the problem of representing 1 as a sum of unit fractions with restricted prime factors in the denominators is closely related to so called "pseudoperfect" numbers. A number is called *pseudoperfect* if it is the sum of some of its divisors. For example, Sierpiński [23] observed that

$$945 = 315 + 189 + 135 + 105 + 63 + 45 + 35 + 27 + 15 + 9 + 7,$$

which is equivalent to a decomposition already stated by Sierpiński in [22],

$$1 = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{15} + \frac{1}{21} + \frac{1}{27} + \frac{1}{35} + \frac{1}{63} + \frac{1}{105} + \frac{1}{135}.$$

Observe that the denominators have the prime factors 3, 5 and 7 only.

2. Statement of results. In this paper we prove the following results.

Theorem 2.1. *For $k \geq 4$ we have*

$$T_o(2k+1) \geq (\sqrt{2})^{(k+1)(k-4)}.$$

Let $p_1, \ldots, p_t$ be distinct primes. Define

$$K(p_1, \ldots, p_t) = \{k : T_k(p_1, \ldots, p_t) \geq 1\}.$$

By Lemma 4.1, if $k, l \in K(p_1, \ldots, p_t)$, then $k+l-1 \in K(p_1, \ldots, p_t)$. Observe that for $l \in K(p_1, \ldots, p_t)$, the infinite arithmetic progression $a(l-1)+1$ is contained in $K(p_1, \ldots, p_t)$.

Theorem 2.2. *Let $p_1, \ldots, p_t$ be distinct primes. Then*

(a) *$K(p_1, \ldots, p_t)$ is a union of finitely many arithmetic progressions;*
(b) *there are two constants $k_0 = k_0(p_1, \ldots, p_t)$ and $c_1 = c_1(p_1, \ldots, p_t) > 1$ such that for all $k > k_0$ with $k \in K(p_1, \ldots, p_t)$ we have*

$$c_1^k \leq T_k(p_1, \ldots, p_t) \leq \sqrt{2}^{tk^2(1+o_k(1))}.$$

It should be remarked that Evertse's [11] important work on S-unit equations treats a related but more general question. The general bound provided by Evertse would only give a weaker upper bound of $(2^{35}k^2)^{k^3 t}$.

If $t = 1$, there are no solutions, as $\sum_{i=1}^{k} 1/p^i < 1$. On the other hand, if the x_i are not assumed to be distinct, then very precise asymptotic results are known: see for example Boyd [3], Elsholtz, Heuberger and Prodinger [9].

Now let $t \geq 2$ and let

$$A = S(p_1, \ldots, p_t) \setminus \{1\} = \{a_1 < a_2 < \cdots\}.$$

Then

$$\sum_{i=1}^{\infty} \frac{1}{a_i} = \left(1 + \frac{1}{p_1} + \frac{1}{p_1^2} + \cdots\right) \cdots \left(1 + \frac{1}{p_t} + \frac{1}{p_t^2} + \cdots\right) - 1$$
$$= \frac{p_1}{p_1 - 1} \cdots \frac{p_t}{p_t - 1} - 1.$$

As we are studying finite sums of unit fractions, and as the denominator 1 is discarded from consideration, a necessary condition for $K(p_1, \ldots, p_t)$ to be nonempty is

$$\frac{p_1}{p_1 - 1} \cdots \frac{p_t}{p_t - 1} > 2. \tag{2.1}$$

It is interesting that this necessary condition (2.1) is also sufficient:

Theorem 2.3. *Let $p_1, \ldots, p_t$ be distinct primes. Then $K(p_1, \ldots, p_t)$ is nonempty (that is, a solution to $\sum_{i=1}^{k} 1/x_i = 1$ of any length exists with $1 < x_1 < \cdots < x_k$ and all x_i in $S(p_1, \ldots, p_t)$) if and only if the inverse sum* *of the elements in* $S(p_1, ..., p_t)$ *is more than* 2*, that is,*

$$\frac{p_1}{p_1 - 1} \cdots \frac{p_t}{p_t - 1} > 2.$$

For a set B of numbers, let

$$P(B) = \Big\{ \sum_{a \in I} a \;\Big|\; I \subseteq B,\, 0 < |I| < \infty \Big\}$$

denote the set of finite subset sums. For a set B of nonzero numbers, let

$$B^{-1} = \{b^{-1} \mid b \in B\}.$$

In order to prove Theorem 2.3, we make use of well known results of Graham [13, Theorem 5] and Birch [2], and observe that 1, or more generally a/b, can be decomposed into a finite sum of distinct reciprocals for a more general type of integer sequences. Graham's original hypotheses are different, we adapt his work to our applications. We prove the following theorem.

