ACTA ARITHMETICA
154.2 (2012)

L-functions at the origin and annihilation of class groups in multiquadratic extensions

by

JONATHAN W. SANDS (Burlington, VT)

I. Introduction. Fix an abelian Galois extension of number fields K/F and let G denote the Galois group. Also fix a finite set S of primes of F which contains all of the infinite primes of F and all of the primes which ramify in K. Since it is fixed throughout, we will often suppress S in the notation. Associated with this data is an equivariant L-function, $\theta_{K/F}(s) = \theta^S_{K/F}(s)$, a meromorphic function of $s \in \mathbb{C}$ with values in the group ring $\mathbb{C}[G]$. When the real part of s is greater than 1 it is defined as a product over the (finite) primes $\mathfrak{p}$ of F that are not in S. Let $\mathrm{N}\mathfrak{p}$ denote the absolute norm of the ideal $\mathfrak{p}$ and $\sigma_{\mathfrak{p}} \in G$ denote the Frobenius automorphism of $\mathfrak{p}$. Then

$$\theta^S_{K/F}(s) = \prod_{\text{prime}\,\mathfrak{p}\notin S} \left(1 - \frac{1}{\mathrm{N}\mathfrak{p}^s}\sigma_{\mathfrak{p}}^{-1}\right)^{-1}.$$

Each component of this function extends meromorphically to all of $\mathbb{C}$, and its behavior at $s = 0$ is connected with the arithmetic of K.

The ring of S-integers $\mathcal{O}^S_F$ of F is defined to be the set of elements of F whose valuation is non-negative at every prime not in S. When $K = F$, the function $\theta^S_{F/F}(s)$ is simply the identity automorphism of F times $\zeta^S_F(s)$, the Dedekind zeta-function of F with Euler factors for the primes in S removed. The function $\zeta^S_F(s)$ may be viewed as the zeta-function of the Dedekind domain $\mathcal{O}^S_F$.

Letting S_K denote the set of primes of K lying above those in S, we define $\mathcal{O}^S_K$ to be the ring of S_K-integers of K. Then Cl^S_K denotes the S_K-class group of K, which may be identified with the group of non-zero fractional ideals of $\mathcal{O}^S_K$ modulo principal fractional ideals. Denote the order of Cl^S_K by h^S_K. Let μ_K denote the group of all roots of unity in K, and w_K de-

2010 *Mathematics Subject Classification*: Primary 11R29, 11R42.
Key words and phrases: Artin L-function, class group, multiquadratic extension, Stickelberger ideal.

DOI: 10.4064/aa154-2-5

note its order. When the Brumer–Stark conjecture holds, it implies that $w_K\theta^S_{K/F}(0)$ annihilates Cl^S_K as a module over the group-ring $\mathbb{Z}[G]$. However, this conjecture is vacuous when $\theta^S_{K/F}(0) = 0$. On the other hand, one knows that for $K = F$, the leading term in the Taylor series at $s = 0$ for ζ^S_F is $\zeta^{S,*}_F = -h^S_F R^S_F/w_F$, where R^S_F is the regulator of the S-units of F. One sees that this quantity still provides an annihilator $-h^S_F$ for Cl^S_F, upon removing the factors R^S_F and w_F which relate to the group of S-units and its torsion subgroup. In this paper, we obtain results on the annihilation of Cl^S_K by what may be considered the leading term of $\theta^S_{K/F}(s)$ at $s = 0$. Indeed, we obtain a non-trivial annihilator associated with each irreducible character of G, regardless of the order of vanishing of the corresponding L-function. Such results are clearly related to the refined Stark conjectures of Rubin and Popescu, but those do not directly concern annihilators for Cl^S_K. The connection between leading terms of equivariant L-functions and annihilators of class groups appears in more recent conjectures of Burns [2] growing out of his work with Flach on the Equivariant Tamagawa Number Conjecture [3], and results of Buckingham [1] which had their origins in ideas of Snaith [6].

To state our results, let $\hat{G}$ denote the group of characters of G and recall that the *S-imprimitive Artin L-function* for a character $\psi \in \hat{G}$ is defined as

$$L^S_{K/F}(s,\psi) = \prod_{\text{prime}\,\mathfrak{p}\notin S} \left(1 - \frac{1}{\mathrm{N}\mathfrak{p}^s}\psi(\sigma_{\mathfrak{p}})\right)^{-1},$$

so that using the idempotents $e_\psi = |G|^{-1}\sum_{\sigma\in G}\psi(\sigma)\sigma^{-1}$, we have

$$\theta^S_{K/F}(s) = \sum_{\psi\in\hat{G}} L^S_{K/F}(s,\psi^{-1})e_\psi.$$

Defining $L^*_{K/F}(\psi) = L^{S,*}_{K/F}(\psi)$ to be the first non-zero coefficient in the Taylor series for $L^S_{K/F}(s,\psi)$ at $s = 0$, one then puts

$$\theta^*_{K/F} = \theta^{S,*}_{K/F} = \sum_{\psi\in\hat{G}} L^{S,*}_{K/F}(\psi^{-1})e_\psi.$$

Next define a regulator as in Burns [2]. For each prime $w \in S_K$, let $|\ |_w$ denote the corresponding normalized absolute value on K. Let $U_K = U^S_K = (\mathcal{O}^S_K)^*$, the multiplicative group of S_K-units in K. Let Y^S_K be the free abelian group on primes in S_K. This has a natural G-action which makes it a $\mathbb{Z}[G]$-module. The submodule $X_K = X^S_K$ is the kernel of the augmentation homomorphism $Y_K = Y^S_K \to \mathbb{Z}$ which sends each element to the sum of its coefficients. Then $\mathbb{R}U^S_K = \mathbb{R}\otimes_{\mathbb{Z}} U^S_K$ is known to be isomorphic to $\mathbb{R}X^S_K = \mathbb{R}\otimes_{\mathbb{Z}} X^S_K$ by the $\mathbb{R}$-linear extension $\lambda_{K,\mathbb{R}} = \lambda^S_{K,\mathbb{R}}$ of the map

