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Zero-density estimate for modular form L-functions in weight aspect

by

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1. Introduction. In the analytic theory of L-functions, it is sometimes possible to circumvent assumption of the Riemann Hypothesis by applying zero density arguments. Briefly, one argues that for a family of L-functions that is sufficiently "spectrally complete", the functions in the family have comparatively few zeros to the right of the 1/2-line. Historically, zero density questions were first considered with respect to the Riemann zeta function $\zeta(1/2+\sigma+it)$ as the parameter t varied, and the first result along these lines could be said to be the Hadamard–de la Vallée-Poussin zero-free region. Later investigations focused on the number

$$N(\sigma,T)=\#\{\rho=1/2+\beta+i\gamma:\zeta(\rho)=0,\,\sigma<\beta,\,0<\gamma<T\},$$

proving that this number decayed in the power of T with increasing $\sigma>0$. A classical result in this direction is due to Ingham [I]:

$$N(\sigma,T)=O(T^{3(1/2-\sigma)/(3/2-\sigma)}\log^5 T).$$

Selberg [S1] made a major contribution to this theory, proving the uniform bound

$$N(\sigma,T)\ll T^{1-\sigma/4}\log T,$$

in $0\le\sigma\le 1/2$. The crucial feature of this estimate is that the power of $\log T$ matches the true order in the number of zeros of ζ up to height T, so that the estimate is still useful even when σ is on the order of $1/\log T$. This formed one of the key analytic ingredients in Selberg's unconditional proof that the real and imaginary parts of $\log\zeta(1/2+it)$ become normally distributed in large intervals $t\in[T,2T]$.

Subsequent to his work on ζ, Selberg [S2] proved an analogous zero density estimate in the family of Dirichlet L-functions to a large modulus q,

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with q rather than t thought of as the varying parameter. Using this estimate, he showed that for fixed t the argument of $L(1/2+it;\chi)$ becomes normally distributed as χ varies modulo q, for $q\to\infty$. More recently Luo [L] has given an analogue of Selberg's bound in t-aspect, replacing ζ with the L-function of a fixed Hecke-eigen cusp form for $\mathrm{SL}_2(\mathbb{Z})$:

$$\begin{aligned} N_f(\sigma,T) &:= \#\{\rho = 1/2+\beta+i\gamma : L(\rho;f)=0,\ \sigma<\beta,\ 0<\gamma<T\} \\ &\ll_f T^{1-\sigma/72}\log T. \end{aligned}$$

Together with earlier work of Bombieri and Hejhal [BH], this established the asymptotic normality of $\log L(1/2+it;f)$ for $t\in[T,2T]$, f fixed with $T\to\infty$.

The purpose of this article is to prove a parallel extension of Selberg's Dirichlet L-function estimate but now for the family of L-functions associated to modular forms of large weight k. As in Selberg's work, an important aspect of our estimate is that it is uniform in k and for T in the range $1/\log k<T<k^\delta$, for some small $\delta>0$. This plays a crucial role in the author's related paper [H], where it is established, unconditionally, that varying f among Hecke-eigenforms of weight k, $\log L(1/2;f)$ is bounded above by a quantity that is asymptotically normal as $k\to\infty$. One further piece of context: Kowalski and Michel [KM] have proven another extension of Selberg's theorem to the family of weight 2 modular forms of large prime level q, and Conrey and Soundarajan [CS] (real Dirichlet L-functions) and Ricotta [R] (Rankin–Selberg L-functions) have given related estimates, each with applications to non-vanishing. Suitably modified, our estimate has similar applications, but we do not pursue them here.

To state our density result more precisely, let S_k denote the space of weight k holomorphic cusp forms for the modular group $\Gamma=\mathrm{SL}(2,\mathbb{Z})$ and let H_k be the basis of forms in S_k that are simultaneous eigenfunctions of all the Hecke operators. Write the Fourier expansion of $f\in H_k$ as

$$f(z)=\sum_{n=1}^{\infty} n^{(k-1)/2}\lambda_f(n)e(nz).$$

We normalize $f\in H_k$ so that $\lambda_f(1)=1$ ([1]). The L-function associated to $f\in H_k$ is

$$L(s;f)=\sum_{n=1}^{\infty}\frac{\lambda_f(n)}{n^s}=\prod_p\left(1-\frac{\lambda_f(p)}{p^s}+\frac{1}{p^{2s}}\right)^{-1},\qquad \Re(s)>1. \tag{1.1}$$

([1]) In particular, in our normalization Deligne's bound [D] reads $|\lambda_f(n)|\le d(n)$, the number of divisors of n.

This is a degree two L-function with completed L-function

$$\Lambda(s;f) = (2\pi)^{-s}\Gamma\left(s + \frac{k-1}{2}\right)L(s;f)$$

satisfying the self-dual functional equation

$$\Lambda(s;f) = i^k \Lambda(1-s;f).$$

In particular, with our normalization the Riemann Hypothesis asserts that all zeros ρ of $\Lambda(s;f)$ satisfy $\Re(\rho) = 1/2$.