Theorem 2.4. *Let* $A = \{a_1 < a_2 < \cdots\}$ *be a sequence of positive integers such that*

(a) A *is complete, i.e. all sufficiently large integers are contained in* $P(A)$;
(b) A *is multiplicative, i.e. for all* i, j *with* $a_i, a_j \in A$*, also* $a_i a_j \in A$;
(c) $\sum_{j=i+1}^{\infty} 1/a_j \geq 1/a_i$ *for all* $i \geq 1$.

Then $p/q \in P(A^{-1})$*, where* $(p, q) = 1$*, if and only if*

(d) $p/q < \sum_{i=1}^{\infty} 1/a_i$;
(e) q *divides some term of* A.

This implies the following corollary:

Corollary 2.5. *Let* $A = \{a_1 < a_2 < \cdots\}$ *be a sequence of integers with* $a_1 > 1$ *such that*

(a) A *is complete;*
(b) A *is multiplicative;*
(c) $\sum_{i=1}^{\infty} 1/a_i > 1$.

Then $1 \in P(A^{-1})$.

We pose the following problem for future research.

Problem 2.6. *Let* $p_1, \ldots, p_t$ *be distinct primes. Is there a constant* V *depending only on* $p_1, \ldots, p_t$ *such that*

$$T_k(p_1, \ldots, p_t) \leq V^k?$$

Finally, we give two special results.

THEOREM 2.7.

(a) $T_k(3,5,7) \geq c_1\sqrt{62}^k$ *for a computable constant* $c_1 > 0$ *and any odd number* $k \geq 11$*;*

(b) $T_k(2,3,5) \geq c_2\sqrt{368}^k$ *for a computable constant* $c_2 > 0$ *and any integer* $k \geq 3$.

3. Proof of Theorem 2.1. In order to prove Theorem 2.1, we establish a relation between $T_o(2k-1)$ and $T_o(2k+1)$, which inductively gives a bound for an arbitrary odd number of fractions. For this purpose we first establish the following lemma.

LEMMA 3.1. *If* n *is odd, then the number of solutions of*

$$\frac{1}{n} = \frac{1}{u} + \frac{1}{v} + \frac{1}{w}, \quad n < u < v < w,\ 2 \nmid uvw,\ d(w) \geq 2d(n) + 1,$$

is at least $\frac{1}{2}d(n) - 1$. *(Here* $d(n)$ *denotes the number of positive divisors of* n.*)*

Proof. Recall that the number of ways to write an integer n as a sum of two squares is $r_2(n) = 4(d_1(n) - d_3(n))$, where $d_i(n)$ is the number of positive divisors d of n with $d \equiv i \pmod 4$ $(i = 1, 3)$ (see [15, Theorem 278 and (16.9.2)] or [18, Theorem 14.3]): As $r_2(n)$ is a nonnegative integer it follows that $d_1(n) \geq d_3(n)$ and $d_1(n) \geq \frac{1}{2}d(n)$.

Let $k > 1$ be a positive divisor of n of the form $4l + 1$. Let

$$u = n + 2, \quad v = \frac{1}{2k}n(n+2)(k+1), \quad w = \frac{1}{2}n(n+2)(k+1).$$

Then

$$\frac{1}{n} = \frac{1}{u} + \frac{1}{v} + \frac{1}{w}, \quad n < u < v < w,\ 2 \nmid uvw.$$

Since $(k+1)/2 > 1$ is an integer and $(n, n+2) = 1$, we have

$$\begin{aligned} d(w) &= d(n(n+2)(k+1)/2) \geq d(n(n+2)) + 1 = d(n)d(n+2) + 1 \\ &\geq 2d(n) + 1. \ \blacksquare \end{aligned}$$

Proof of Theorem 2.1. Let $T'_o(2k+1)$ denote the number of solutions of $\sum_{i=1}^{2k+1} 1/x_i = 1$ in odd numbers $1 < x_1 < \cdots < x_{2k+1}$ with $d(x_{2k+1}) > 2^k$. Suppose that $1 < x_1 < \cdots < x_{2k-1}$ $(k \geq 5)$ is a solution of $\sum_{i=1}^{2k-1} 1/x_i = 1$ in odd numbers with $d(x_{2k-1}) > 2^{k-1}$. By Lemma 3.1 the number of solutions of

$$\frac{1}{x_{2k-1}} = \frac{1}{u} + \frac{1}{v} + \frac{1}{w}, \quad x_{2k-1} < u < v < w,\ 2 \nmid uvw,\ d(w) \geq 2d(x_{2k-1}) + 1,$$