$\lambda_K = \lambda_K^S : U_K^S \to \mathbb{R}X_K^S$ defined by

$$\lambda_K(u) = -\sum_{w\in S_K} \log|u|_w \cdot w.$$

Any $\mathbb{Z}[G]$-module homomorphism $f : M \to N$, determines an $\mathbb{R}[G]$-module homomorphism

$$f_{\mathbb{R}} : \mathbb{R}M \to \mathbb{R}N$$

by extension of scalars. In particular, suppose that we fix a $\mathbb{Z}[G]$-module homomorphism $f : U_K^S \to X_K^S$. Since $\mathbb{R}[G]$ is a semisimple commutative ring and $\mathbb{R}U_K^S$ is finitely generated as a module over this ring, there exists a complementary $\mathbb{R}[G]$-module P such that $\mathbb{R}U_K^S \oplus P$ is a finitely generated free module. Using the identity map 1_P on P, one then obtains a well-defined regulator of f in $\mathbb{R}[G]$:

$$R(f) = \det_{\mathbb{R}[G]}(\lambda_{K,\mathbb{R}}^{-1} \circ f_{\mathbb{R}}) = \det_{\mathbb{R}[G]}((\lambda_{K,\mathbb{R}}^{-1} \circ f_{\mathbb{R}}) \oplus 1_P).$$

Let $r^S(\psi) = r(\psi)$ denote the dimension of the $\mathbb{R}$-vector space $e_\psi \mathbb{R}U_K^S$.

Our main result is the following theorem, proved in a slightly stronger form as Theorem 4.5 at the end of this paper. Remark 4.6 indicates how it may be strengthened further.

Main Theorem. *Let K be a composite of a finite number of quadratic extensions of a number field F. Let S contain the infinite primes of F and those which ramify in K/F. Suppose that $f : U_K^S \to X_K^S$ is a $\mathbb{Z}[G]$-module homomorphism with* $\ker(f)$ *finite. Let $\alpha \in \mathbb{Z}[G]$ annihilate μ_K, and let ψ be an irreducible character of G. Then $|G|^{r^S(\psi)+1}\alpha R(f)\theta_{K/F}^{S,*}e_\psi$ lies in $\mathbb{Z}[G]$ and annihilates* Cl_K^S.

Remark 1.1. Burns [2] obtains more general results of this form, considering components of the units and of X_K^S for each character separately. His Conjecture 2.6.1 and evidence for it (which includes the multiquadratic extensions considered here) then involves an additional factor of $|G|^2$ in the resulting annihilator. Macias Castillo [5] obtains stronger results specifically for multiquadratic extensions such as those considered here, but not for all characters. We have chosen to show what can be done working with U_K^S; Burns and Macias Castillo (and others) formulate their results in terms of certain torsion-free subgroups of U_K^S. In a subsequent paper, we will detail the connections between their work and ours more fully.

Remark 1.2. The principal Stark conjecture [7] states that

$$|G|^{r^S(\psi)+1}\alpha R(f)\theta_{K/F}^{S,*}e_\psi$$

lies in $\mathbb{Q}[G]$, and is already known in the case of multiquadratic extensions.

II. Computing $\theta^*_{K/F} = \theta^{S,*}_{K/F}$**.** From now on, we will omit the set of primes S from our notation. So $Y_K = Y_K^S$, $X_K = X_K^S$, $U_K = U_K^S$, $h_F = h_F^S$, $r(\psi) = r^S(\psi)$, $R_F = R_F^S$, $L_{K/F} = L_{K/F}^S$, $\zeta_F = \zeta_F^S$, and $\theta_{K/F} = \theta_{K/F}^S$, etc.

PROPOSITION 2.1. *For the principal character* ψ_0 *of* $\mathrm{Gal}(K/F)$*, we have*

$$\theta^*_{K/F} e_{\psi_0} = \frac{-h_F R_F}{w_F} e_{\psi_0}.$$

Proof. Since ψ_0 is the inflation of the trivial character on $\mathrm{Gal}(F/F)$, the functorial properties of Artin L-functions give

$$\theta^*_{K/F} e_{\psi_0} = L^*_{K/F}(\psi_0) e_{\psi_0} = \zeta^*_F e_{\psi_0}.$$

The result then follows from the analytic class number formula.

Now assume that $G = \mathrm{Gal}(K/F)$ has exponent 2, and let ψ be a non-trivial character of G. The image of ψ is a non-trivial cyclic group of exponent 2, hence of order 2. So $\ker(\psi)$ has index 2 in G. Let E_ψ denote the fixed field of $\ker(\psi)$, a relative quadratic extension of F. Let $C_{E_\psi/F}$ denote the cokernel of the natural map from Cl_F to Cl_{E_ψ} that is induced by extension of ideals. Let τ_ψ denote the generator of $\mathrm{Gal}(E_\psi/F)$. We will have occasion to fix a lift of τ_ψ to an element of G, which we also denote by τ_ψ. If M is a $\mathbb{Z}[G]$-module and $\alpha \in \mathbb{Z}[G]$, we let M^α denote the image of M under multiplication by α, and M_α denote the kernel of multiplication by α.

PROPOSITION 2.2.

$$\theta^*_{K/F} e_\psi = \frac{|C_{E_\psi/F}|}{((U_{E_\psi})_{1+\tau_\psi} : (U_{E_\psi})^{1-\tau_\psi})} \frac{R_{E_\psi}}{R_F} \frac{w_F}{w_{E_\psi}} e_\psi.$$

Proof. First, ψ is induced from the non-trivial character of $\mathrm{Gal}(E_\psi/F)$, and this character is the difference between the regular representation of $\mathrm{Gal}(E_\psi/F)$ and the trivial character. The functorial properties of Artin L-functions and the analytic class number formula then give

$$\theta^*_{K/F} e_\psi = L^*_{K/F}(\psi) e_\psi = \frac{\zeta^*_{E_\psi}(0)}{\zeta^*_F(0)} e_\psi = \frac{h_{E_\psi}}{h_F} \frac{R_{E_\psi}}{R_F} \frac{w_F}{w_{E_\psi}} e_\psi.$$

A computation of Tate ([7, Thm. IV.5.4]) then shows that

$$\frac{h_{E_\psi}}{h_F} = \frac{|C_{E_\psi/F}|}{((U_{E_\psi})_{1+\tau_\psi} : (U_{E_\psi})^{1-\tau_\psi})},$$

and this completes the proof.

III. Computing $R(f)$

LEMMA 3.1. *Suppose that* ϕ *is an endomorphism of a finitely generated projective* R*-module* M*.*

(a) *If R' is an overring of R, let $M' = R' \otimes_R M$ and $\phi' = 1_{R'} \otimes_R \phi$, an endomorphism of M'. Then* $\det_{R'}(\phi') = \det_R(\phi)$.