is at least $\frac{1}{2}d(x_{2k-1}) - 1$. Since

$$d(w) \ge 2d(x_{2k-1}) + 1 > 2^k, \quad \frac{1}{2}d(x_{2k-1}) - 1 \ge \frac{1}{2}(2^{k-1}+1) - 1 = 2^{k-2} - \frac{1}{2},$$

the number of solutions of

$$\frac{1}{x_{2k-1}} = \frac{1}{u} + \frac{1}{v} + \frac{1}{w}, \quad x_{2k-1} < u < v < w,\ 2 \nmid uvw,\ d(w) > 2^k$$

is at least 2^{k-2}. Hence

$$T_o'(2k+1) \ge 2^{k-2} T_o'(2k-1).$$

By [20], [21] (see also [5]) there exist nine odd numbers $1 < x_1 < \cdots < x_9$ with $x_9 = 10395$ and

$$\sum_{i=1}^{9} \frac{1}{x_i} = 1.$$

Since $d(10395) = 32$, we have $T_o'(9) \ge 1$. Thus

$$\begin{aligned} T_o'(2k+1) &\ge 2^{k-2} T_o'(2k-1) \ge \cdots \ge 2^{(k-2)+(k-3)+\cdots+(5-2)} T_o'(9) \\ &\ge 2^{\frac{1}{2}(k+1)(k-4)}. \end{aligned}$$

Hence $T_o(2k+1) \ge (\sqrt{2})^{(k+1)(k-4)}$. ■

4. Proof of Theorem 2.2. For distinct primes $p_1, \ldots, p_t$, we define $\mathcal{T}_k(p_1, \ldots, p_t)$ to be the set of all solutions $(x_1, \ldots, x_k)$ of

$$\sum_{i=1}^{k} \frac{1}{x_i} = 1, \quad 1 < x_1 < \cdots < x_k,\ x_i \in S(p_1, \ldots, p_t).$$

Define

$$(x_1, \ldots, x_k) * (y_1, \ldots, y_l) = (x_1, \ldots, x_{k-1}, x_k y_1, \ldots, x_k y_l)$$

and

$$(a_1, \ldots, a_k)^i = (a_1, \ldots, a_k)^{i-1} * (a_1, \ldots, a_k), \quad i \ge 2.$$

It is clear that if $(x_1, \ldots, x_k) \in \mathcal{T}_k(p_1, \ldots, p_t)$ and $(y_1, \ldots, y_l) \in \mathcal{T}_l(p_1, \ldots, p_t)$, then

$$(x_1, \ldots, x_k) * (y_1, \ldots, y_l) \in \mathcal{T}_{k+l-1}(p_1, \ldots, p_t). \tag{4.1}$$

The following lemma gives a recursive lower bound:

Lemma 4.1. *Let $p_1, \ldots, p_t$ be distinct primes. Then, for any two positive integers k and l, we have*

$$T_{k+l-1}(p_1, \ldots, p_t) \ge T_k(p_1, \ldots, p_t) T_l(p_1, \ldots, p_t).$$

Proof. We define a map

$$f : \mathcal{T}_k(p_1, \ldots, p_t) \times \mathcal{T}_l(p_1, \ldots, p_t) \to \mathcal{T}_{k+l-1}(p_1, \ldots, p_t)$$

as follows:

$$(x_1,\dots,x_k)\times(y_1,\dots,y_l)\mapsto(x_1,\dots,x_k)*(y_1,\dots,y_l).$$

It is clear that f is injective. Now Lemma 4.1 follows immediately. ∎

LEMMA 4.2. *Let $p_1,\dots,p_t$ be distinct primes. If we have $(x_1,\dots,x_k)\in\mathcal{T}_k(p_1,\dots,p_t)$ and $(y_1,\dots,y_l)\in\mathcal{T}_l(p_1,\dots,p_t)$ with $x_k^{l-1}\neq y_l^{k-1}$, then*

$$T_{(k-1)(l-1)+1}(p_1,\dots,p_t)\geq 2.$$

Proof. By (4.1) we have

$$(x_1,\dots,x_k)^{l-1},(y_1,\dots,y_l)^{k-1}\in\mathcal{T}_{(k-1)(l-1)+1}(p_1,\dots,p_t).$$

Since x_k^{l-1},y_l^{k-1} are the largest elements of $(x_1,\dots,x_k)^{l-1},(y_1,\dots,y_l)^{k-1}$ respectively, by $x_k^{l-1}\neq y_l^{k-1}$ we have

$$(x_1,\dots,x_k)^{l-1}\neq(y_1,\dots,y_l)^{k-1}.$$

Hence $T_{(k-1)(l-1)+1}(p_1,\dots,p_t)\geq 2$. ∎

The following lemma is an extension of a well known theorem of Birch [2]. The possibility for this extension was already mentioned by Davenport and Birch (see [2] and [16]). Hegyvári [16] gave an explicit value on $C(p,q)$. The upper bound on $C(p,q)$ was recently improved by Fang [12] and further improved by Chen and Fang [7].

LEMMA 4.3 (Hegyvári [16]). *For any integers p,q with $p,q>1$ and $(p,q)=1$, there exists $C=C(p,q)$ such that the set*

$$Y_C=\{p^\alpha q^\beta\mid\alpha,\beta\in\mathbb{N}_0,\ 0\leq\beta\leq C\}$$

is complete. That is, every sufficiently large integer is the sum of distinct terms taken from Y_C.

LEMMA 4.4. *Let $p_1,\dots,p_t$ be distinct primes. If $T_k(p_1,\dots,p_t)\geq 1$ for some k, then $T_l(p_1,\dots,p_t)\geq 2$ for some l.*

Proof. Let $(x_1,\dots,x_k)\in\mathcal{T}_k(p_1,\dots,p_t)$. It is clear that x_k is not a prime power. Therefore, there exist two distinct primes $p,q\in\{p_1,\dots,p_t\}$ with $pq\mid x_k$. Let C be as in Lemma 4.3. Take a large $v>C$ such that q^v is the sum of distinct terms taken from Y_C. Assume that

$$q^v=\sum_{i=1}^{t}p^{\alpha_i}q^{\beta_i},\quad p^{\alpha_1}q^{\beta_1}<\dots<p^{\alpha_t}q^{\beta_t},$$

where $\alpha_i,\beta_i\in\mathbb{N}_0$ and $0\leq\beta_i\leq C$. Since $v>C$, we have $t\geq 2$ and $v>\beta_i$ $(1\leq i\leq t)$. Let $u=\max\{v,\alpha_1,\dots,\alpha_t\}$. Write

$$(x_1,\dots,x_k)^u=(y_1,\dots,y_{u(k-1)+1}).$$

Then $y_{u(k-1)+1} = x_k^u$. It is clear that

$$(y_1, \dots, y_{u(k-1)}, y_{u(k-1)+1} q^{v-\beta_t} p^{-\alpha_t}, \dots, y_{u(k-1)+1} q^{v-\beta_1} p^{-\alpha_1}) \in \mathcal{T}_{u(k-1)+t}(p_1, \dots, p_t).$$

In order to prove Lemma 4.4, it is enough by Lemma 4.2 to prove that

$$y_{u(k-1)+1}^{u(k-1)+t-1} \neq (y_{u(k-1)+1} q^{v-\beta_1} p^{-\alpha_1})^{u(k-1)},$$

or equivalently

$$y_{u(k-1)+1}^{t-1} p^{u(k-1)\alpha_1} \neq q^{u(k-1)(v-\beta_1)}.$$

This follows from $t \geq 2$, $u(k-1)\alpha_1 \geq 0$ and $pq \mid y_{u(k-1)+1}^{t-1}$. ∎

Proof of Theorem 2.2. If $K(p_1, \dots, p_t)$ is empty, then Theorem 2.2 is true trivially. So we assume that $K(p_1, \dots, p_t)$ is not empty.

We first prove (a). By Lemma 4.4 there exists an m_0 with $T_{m_0}(p_1, \dots, p_t) \geq 2$. For each integer $0 \leq i < m_0 - 1$, let k_i be the least positive integer k (if any) such that $k \equiv i \pmod{m_0 - 1}$ and $k \in K(p_1, \dots, p_t)$. By Lemma 4.1 we have

$$T_{(m_0-1)l+k_i}(p_1, \dots, p_t) \geq (T_{m_0}(p_1, \dots, p_t))^l T_{k_i}(p_1, \dots, p_t) \geq 2^l. \tag{4.2}$$

Hence

$$K(p_1, \dots, p_t) = \bigcup_{i=0,\, k_i \text{ exists}}^{m_0-2} \{(m_0 - 1)l + k_i : l = 1, 2, \dots\}, \tag{4.3}$$

which proves part (a).