(b) *If $R = R_1 \oplus R_2$, then consequently $M = M_1 \oplus M_2$ where M_1 is a finitely generated projective R_1-module and M_2 is a finitely generated projective R_2-module, and $\phi = \phi_1 \oplus \phi_2$ for ϕ_1 an endomorphism of M_1 and ϕ_2 an endomorphism of M_2. Then* $\det_R(\phi) = (\det_{R_1}(\phi_1), \det_{R_2}(\phi_2)) \in R_1 \oplus R_2 = R$. *Using $1 = e_1 + e_2$ where e_1 and e_2 are idempotents of R lying in R_1 and R_2 respectively, this may be written as* $\det_R(\phi) = \det_{R_1}(\phi_1)e_1 + \det_{R_2}(\phi_2)e_2$.

Proof. (a) Choose P so that $M \oplus P$ is a finitely generated free R-module with basis $\{b_1, \ldots, b_k\}$, and let $P' = R' \otimes_R P$. Then $M' \oplus P' \cong R' \otimes_R (M \oplus P)$ is a finitely generated free R'-module with basis $\{b'_1 = 1 \otimes b_1, \ldots, b'_k = 1 \otimes b_k\}$. Using these bases, it is clear that the matrix of $\phi \oplus 1_P$ is the same as the matrix of $\phi' \oplus 1_{P'}$, as the latter may be identified with $1_{R'} \otimes (\phi \oplus 1_P)$. Thus $\det_{R'}(\phi') = \det_{R'}(\phi' \oplus 1_{P'}) = \det_R(\phi \oplus 1_P) = \det_R(\phi)$.

(b) Note that $M_1 = e_1 M$ and $M_2 = e_2 M$. After choosing P so that $M \oplus P$ is a finitely generated free R-module, we see that $e_1(M \oplus P) = e_1 M \oplus e_1 P = M_1 \oplus e_1 P$ is a finitely generated free R_1-module, making M_1 a finitely generated projective R_1-module, and similarly M_2 is a finitely generated projective R_2-module. Choosing a basis $\{b_1, \ldots, b_k\}$ for $M \oplus P$ over R clearly gives a basis $\{e_1 b_1, \ldots, e_1 b_k\}$ for $e_1 M \oplus e_1 P$ over R_1, and the case of $e_2 M \oplus e_2 P$ is similar. Now if $(r_{i,j}) = (e_1 r_{i,j}) + (e_2 r_{i,j})$ is the matrix of $\phi \oplus 1_P$, then $(e_1 r_{i,j})$ is the matrix of $\phi_1 \oplus 1_{P_1}$ over R_1, and similarly for $\phi_2 \oplus 1_{P_2}$. Thus

$$\begin{aligned} \det_R(\phi) &= \det(r_{i,j}) = (e_1 + e_2)\det(r_{i,j}) \\ &= \det(e_1 r_{i,j}) + \det(e_2 r_{i,j}) = \det_{R_1}(\phi_1)e_1 + \det_{R_2}(\phi_2). \end{aligned}$$

Proposition 3.2.

(a) *The following are equivalent:*

(1) $\ker(f)$ *is finite,*

(2) $\ker(f) = \mu_K$,

(3) $\operatorname{coker}(f)$ *is finite,*

(4) $f_{\mathbb{R}}$ *is an isomorphism,*

(5) $R(f) \in \mathbb{R}[G]^*$.

(b) *We have the following equalities, the last one requiring that one of the equivalent conditions in* (a) *hold* (*note that* $\mathbb{C}[G]e_\psi = \mathbb{C}e_\psi \cong \mathbb{C}$):

$$\begin{aligned} R(f) &= \det_{\mathbb{R}[G]}(f_{\mathbb{R}} \circ \lambda_{\mathbb{R}}^{-1}) = \det_{\mathbb{C}[G]}(f_{\mathbb{C}} \circ \lambda_{\mathbb{C}}^{-1}) \\ &= \sum_{\psi \in \hat{G}} \det_{\mathbb{C}[G]e_\psi}(f_{\mathbb{C}} \circ \lambda_{\mathbb{C}}^{-1}|_{e_\psi \mathbb{C} X_K^S}) = \sum_{\psi \in \hat{G}} \det_{\mathbb{C}[G]e_\psi}(\lambda_{\mathbb{C}} \circ f_{\mathbb{C}}^{-1}|_{e_\psi \mathbb{C} X_K^S})^{-1}. \end{aligned}$$

(c) *When G has exponent* 2, *we have*

$$R(f)=\sum_{\psi\in\hat{G}}\det_{\mathbb{R}[G]e_\psi}(f_{\mathbb{R}}\circ\lambda_{\mathbb{R}}^{-1}|_{e_\psi\mathbb{R}X_K^S})=\sum_{\psi\in\hat{G}}\det_{\mathbb{R}[G]e_\psi}(\lambda_{\mathbb{R}}\circ f_{\mathbb{R}}^{-1}|_{e_\psi\mathbb{R}X_K^S})^{-1}.$$

Proof. (a) These are clear because μ_K is the torsion subgroup of U_K^S, while U_K^S/μ_K and X_K^S are free abelian groups of the same rank.

(b) This follows from Lemma 3.1.

(c) This follows from part (b) and Lemma 3.1(a).

When G has exponent 2, it remains for us to compute

$$\det_{\mathbb{R}[G]e_\psi}(\lambda_{\mathbb{R}}\circ f_{\mathbb{R}}^{-1}|_{e_\psi\mathbb{R}X_K})$$

for each $\psi\in\hat{G}$. To do this, suppose that E is an intermediate field between F and K, and $H=\mathrm{Gal}(K/E)$. Let $N_H=\sum_{\sigma\in H}\sigma$. For $w\in S_E$, let $\tilde{w}\in S_K$ be a choice of a prime above w in K. There is a natural injective $\mathbb{Z}[G]$-module map $Y_E\to Y_K$ which sends each $w\in S_E$ to $N_H\tilde{w}$. We let $\gamma_{K/E}:X_E\to X_K$ denote the restriction of this map to X_E. Similarly, let $\pi_{K/E}$ be the restriction to X_K of the $\mathbb{Z}[G]$-module map which sends each prime $\tilde{w}\in S_K$ to the corresponding prime w of E, and note that the image of $\pi_{K/E}$ lies in X_E. It is easy to see that $\gamma_{K/E}$ gives an isomorphism between X_E^S and $N_H(X_K^S)$, and that for $u\in U_E$, we have $\lambda_K(u)=\gamma_{K/E,\mathbb{R}}(\lambda_E(u))$.