We now prove the lower bound of part (b). Let $c_1 = \min 2^{1/(m_0-1+k_i)}$ and $t_0 = \max k_i$, where the minimum and maximum are taken over all i such that k_i exists. If $k \in K(p_1, \dots, p_t)$ and $k > t_0$, then, by (4.3), there exists an i with $0 \leq i < m_0 - 1$ and a positive integer l such that $k = (m_0 - 1)l + k_i$. By (4.2) we have

$$T_k(p_1, \dots, p_t) = T_{(m_0-1)l+k_i}(p_1, \dots, p_t) \geq 2^l \geq 2^{((m_0-1)l+k_i)/(m_0-1+k_i)} \geq c_1^k.$$

To prove the upper bound, let

$$\sum_{i=1}^{k} \frac{1}{x_i} = 1, \quad 1 < x_1 < \dots < x_k,\ x_i \in S(p_1, \dots, p_t)\ (1 \leq i \leq k).$$

Define $u_1 = 1$ and $u_{n+1} = u_n(u_n + 1)$ for $n \geq 1$. Then $u_n < 2^{2^n}$ for $n \geq 1$. As in the proof of [19, p. 218] we have

$$x_j \leq (k - j + 1)u_j < k2^{2^j}, \quad j = 1, \dots, k.$$

Let $x_j = p_1^{\alpha_{j1}} \cdots p_t^{\alpha_{jt}}$. Then $\alpha_{ji} \le 2\log k + 2^j$. Thus

$$T_k(p_1,\ldots,p_t) \le \prod_{2^j \le 2\log k} (4\log k)^t \prod_{j\le k,\, 2^j > 2\log k} (2^{j+1})^t = \sqrt{2}^{tk^2(1+o(1))}. \ \blacksquare$$

5. Proofs of Theorems 2.3, 2.4 and Corollary 2.5. In order to prove Theorem 2.4, we need a well known result of Graham.

For a sequence $S = (s_1, s_2, \ldots)$ of positive integers, $M(S)$ is defined to be the increasing sequence of all products $\prod_{i=1}^m s_{k_i}$, where $m = 1, 2, \ldots$ and $k_1 < \cdots < k_m$. Thus all the terms of $M(S)$ are distinct.

For a sequence of real numbers, a real number α is said to be S-*accessible* if, for any $\varepsilon > 0$, there exists $\beta \in P(S)$ such that $0 \le \beta - \alpha < \varepsilon$.

S is said to be *complete* if all sufficiently large integers belong to $P(S)$.

Theorem A ([13, Theorem 5]). *Let $S = (s_1, s_2, \ldots)$ be a sequence of positive integers such that*

(1) *$M(S)$ is complete,*
(2) *s_{n+1}/s_n is bounded.*

Then

$$p/q \in P((M(S))^{-1})$$

(*where* $(p,q) = 1$) *if and only if*

(3) *p/q is $(M(S))^{-1}$-accessible,*
(4) *q divides some term of $M(S)$.*

With this preparation, we can prove our Theorem 2.4.

Proof of Theorem 2.4. By (b) we have $M(A) = A$. By (a) and $M(A) = A$ we know that condition (1) of Theorem A is true. By (b) we have $a_2 a_n \in A$. As $a_2 > a_1 \ge 1$ we have $a_2 a_n > a_n$. Thus $a_{n+1} \le a_2 a_n$. So $a_{n+1}/a_n \le a_2$. Hence condition (2) of Theorem A holds.

If $p/q \in P(A^{-1})$, where $(p,q) = 1$, then (d) is true and by Theorem A, q divides some term of A, i.e. (e) holds.

Now we assume that (d) and (e) are true. From (e) we know that condition (4) of Theorem A holds. In order to prove that $p/q \in P(A^{-1})$, by Theorem A, it is enough to prove that condition (3) of Theorem A holds, i.e., p/q is A^{-1}-accessible.

Suppose that $p/q \notin P(A^{-1})$ (this prevents equality in the following arguments). We will show that p/q is A^{-1}-accessible. Then by Theorem A we have $p/q \in P(A^{-1})$, a contradiction.

If

$$\sum_{i=1}^{\infty} \frac{1}{a_i} = +\infty,$$

let $a_0 = 0$. Otherwise the infinite sum is convergent and we define the real number a_0 by

$$\frac{1}{a_0} = \sum_{i=1}^{\infty} \frac{1}{a_i}.$$

Let i_1 be the integer i such that

$$\frac{1}{a_i} < \frac{p}{q} < \frac{1}{a_{i-1}}. \tag{5.1}$$

By (d) we have $i_1 \geq 1$. Moreover, by (c) and (d),

$$\sum_{i=i_1}^{\infty} \frac{1}{a_i} \geq \frac{1}{a_{i_1-1}} > \frac{p}{q}. \tag{5.2}$$

Thus by (5.1) and (5.2) we obtain

$$0 < \frac{p}{q} - \frac{1}{a_{i_1}} < \frac{1}{a_{i_1-1}} - \frac{1}{a_{i_1}} \leq \sum_{i=i_1+1}^{\infty} \frac{1}{a_i}.$$

Suppose that we have found a sequence $\{i_k\}_{k=1}^{n}$ such that $1 \leq i_1 < \cdots < i_n$ and

$$0 < \frac{p}{q} - \sum_{l=1}^{k} \frac{1}{a_{i_l}} < \sum_{i=i_k+1}^{\infty} \frac{1}{a_i}, \quad k = 1, \ldots, n.$$

If

$$\frac{1}{a_{i_n+1}} < \frac{p}{q} - \sum_{l=1}^{n} \frac{1}{a_{i_l}},$$

let $i_{n+1} = i_n + 1$; then

$$0 < \frac{p}{q} - \sum_{l=1}^{n+1} \frac{1}{a_{i_l}} < \sum_{i=i_{n+1}+1}^{\infty} \frac{1}{a_i}.$$

If

$$\frac{p}{q} - \sum_{l=1}^{n} \frac{1}{a_{i_l}} < \frac{1}{a_{i_n+1}},$$

let i_{n+1} be the integer i with

$$\frac{1}{a_i} < \frac{p}{q} - \sum_{l=1}^{n} \frac{1}{a_{i_l}} < \frac{1}{a_{i-1}};$$

then $i_{n+1} > i_n + 1$ and

$$0 < \frac{p}{q} - \sum_{l=1}^{n+1} \frac{1}{a_{i_l}} < \frac{1}{a_{i_{n+1}-1}} - \frac{1}{a_{i_{n+1}}} \leq \sum_{i=i_{n+1}+1}^{\infty} \frac{1}{a_i}.$$

Thus we can find a sequence $\{i_k\}_{k=1}^{\infty}$ such that $1 \le i_1 < i_2 < \cdots$ and

$$0 < \frac{p}{q} - \sum_{l=1}^{k} \frac{1}{a_{i_l}} < \sum_{i=i_k+1}^{\infty} \frac{1}{a_i}, \quad k = 1, 2, \ldots.$$

Let j_k be the least j with $j \ge i_k + 1$ such that

$$0 < \frac{p}{q} - \sum_{l=1}^{k} \frac{1}{a_{i_l}} < \sum_{i=i_k+1}^{j} \frac{1}{a_i}.$$

Then

$$0 < \sum_{i=i_k+1}^{j_k} \frac{1}{a_i} - \left(\frac{p}{q} - \sum_{l=1}^{k} \frac{1}{a_{i_l}}\right) < \frac{1}{a_{j_k}},$$

that is,

$$0 < \sum_{l=1}^{k} \frac{1}{a_{i_l}} + \sum_{i=i_k+1}^{j_k} \frac{1}{a_i} - \frac{p}{q} < \frac{1}{a_{j_k}}.$$

Since $a_{j_k} \to \infty$, it follows that p/q is A^{-1}-accessible. ∎

Proof of Corollary 2.5. By Theorem 2.4 it is enough to prove that

$$\sum_{j=i+1}^{\infty} \frac{1}{a_j} \ge \frac{1}{a_i} \qquad \text{for all } i \ge 1.$$

Since $a_i < a_i a_1 < a_i a_2 < \cdots$, we have $\sum_{j=i+1}^{\infty} \frac{1}{a_j} \ge \sum_{j=1}^{\infty} \frac{1}{a_i a_j} > \frac{1}{a_i}$. ∎

Proof of Theorem 2.3. Let

$$A = S(p_1, \ldots, p_t) \setminus \{1\} = \{a_1 < a_2 < \cdots\}.$$

The necessity of the condition was explained as motivation just before the statement of Theorem 2.3. We only need to prove the sufficiency. Assume that

$$\frac{p_1}{p_1 - 1} \cdots \frac{p_t}{p_t - 1} > 2.$$

Then $t \ge 2$ and

$$\sum_{i=1}^{\infty} \frac{1}{a_i} > 1. \tag{5.3}$$

Since $t \ge 2$, by Lemma 4.3, A is complete. It is clear that (b) in Corollary 2.5 is true. By Corollary 2.5 we have $1 \in P(A^{-1})$. ∎