LEMMA 3.3. *Suppose that* $\ker(f)$ *is finite. Let* $\pi_{G/H}:\mathbb{R}[G]\to\mathbb{R}[G/H]$ *be the natural projection map. If* χ *is a first degree character of* $\overline{G}=G/H$ *and* $\psi\in\hat{G}$ *is its inflation, recall that* $r(\chi)$ *denotes the dimension of* $e_\chi\mathbb{R}X_E$ *as a real vector space. Then*

$$\pi_{G/H}(R(f)e_\psi)=|H|^{-r(\chi)}R(\pi_{K/E}\circ f|_{U_E})e_\chi.$$

Proof. (See [7, I.6.4(3)].) By Proposition 3.2(c),

$$\pi_{G/H}(R(f)^{-1}e_\psi)=\pi_{G/H}(\det_{\mathbb{R}[G]e_\psi}(\lambda_{K,\mathbb{R}}\circ f_{\mathbb{R}}^{-1}|_{e_\psi\mathbb{R}X_K})).$$

Since $\gamma_{K/E}(X_E)\subset N_H(X_K)$, and $f_{\mathbb{R}}^{-1}$ is an $\mathbb{R}[G]$-homomorphism, we see that the image of $f_{\mathbb{R}}^{-1}\circ\gamma_{K/E,\mathbb{R}}^S$ is contained in $N_H(\mathbb{R}U_K)\subset\mathbb{R}U_E$. Thus we may follow this map with $\gamma_{K/E,\mathbb{R}}\circ\lambda_{E,\mathbb{R}}=\lambda_{K,\mathbb{R}}|_{\mathbb{R}U_E}$ and obtain

$$\gamma_{K/E,\mathbb{R}}\circ\lambda_{E,\mathbb{R}}\circ f_{\mathbb{R}}^{-1}\circ\gamma_{K/E,\mathbb{R}}=\lambda_{K,\mathbb{R}}^S|_{\mathbb{R}U_E}\circ f_{\mathbb{R}}^{-1}\circ\gamma_{K/E,\mathbb{R}}.$$

Restricting the isomorphism $\gamma_{K/E,\mathbb{R}}:\mathbb{R}X_E\to N_H(\mathbb{R}X_K)$ gives an isomorphism between $e_\chi\mathbb{R}X_E=e_\psi\mathbb{R}X_E$ and $e_\psi N_H(\mathbb{R}X_K)=|H|e_\psi\mathbb{R}X_K=e_\psi\mathbb{R}X_K$. So, restricting the functions in the last displayed equation to $e_\chi\mathbb{R}X_E$ and noting that $\pi_{G/H}(e_\psi)=e_\chi$, we get

$$\det_{\mathbb{R}[\overline{G}]e_\chi}(\lambda_{E,\mathbb{R}}\circ(f_{\mathbb{R}}^{-1}\circ\gamma_{K/E,\mathbb{R}})|_{e_\chi\mathbb{R}X_E})=\pi_{G/H}(\det_{\mathbb{R}[G]e_\psi}(\lambda_{K,\mathbb{R}}\circ f_{\mathbb{R}}^{-1}|_{e_\psi\mathbb{R}X_K})).$$

Since $\gamma_{K/E,\mathbb{R}}|_{e_\chi\mathbb{R}X_E} : e_\chi\mathbb{R}X_E \to e_\psi\mathbb{R}X_K$ has the inverse $|H|^{-1}\pi_{K/E,\mathbb{R}}|_{e_\psi\mathbb{R}X_K}$, we deduce from Proposition 3.2(c) again that

$$\begin{aligned}\det{}_{\mathbb{R}[\overline{G}]e_\chi}(\lambda_{E,\mathbb{R}} \circ (f_{\mathbb{R}}^{-1} \circ \gamma_{K/E,\mathbb{R}})|_{e_\chi\mathbb{R}X_E})\\ &= \det{}_{\mathbb{R}[\overline{G}]e_\chi}\left(\lambda_{E,\mathbb{R}} \circ \left(\frac{1}{|H|}\pi_{K/E} \circ f|_{U_E}\right)_{\mathbb{R}}^{-1}\Bigg|_{e_\chi\mathbb{R}X_E}\right)\\ &= |H|^{r(\chi)}R(\pi_{K/E} \circ f|_{U_E})^{-1}e_\chi.\end{aligned}$$

Combining the displayed equations gives the result.

LEMMA 3.4. *Suppose that E/F is relative quadratic and τ is the non-trivial automorphism of E over F. Let χ be the non-trivial character of $\overline{G} = \mathrm{Gal}(E/F) = \langle\tau\rangle$. If $\overline{f} : U_E \to X_E$ is a $\mathbb{Z}[\overline{G}]$-module homomorphism with finite kernel, then*

$$R(\overline{f})e_\chi = ((X_E)_{1+\tau} : \overline{f}((U_E)^{1-\tau}))\frac{R_F}{R_E}\,\frac{w_E}{w_F}\,\frac{2^{|S|-1-r(\chi)}}{|\mu_E \cap (U_E)^{1-\tau}|}.$$

Proof. Let $M = (X_E : \overline{f}(U_E))$, and let $\overline{f}_0 : U_E/\mu_E \to \overline{f}(U_E)$ be the induced isomorphism. Then the composite

$$g : X_E \xrightarrow{M} \overline{f}(U_E) \xrightarrow{\overline{f}_0^{-1}} U_E/\mu_E \xrightarrow{w_E} U_E$$

is an injective $\mathbb{Z}[G]$-module map. For such a map, Tate ([7, I.6.3]) defines $R(\chi, g)$, and it is easy to see that the definition is equivalent to

$$R(\chi, g)e_\chi = \det{}_{e_\chi\mathbb{R}[\overline{G}]}(\lambda_{E,\mathbb{R}} \circ g_{\mathbb{R}}|_{e_\chi\mathbb{R}X_E}).$$

By Proposition 3.2(a), $\overline{f}_{\mathbb{R}}$ is an isomorphism, and it is then clear from our definition of g that $g_{\mathbb{R}} = Mw_E\overline{f}_{\mathbb{R}}^{-1}$. Since $r(\chi)$ equals the dimension of $e_\chi\mathbb{R}X_E$ as a real vector space, we see from Proposition 3.2(b) that

$$R(\chi, g)e_\chi = (Mw_E)^{r(\chi)}\det{}_{\mathbb{R}[\overline{G}]e_\chi}(\lambda_{E,\mathbb{R}} \circ \overline{f}_{\mathbb{R}}^{-1} \circ |_{e_\chi\mathbb{R}X_E}) = (Mw_E)^{r(\chi)}R(\overline{f})^{-1}e_\chi.$$

On the other hand, the proof of [7, Prop. II.2.1] gives

$$R(\chi, g) = \frac{w_F}{w_E}\,\frac{R_E}{R_F}\,\frac{((U_E)^{1-\tau} : g((X_E)_{1+\tau})^2)}{2^{|S|-1}}.$$

As an abelian group, $(U_E)^{1-\tau}$ is the direct product of its torsion subgroup $(U_E)^{1-\tau} \cap \mu_E$ and a free abelian group of rank $r(\chi)$. Using this and the definition of g, we have

$$\begin{aligned}((U_E)^{1-\tau} : g((X_E)_{1+\tau})^2) &= \frac{((U_E)^{1-\tau} : ((U_E)^{1-\tau})^{2Mw_E})}{(g((X_E)_{1+\tau})^2 : ((U_E)^{1-\tau})^{2Mw_E})}\\ &= \frac{|(U_E)^{1-\tau} \cap \mu_E|(2Mw_E)^{r_s(\chi)}}{(\overline{f}^{-1}(M(X_E)_{1+\tau})^{2w_E} : ((U_E)^{1-\tau})^{2Mw_E})}.\end{aligned}$$

Now $\overline{f}^{-1}(M(X_E)_{1+\tau})^{2w_E}$ is torsion-free and hence $\overline{f}$ is injective on this submodule, so we have

$$\overline{f}^{-1}(M(X_E)_{1+\tau})^{2w_E}/((U_E)^{1-\tau})^{2Mw_E} \cong 2Mw_E(X_E)_{1+\tau}/2Mw_E\overline{f}((U_E)^{1-\tau}).$$

Then since X_E is $\mathbb{Z}$-torsion-free,

$$\begin{aligned}(&\overline{f}^{-1}(M(X_E)_{1+\tau})^{2w_E} : ((U_E)^{1-\tau})^{2Mw_E})\\ &= (2Mw_E(X_E)_{1+\tau} : 2Mw_E f((U_E)^{1-\tau})) = ((X_E)_{1+\tau} : \overline{f}((U_E)^{1-\tau})).\end{aligned}$$

Combining the displayed equations gives the result.

PROPOSITION 3.5. *Suppose that $G = \mathrm{Gal}(K/F)$ has exponent 2, ψ is a non-trivial character of G, and $f : U_K \to X_K$ is a $\mathbb{Z}[G]$-module homomorphism with finite kernel. Then*

$$R(f)e_\psi = \frac{2^{|S|-1}}{|G|^{r(\psi)}}\,\frac{w_{E_\psi}}{w_F}\,\frac{R_F}{R_{E_\psi}}\,\frac{((X_{E_\psi})_{1+\tau_\psi} : (\pi_{K/E_\psi}\circ f)((U_{E_\psi})^{1-\tau_\psi}))}{|(U_{E_\psi})^{1-\tau_\psi}\cap \mu_{E_\psi}|}e_\psi.$$

Proof. Let $E = E_\psi$ and $H = \ker(\psi) = \mathrm{Gal}(K/E)$. Then ψ is the inflation of the non-trivial character χ on $G/H \cong \mathrm{Gal}(E/F) = \overline{G}$. Since $\pi_{G/H}$ restricts to an $\mathbb{R}$-module isomorphism from $\mathbb{R}[G]e_\psi = \mathbb{R}e_\psi$ to $\mathbb{R}[\overline{G}]e_\chi = \mathbb{R}e_\chi$ with $\pi_{G/H}(e_\psi) = e_\chi$, the result follows directly from Lemmas 3.3 and 3.4.

LEMMA 3.6. *For the trivial extension F/F, with identity automorphism σ_0, and $\overline{f} : U_F \to X_F$ with finite kernel, we have*

$$R(\overline{f}) = \pm\frac{(X_F : \overline{f}(U_F))}{R_F}\sigma_0.$$

Proof. Let $M = (X_F : \overline{f}(U_F))$, and let $\overline{f}_0 : U_F/\mu_F \to \overline{f}(U_F)$ be the induced isomorphism. Then the composite

$$g : X_F \xrightarrow{M} \overline{f}(U_F) \xrightarrow{\overline{f}_0^{-1}} U_F/\mu_F \xrightarrow{w_F} U_F$$

is an injective $\mathbb{Z}$-module map. Therefore, as in the proof of Lemma 3.4,

$$\begin{aligned}R(1,g) &= \det_{\mathbb{R}}(\lambda_{F,\mathbb{R}}\circ g_{\mathbb{R}}) = (Mw_F)^{|S|-1}\det_{\mathbb{R}}(\lambda_{F,\mathbb{R}}\circ \overline{f}_{\mathbb{R}}^{-1})\\ &= (Mw_F)^{|S|-1}R(\overline{f})^{-1}.\end{aligned}$$

On the other hand, the proof of [7, Prop. II.1.1] gives

$$R(1,g) = \pm\frac{R_F}{w_F}(U_F : g(X_F)).$$

As an abelian group, U_F is the direct product of its torsion subgroup μ_F and a free abelian group of rank $|S|-1$. Using this and the definition of g, we have

$$(U_F : g(X_F)) = \frac{(U_F : (U_F)^{Mw_F})}{(g(X_F) : (U_F)^{Mw_F})} = \frac{w_F(Mw_F)^{|S|-1}}{(\overline{f}^{-1}(MX_F)^{w_F} : (U_F)^{Mw_F})}.$$

Now $\overline{f}^{-1}(MX_F)^{w_F}$ is $\mathbb{Z}$-torsion-free and hence $\overline{f}$ is injective on this submodule, so we have $\overline{f}^{-1}(MX_F)^{w_F}/(U_F)^{Mw_F} \cong Mw_F(X_F)/Mw_F\overline{f}(U_F)$. Then since X_F is $\mathbb{Z}$-torsion-free,

$$(\overline{f}^{-1}(MX_F)^{w_F} : (U_F)^{Mw_F}) = (Mw_FX_F : Mw_F\overline{f}(U_F)) = (X_F : \overline{f}(U_F)).$$

Combining the displayed equations gives the result.