6. Proof of Theorem 2.7. Let $A_k(M)$ denote the set of solutions of $\sum_{i=1}^{k} 1/x_i = 1$ in distinct integers $1 < x_1 < \cdots < x_k$ with $M | x_k$ and $x_i \in \{2^\alpha 3^\beta 5^\gamma\}$ $(1 \le i \le k)$. Let $B_k(M)$ denote the set of solutions of $\sum_{i=1}^{k} 1/x_i = 1$ in distinct integers $1 < x_1 < \cdots < x_k$ with $M \mid x_k$ and

$x_i \in \{3^\alpha 5^\beta 7^\gamma\}$ $(1 \le i \le k)$. It is clear that $T_k(2,3,5) \ge |A_k(M)|$ for any $M \in \{2^\alpha 3^\beta 5^\gamma\}$ and $T_k(3,5,7) \ge |B_k(M)|$ for any $M \in \{3^\alpha 5^\beta 7^\gamma\}$. In order to obtain good lower bounds on $T_k(2,3,5)$ and $T_k(3,5,7)$, we choose two suitable constants M_1 and M_2 such that $|A_k(M_1)|$ and $|B_k(M_2)|$ have good lower bounds to start with. We will establish recursive relations between $|A_{k+2}(M)|$ and $|A_k(M)|$, and between $|B_{k+2}(M)|$ and $|B_k(M)|$, which inductively prove the desired result. (Observe that $B_{2k}(M) = \emptyset$.)

We start with the following lemma.

Lemma 6.1. *Let m_i, a_i, b_i, c_i, d_i be nonzero integers with $0 < a_i < b_i < c_i$, $a_i + b_i + c_i = d_i$ and $(a_i, b_i, c_i) = 1$ $(i = 1, 2)$. If*

$$\left\{\frac{d_1 m_1}{a_1}, \frac{d_1 m_1}{b_1}, \frac{d_1 m_1}{c_1}\right\} = \left\{\frac{d_2 m_2}{a_2}, \frac{d_2 m_2}{b_2}, \frac{d_2 m_2}{c_2}\right\}, \tag{6.1}$$

then $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$ and $m_1 = m_2$.

Proof. Since $0 < a_i < b_i < c_i$ $(i = 1, 2)$, by (6.1) we have

$$\frac{d_1 m_1}{a_1} = \frac{d_2 m_2}{a_2}, \quad \frac{d_1 m_1}{b_1} = \frac{d_2 m_2}{b_2}, \quad \frac{d_1 m_1}{c_1} = \frac{d_2 m_2}{c_2}.$$

Thus

$$a_2 d_1 m_1 = a_1 d_2 m_2, \quad b_2 d_1 m_1 = b_1 d_2 m_2, \quad c_2 d_1 m_1 = c_1 d_2 m_2. \tag{6.2}$$

Hence

$$(a_2 d_1 m_1, b_2 d_1 m_1, c_2 d_1 m_1) = (a_1 d_2 m_2, b_1 d_2 m_2, c_1 d_2 m_2).$$

Since $(a_i, b_i, c_i) = 1$ $(i = 1, 2)$, we have $d_1 m_1 = d_2 m_2$. By (6.2) we have $a_1 = a_2$, $b_1 = b_2$ and $c_1 = c_2$. Thus $d_1 = d_2$. By $d_1 m_1 = d_2 m_2$ we have $m_1 = m_2$. ∎

The two lemmas below establish recursive relations between $|A_{k+2}(M)|$ and $|A_k(M)|$, and between $|B_{k+2}(M)|$ and $|B_k(M)|$.

Lemma 6.2. *Let $M_2 = 3^{20} \times 5^{20} \times 7^{20}$. Then*

$$|B_{k+2}(M_2)| \ge 62 |B_k(M_2)|.$$

Proof. If $|B_k(M_2)| = 0$, then the conclusion is clear. So we assume that $|B_k(M_2)| > 0$. By Lemma 6.1 we only need to find 62 four-tuples (a, b, c, d) to each $(x_1, \dots, x_k) \in B_k(M_2)$ with $a, b, c, d \in \{3^\alpha 5^\beta 7^\gamma\}$, $a + b + c = d$, $a < b < c$, $(a, b, c) = 1$, $a \mid dx_k$, $b \mid dx_k$, $c \mid dx_k$ and $M_2 \mid \frac{dx_k}{a}$. The reason is that

$$\frac{1}{x_k} = \frac{1}{dx_k/c} + \frac{1}{dx_k/b} + \frac{1}{dx_k/a}$$

and

$$x_k < dx_k/c < dx_k/b < dx_k/a.$$

By a simple Mathematica program we find that there are 62 four-tuples (a,b,c,d) with $a,b,c,d \in \{3^\alpha 5^\beta 7^\gamma : 0 \le \alpha \le 14,\ 0 \le \beta,\ \gamma \le 8\}$, $a+b+c=d$, $a<b<c$, $(a,b,c)=1$ and $a \mid d$. Since $M_2 = 3^{20} \times 5^{20} \times 7^{20}$ and $M_2 \mid x_k$, the conclusion follows immediately. ∎