PROPOSITION 3.7. *Suppose that* $G = \mathrm{Gal}(K/F)$ *has exponent* 2, ψ_0 *is the trivial character of* G, *and* $f : U_K \to X_K$ *is a* $\mathbb{Z}[G]$*-module homomorphism with finite kernel. Then*

$$R(f)e_{\psi_0} = \frac{(X_F : \pi_{K/F} \circ f(U_F))}{|G|^{|S|-1}R_F}e_{\psi_0}.$$

Proof. Since ψ_0 is the inflation of the trivial character χ_0 on $\mathrm{Gal}(F/F)$, and $\pi_{G/G}$ restricts to an $\mathbb{R}$-module isomorphism from $\mathbb{R}[G]e_{\psi_0} = \mathbb{R}e_{\psi_0}$ to $\mathbb{R}\sigma_0$ with $\pi_{G/G}(e_{\psi_0}) = \sigma_0$, the result follows from Lemmas 3.3 and 3.6.

IV. Class group annihilators

PROPOSITION 4.1. *Suppose that* $G = \mathrm{Gal}(K/F)$ *has exponent* 2 *and that* $f : U_K \to X_K$ *is a* $\mathbb{Z}[G]$*-module homomorphism with finite kernel. Then*

$$\begin{aligned} R(f)\theta^*_{K/F} = {} & \frac{h_F(X_F : \pi_{K/F}(f(U_F)))}{w_F|G|^{|S|-1}}e_{\psi_0} \\ & + \sum_{\psi\neq\psi_0} \frac{2^{|S|-1}|C_{E_\psi/F}|}{|G|^{r^S(\psi)}}\, \frac{((X_{E_\psi})_{1+\tau_\psi} : \pi_{K/E_\psi}(f((U_{E_\psi})_{1+\tau_\psi})))}{|(\mu_{E_\psi})_{1+\tau_\psi}|}e_\psi. \end{aligned}$$

Proof. Combining Propositions 2.1 and 3.7 gives the coefficient of e_{ψ_0}. Using Propositions 2.2 and 3.5 for $\psi \neq \psi_0$ yields

$$\begin{aligned} & R(f)\theta^*_{K/F}e_\psi \\ & = \frac{2^{|S|-1}}{|G|^{r^S(\psi)}}\, \frac{|C_{E_\psi/F}|}{|(U_{E_\psi})^{1-\tau_\psi} \cap \mu_{E_\psi}|}\, \frac{((X_{E_\psi})_{1+\tau_\psi} : \pi_{K/E_\psi}(f((U_{E_\psi})^{1-\tau_\psi})))}{((U_{E_\psi})_{1+\tau_\psi} : (U_{E_\psi})^{1-\tau_\psi})}e_\psi. \end{aligned}$$

Then

$$\begin{aligned} & ((X_{E_\psi})_{1+\tau_\psi} : \pi_{K/E_\psi}(f((U_{E_\psi})^{1-\tau_\psi}))) \\ & \qquad = ((X_{E_\psi})_{1+\tau_\psi} : \pi_{K/E_\psi}(f((U_{E_\psi})_{1+\tau_\psi}))) \\ & \qquad\quad \times (\pi_{K/E_\psi}(f((U_{E_\psi})_{1+\tau_\psi})) : \pi_{K/E_\psi}(f((U_{E_\psi})^{1-\tau_\psi}))). \end{aligned}$$

Now consider the kernel of $\pi_{K/E_\psi} \circ f$ restricted to U_{E_ψ}. So let $u \in U_{E_\psi}$ and $f(u) = \sum_{w\in S_K} n_w w$. Since $\sigma(u) = u$ for $\sigma \in H = \mathrm{Gal}(K/E_\psi)$, we have $n_w = n_{\sigma(w)}$ for each w. Fix a set of representatives $\{w_i\}$, one for each distinct orbit of S_K under the action of H, and write $w_i \sim w$ if w_i and w

lie in the same orbit with cardinality d_i. Then

$$f(u) = \sum_i \sum_{w\sim w_i} n_w w = \sum_i \sum_{w\sim w_i} n_{w_i} w = \sum_i n_{w_i} \sum_{w\sim w_i} w$$

and

$$\begin{aligned}\pi_{K/E_\psi}(f(u)) &= \sum_i n_{w_i} \sum_{w\sim w_i} \pi_{K/E_\psi}(w) = \sum_i n_{w_i} \sum_{w\sim w_i} \pi_{K/E_\psi}(w_i) \\ &= \sum_i n_{w_i} d_i \pi_{K/E_\psi}(w_i).\end{aligned}$$

Since the elements $\pi_{K/E_\psi}(w_i)$ are distinct, the above is zero if and only if each n_{w_i} is zero and hence $f(u) = 0$. Our assumption on f implies that this holds if and only if $u \in \mu_K$. So the kernel of $\pi_{K/E_\psi} \circ f$ restricted to U_{E_ψ} is clearly μ_{E_ψ}. Thus $\pi_{K/E_\psi} \circ f$ induces a homomorphism from $(U_{E_\psi})_{1+\tau_\psi}/(U_{E_\psi})^{1-\tau_\psi}$ onto $\pi_{K/E_\psi}(f((U_{E_\psi})_{1+\tau_\psi}))/\pi_{K/E_\psi}(f((U_{E_\psi})^{1-\tau_\psi}))$ with kernel $(\mu_{E_\psi})_{1+\tau_\psi}/(U_{E_\psi})^{1-\tau_\psi} \cap (\mu_{E_\psi})_{1+\tau_\psi}$. Consequently,

$$\begin{aligned}&\frac{((U_{E_\psi})_{1+\tau_\psi} : (U_{E_\psi})^{1-\tau_\psi})}{((\mu_{E_\psi})_{1+\tau_\psi} : (U_{E_\psi})^{1-\tau_\psi} \cap (\mu_{E_\psi})_{1+\tau_\psi})} \\ &\qquad = (\pi_{K/E_\psi}(f((U_{E_\psi})_{1+\tau_\psi})) : \pi_{K/E_\psi}(f((U_{E_\psi})^{1-\tau_\psi}))).\end{aligned}$$

Combining the displayed equations then gives the result.

LEMMA 4.2. *Suppose that $\alpha \in \mathrm{Ann}_{Z[G]}(\mu_K)$ and that G is the direct product of its subgroups H and J. Let M be the fixed field of H, and identify J with* Gal(M/F) *by restriction. Then $\alpha N_H = \beta N_H$ for some $\beta \in \mathrm{Ann}_{\mathbb{Z}[J]}(\mu_M)$.*

Proof. Write

$$\alpha = \sum_{\rho\in J} \sum_{\sigma\in H} n_{\rho\sigma} \rho\sigma \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K).$$

Restricting to M, we define

$$\beta = \sum_{\rho\in J} \Big(\sum_{\sigma\in H} n_{\rho\sigma}\Big)\rho \in \mathrm{Ann}_{\mathbb{Z}[J]}(\mu_M).$$

Note that

$$(\alpha - \beta) = \sum_{\rho\in J} \sum_{\sigma\in H} n_{\rho\sigma} \rho(\sigma - 1).$$

Since $(\sigma - 1)N_H = 0$ for each $\sigma \in H$, we have $(\alpha - \beta)N_H = 0$ and thus $\alpha N_H = \beta N_H$, as desired.

COROLLARY 4.3. *Suppose that $\alpha \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$. Then:*

(1) *$\alpha N_G = c w_F N_G$ for some $c \in \mathbb{Z}$.*

(2) *Suppose that E is a quadratic extension of F in K, with $H = \mathrm{Gal}(K/E)$, and $H \not\supset J = \langle\tau\rangle$ of order 2, so that G is the direct product of H and J. Then $\alpha N_H(1-\tau) = d|(\mu_E)_{1+\tau}|N_H(1-\tau)$ for some integer d.*

Proof. (1) Applying Lemma 4.2 with $H = G$ and J trivial gives $\alpha N_G = \beta N_G$ with $\beta \in \mathrm{Ann}_{\mathbb{Z}}(\mu_F) = w_F\mathbb{Z}$. So $\beta = cw_F$, giving the desired result.

(2) First, applying Lemma 4.2 with $M = E$ gives

$$\alpha N_H = \beta N_H$$

with $\beta \in \mathrm{Ann}_{\mathbb{Z}[J]}(\mu_E)$. Now $\mathbb{Z}[J] = \mathbb{Z} + \mathbb{Z}\tau$, so $\beta = m + n\tau$ with $m, n \in \mathbb{Z}$. Since β annihilates $(\mu_E)_{1+\tau}$ on which τ acts as -1, we have

$$1 = ((\mu_E)_{1+\tau})^{\beta} = ((\mu_E)_{1+\tau})^{m+n\tau} = ((\mu_E)_{1+\tau})^{m-n}.$$

Therefore $m - n \in \mathrm{Ann}_{\mathbb{Z}}((\mu_E)_{1+\tau}) = |(\mu_E)_{1+\tau}|\mathbb{Z}$, and $m - n = d|(\mu_E)_{1+\tau}|$. Finally,

$$\beta(1-\tau) = (m+n\tau)(1-\tau) = (m-n)(1-\tau) = d|(\mu_E)_{1+\tau}|(1-\tau).$$

Combining this with the first displayed equation gives the result.

PROPOSITION 4.4. *If $\psi \neq \psi_0$ and the integer b is an exponent for $C_{E_\psi/F}$, then $b|G|e_\psi$ annihilates Cl_K^S. Indeed, if $\mathfrak{a}$ is an ideal of $\mathcal{O}_K^S$, then $\mathfrak{a}^{b|G|e_\psi} = \delta\mathcal{O}_K^S$ for some $\delta \in (E_\psi)_{1+\tau_\psi}$.*

Proof. Let $H = \mathrm{Gal}(K/E_\psi)$ and let τ_ψ be a fixed lift of a generator of $\mathrm{Gal}(E_\psi/F)$ to G. Then

$$b|G|e_\psi = bN_H(1-\tau_\psi).$$

Any element of Cl_K^S is represented by an ideal $\mathfrak{a}_K$ of $\mathcal{O}_K^S$. Then

$$\mathfrak{a}_K^{N_H} = \mathfrak{a}_E\mathcal{O}_K^S$$

for some ideal $\mathfrak{a}_E$ of $\mathcal{O}_{E_\psi}^S$, while

$$\mathfrak{a}_E^b = \gamma\mathfrak{a}_F\mathcal{O}_{E_\psi}^S$$

for some ideal $\mathfrak{a}_F$ of $\mathcal{O}_F^S$ and $0 \neq \gamma \in E_\psi$, since b annihilates $\mathrm{Cl}_{E_\psi}^S$ modulo the image of Cl_F^S. Finally,

$$(\gamma\mathfrak{a}_F)^{1-\tau_\psi} = \gamma^{1-\tau_\psi}\mathfrak{a}_F^{1-\tau_\psi} = \gamma^{1-\tau_\psi}\mathcal{O}_E^S.$$

Since $\delta = \gamma^{1-\tau_\psi} \in (E_\psi)_{1+\tau_\psi}$, combining the displayed equations gives the result.

THEOREM 4.5. *Let K be a composite of a finite number of quadratic extensions of a number field F. Let S contain the infinite primes of F and those which ramify in K/F. Suppose $\ker(f)$ is finite and $\alpha \in \mathbb{Z}[G]$ annihilates μ_K.*

Let ψ be an irreducible character of G. Then $|G|^{r^S(\psi)+1}\alpha R(f)\theta_{K/F}^{S,*}e_\psi$ lies in $\mathbb{Z}[G]$ and annihilates Cl_K^S. Indeed, if $\mathfrak{a}$ is an ideal of $\mathcal{O}_K^S$, then

$$\mathfrak{a}^{|G|^{r^S(\psi)+1}\alpha R(f)\theta_{K/F}^{S,*}e_\psi} = \delta\mathcal{O}_K^S$$

for some $\delta \in F$ when $\psi = \psi_0$, and for some δ satisfying $\delta \in (E_\psi)_{1+\tau_\psi}$ when $\psi \neq \psi_0$.