Lemma 6.3. *Let $M_1 = 2^{20} \times 3^{20} \times 5^{20}$. Then*

$$|A_{k+2}(M_1)| \ge 368 |A_k(M_1)|.$$

Proof. By a simple Mathematica program we find that there are 368 four-tuples (a,b,c,d) with $a,b,c,d \in \{2^\alpha 3^\beta 5^\gamma : 0 \le \alpha \le 15,\ 0 \le \beta \le 10,\ 0 \le \gamma \le 8\}$, $a+b+c=d$, $a<b<c$, $(a,b,c)=1$ and $a \mid d$. The proof is now similar to the proof of Lemma 6.2. ∎

Proof of Theorem 1. (a) By [22] (see also [6]) there exist 11 odd numbers $1 < x_1 < \cdots < x_{11}$ with $x_{11} = 135$, $x_i \in \{3^\alpha 5^\beta 7^\gamma\}$ such that

$$\sum_{i=1}^{11} \frac{1}{x_i} = 1$$

(see the introduction). Since

$$\frac{1}{x_{11}} = \frac{1}{105x_{11}/(3^2 \times 7)} + \frac{1}{105x_{11}/3^3} + \frac{1}{105x_{11}/3^2} + \frac{1}{105x_{11}/5} + \frac{1}{105x_{11}},$$

there exist 15 odd numbers $1 < y_1 < \cdots < y_{15}$ with $y_{15} = 105x_{11}$, $y_i \in \{3^\alpha 5^\beta 7^\gamma\}$ such that

$$\sum_{i=1}^{15} \frac{1}{y_i} = 1.$$

Continuing this procedure, there exists an odd number k_0 such that $|B_{k_0}(M_2)| \ge 1$. By Lemma 6.2 we have

$$|B_k(M_2)| \ge \sqrt{62}^{\,k-k_0} |B_{k_0}(M_2)| \ge \sqrt{62}^{\,k-k_0}, \quad k \ge k_0,\ 2 \nmid k.$$

So

$$T_k(3,5,7) \ge \sqrt{62}^{\,k-k_0}, \quad k \ge k_0,\ 2 \nmid k.$$

Since

$$\frac{1}{x_{11}} = \frac{1}{5x_{11}/3} + \frac{1}{3x_{11}} + \frac{1}{15x_{11}},$$

there exist 13 odd numbers $1 < z_1 < \cdots < z_{13}$ with $z_{13} = 15x_{11}$, $z_i \in \{3^\alpha 5^\beta 7^\gamma\}$ such that

$$\sum_{i=1}^{13} \frac{1}{z_i} = 1.$$

Continuing this procedure, we have $T_k(3,5,7) \ge 1$ for all odd numbers $k \ge 11$. Hence there exists a positive constant c_1 such that $T_k(3,5,7) \ge c_1 \sqrt{62}^{\,k}$ for all odd numbers $k \ge 11$.

(b) Since

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{30}+\frac{1}{60}+\frac{1}{120}+\frac{1}{240}=1,$$
$$\frac{1}{a}=\frac{1}{30a/24}+\frac{1}{30a/3}+\frac{1}{30a/2}+\frac{1}{30a}$$

and

$$\frac{1}{a}=\frac{1}{3a/2}+\frac{1}{3a},$$

there exists an integer k_0 such that

$$|A_{2k_0}(M_1)|\geq 1,\quad |A_{2k_0+1}(M_1)|\geq 1.$$

By Lemma 6.3 we have

$$|A_{2k}(M_1)|\geq \sqrt{368}^{2k-2k_0}|A_{2k_0}(M_1)|\geq 368^{k-k_0},\qquad k\geq k_0,$$
$$|A_{2k+1}(M_1)|\geq \sqrt{368}^{2k-2k_0}|A_{2k_0+1}(M_1)|\geq 368^{k-k_0},\quad k\geq k_0.$$

Since

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{3^2}+\cdots+\frac{1}{3^k}+\frac{1}{2\times 3^k}=1,$$

we have $T_k(2,3,5)\geq 1$ for all $k\geq 3$. So there exists a positive constant c_2 such that $T_k(2,3,5)\geq c_2\sqrt{368}^k$ for all integers $k\geq 3$. ∎

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