Proof. First consider $\psi = \psi_0$. Note that $|G|e_{\psi_0} = N_G$ and $r^S(\psi_0) = |S| - 1$. Using Proposition 4.1 and Corollary 4.3(1) yields

$$\begin{aligned}
&|G|^{r^S(\psi_0)+1}\alpha R(f)\theta_{K/F}^{S,*}e_{\psi_0}\\
&\quad = |G|^{|S|-1}R(f)\theta_{K/F}^{S,*}e_{\psi_0}\alpha|G|e_{\psi_0} = \frac{h_F(X_F : \pi_{K/F}(f(U_F)))}{w_F}\alpha N_G\\
&\quad = \frac{h_F(X_F : \pi_{K/F}(f(U_F)))}{w_F}cw_F N_G = h_F(X_F : \pi_{K/F}(f(U_F)))cN_G,
\end{aligned}$$

which clearly lies in $\mathbb{Z}[G]$. Now any element of Cl_K^S is represented by an ideal $\mathfrak{a}_K$ of $\mathcal{O}_K^S$, and

$$\mathfrak{a}_K^{N_G} = \mathfrak{a}_F\mathcal{O}_K^S$$

for some ideal $\mathfrak{a}_F$ of $\mathcal{O}_F^S$. Then

$$\mathfrak{a}_F^{h_F} = \gamma\mathcal{O}_F^S,$$

for some $\gamma \in F$. Thus the result follows from the displayed equations, with $\delta = \gamma^{(X_F:\pi_{K/F}(f(U_F)))c}$.

Next consider $\psi \neq \psi_0$. Put $H = \mathrm{Gal}(K/E_\psi)$ and let τ_ψ be a fixed lift of a generator of $\mathrm{Gal}(E_\psi/F)$ to G. Then $|G|e_\psi = N_H(1-\tau_\psi)$. Using Proposition 4.1 and Corollary 4.3(2) yields

$$\begin{aligned}
&|G|^{r^S(\psi)+1}\alpha R(f)\theta_{K/F}^{S,*}e_\psi = |G|^{r^S(\psi)}R(f)\theta_{K/F}^{S,*}e_\psi\alpha|G|e_\psi\\
&\quad = 2^{|S|-1}|C_{E_\psi/F}|\frac{((X_{E_\psi})_{1+\tau_\psi} : \pi_{K/E_\psi}(f((U_{E_\psi})_{1+\tau_\psi})))}{|(\mu_{E_\psi})_{1+\tau_\psi}|}e_\psi\alpha|G|e_\psi\\
&\quad = 2^{|S|-1}|C_{E_\psi/F}|\frac{((X_{E_\psi})_{1+\tau_\psi} : \pi_{K/E_\psi}(f((U_{E_\psi})_{1+\tau_\psi})))}{|(\mu_{E_\psi})_{1+\tau_\psi}|}e_\psi d|(\mu_{E_\psi})_{1+\tau_\psi}|\,|G|e_\psi\\
&\quad = 2^{|S|-1}|C_{E_\psi/F}|((X_{E_\psi})_{1+\tau_\psi} : \pi_{K/E_\psi}(f((U_{E_\psi})_{1+\tau_\psi})))d|G|e_\psi.
\end{aligned}$$

Since this is an integer multiple of $|C_{E_\psi/F}|\,|G|e_\psi = |C_{E_\psi/F}|N_H(1-\tau_\psi)$, the result follows from Proposition 4.4.

Remark 4.6. It is clear from the proof of Theorem 4.5 that in fact $(|G|^{r^S(\psi)+1}/2^{|S|-1})\alpha R(f)\theta_{K/F}^{S,*}e_\psi$ annihilates Cl_K^S when $\psi \neq \psi_0$. Furthermore, in this situation, if r_F^S denotes the 2-rank of Cl_F^S, one can show by an argument similar to that in [4, Proposition 2] that the 2-rank of $C_{E_\psi/F}$ is always at least $r_F^S - 1$. Thus $|C_{E_\psi/F}|/2^{r_F^S-2}$ suffices as an exponent for $C_{E_\psi/F}$, and this allows one to modify the proof of Theorem 4.5 to conclude that $(|G|^{r^S(\psi)+1}/2^{|S|+r_F^S-3})\alpha R(f)\theta_{K/F}^{S,*}e_\psi$ annihilates Cl_K^S when $\psi \neq \psi_0$, and that $(|G|^{r^S(\psi)+1}/2^{r_F^S-1})\alpha R(f)\theta_{K/F}^{S,*}e_\psi$ does so when $\psi = \psi_0$. Finally, [4, Corollary 2] shows that $2^{|S|+r_F^S}$ is an integer multiple of $|G|$, so that for $\psi \neq \psi_0$, we see that $2^3|G|^{r^S(\psi)}\alpha R(f)\theta_{K/F}^{S,*}e_\psi$ annihilates Cl_F^S.

REMARK 4.7. By analogy with the Brumer–Stark conjecture, one may also be interested in further properties of the generator δ in Theorem 4.5. The conditions given there guarantee that $K(\sqrt{\delta})/F$ is an abelian Galois extension in all cases. If F has a real embedding, and $\psi \neq \psi_0$, the condition $\delta \in (E_\psi)_{1+\tau_\psi}$ suffices to imply that $K(\delta^{1/w_{E_\psi}})/F$ is an abelian Galois extension, by application of [7, Proposition IV.1.2]. Indeed, $E_\psi(\delta^{1/w_{E_\psi}})/F$ is abelian by the criterion there since $1+\tau_\psi$ annihilates μ_{E_ψ} in this case and $\delta^{1+\tau_\psi} = 1$, which is a w_{E_ψ}-power.

References

[1] P. Buckingham, *The canonical fractional Galois ideal at $s = 0$*, J. Number Theory 128 (2008), 1749–1768.

[2] D. Burns, *On derivatives of Artin L-series*, Invent. Math. 186 (2011), 291–371.

[3] D. Burns and M. Flach, *Tamagawa numbers for motives with (non-commutative) coefficients*, Doc. Math. 6 (2001), 501–570.

[4] D. S. Dummit, J. W. Sands, and B. Tangedal, *Stark's conjecture in multi-quadratic extensions, revisited*, J. Théor. Nombres Bordeaux 15 (2003), 83–97.

[5] D. Macias Castillo, *On higher-order Stickelberger-type theorems for multi-quadratic extensions*, Int. J. Number Theory 8 (2012), 95–110.

[6] V. P. Snaith, *Stark's conjecture and new Stickelberger phenomena*, Canad. J. Math. 58 (2006), 419-448.

[7] J. Tate, *Les conjectures de Stark sur les fonctions L d'Artin en $s = 0$*, Birkhäuser, Boston, 1984.

Jonathan W. Sands
Department of Mathematics and Statistics
University of Vermont
Burlington, VT 05401, U.S.A.
E-mail: Jonathan.Sands@uvm.edu

Received on 30.8.2011
and in revised form on 16.12.2011 (6811)