For $f \in H_k$ and T growing, but small compared to $\sqrt{k}$, the number of zeros ρ of $L(s;f)$ with $0 < \Im(\rho) < T$ is $\sim \frac{T}{2\pi}\log k$. Thus the density of zeros of $L(s;f)$ near the central point $s = \frac{1}{2}$ is $(\log k)/2\pi$. Our main result says that among the family $\mathcal{F}_k$ of L-functions associated to forms in H_k, there are very few L-functions with zeros with small imaginary part and real part to the right of $1/2 + C/\log k$.

Main Theorem 1.1. *Let* $2/\log k < \sigma < 1/2$. *For some sufficiently small* $\delta, \theta > 0$ *we have, uniformly in* $10/\log k < T < k^{\delta}$,

$$\begin{aligned} N(\sigma,T) &:= \frac{1}{|H_k|}\sum_{f\in H_k} \#\{L(1/2+\beta+i\gamma) = 0 : \sigma < \beta,\ |\gamma| < T\} \\ &= O(Tk^{-\theta\sigma}\log k). \end{aligned}$$

The main new analytic ingredient of our theorem is the following asymptotic evaluation of the harmonic twisted second moment of $L(s;f)$, which may be of independent interest.

Theorem 1.2. *Let* $\sigma > 0$, $0 \neq |t| < k^{1/4}$ *and* $l < k^{1/3}$ *be square-free. Denote by* $\tau_\nu(n) = \sum_{n_1n_2=n}(n_1/n_2)^\nu$ *the generalized divisor function. We have the following formula for the harmonic twisted second moment:*

$$\begin{aligned} &\sum_{f\in H_k}^{h} \lambda_f(l)|L(1/2+\sigma+it;f)|^2 \\ &\quad = \zeta(1+2\sigma)\frac{\tau_{it}(l)}{l^{1/2+\sigma}} + \zeta(1-2\sigma)\left(\frac{k}{4\pi}\right)^{-4\sigma}\frac{\tau_{it}(l)}{l^{1/2-\sigma}} \\ &\qquad + i^k\zeta(1+2it)\left(\frac{k}{4\pi}\right)^{-2\sigma+2it}\frac{\tau_\sigma(l)}{l^{1/2+it}} + i^k\zeta(1-2it)\left(\frac{k}{4\pi}\right)^{-2\sigma-2it}\frac{\tau_\sigma(l)}{l^{1/2-it}} \\ &\qquad + O(l^{3/4}k^{-1/2-2\sigma+\epsilon}). \end{aligned}$$

The harmonic average $(\sum^h)$ means that forms $f \in H_k$ are counted with the weight $w_f = (4\pi)^{1-k}\Gamma(k-1)/\langle f,f\rangle$, which appears in the Petersson trace formula. Harmonic averages similar to this one have an extensive history; see for instance [Ku], [Fa], [Fo] and references therein. Our proof is most noteworthy for the fact that the evaluation of main terms goes "beyond the diagonal" and yet is not too difficult. After applying the Petersson trace formula and Voronoi summation to the resulting sums of Kloosterman sums, the off-diagonal main term arises as the Fourier transform of the relevant function at zero, and the remaining integrals against Bessel functions are error terms. The analysis of these error terms involves integrating against the Bessel function $J_{k-1}(x)$ near its transition region, and this is bounded in a similar way to an analysis of the twisted first moment of $L(1/2, \mathrm{sym}^2 f)$ in [Kh].

2. Outline of proof. The method of proof of Theorem 1.1 is the same as in Selberg's original work on Dirichlet L-functions; in particular, we appeal to the following version of the argument principle introduced there.

LEMMA 2.1. *Let ω be an entire function, non-zero in the half-plane $\Re(s) > W$. Let $\mathcal{B}$ be the rectangular box $|\Im(s)| \le H$, $W_0 \le \Re(s) \le W_1$ with $W < W_0 < W_1$. Then*

$$\begin{aligned}
4H \sum_{\substack{\beta+i\gamma\in\mathcal{B}\\ \omega(\beta+i\gamma)=0}} \cos\left(\frac{\pi\gamma}{2H}\right)\sinh\left(\frac{\pi(\beta-W_0)}{2H}\right)
&= \int_{-H}^{H} \cos\left(\frac{\pi t}{2H}\right)\log|\omega(W_0+it)|\,dt\\
&\quad - \Re\int_{-H}^{H}\cos\left(\pi\frac{W_1-W_0+it}{2iH}\right)\log\omega(W_1+it)\,dt\\
&\quad + \int_{W_0}^{W_1}\sinh\left(\frac{\pi(\alpha-W_0)}{2H}\right)\log|\omega(\alpha+iH)\omega(\alpha-iH)|\,d\alpha.
\end{aligned}$$

The fundamental proposition that we prove is the following.

PROPOSITION 2.2. *There exist Dirichlet polynomials $\{M(s;f)\}_{f\in H_k}$ which satisfy $M(\overline{s}) = \overline{M(s)}$ and are such that for sufficiently small positive δ and θ, uniformly in $|t| < k^{\delta}$, $1/\log k \le \sigma \le 1$,*

$$\frac{1}{|H_k|}\sum_{f\in H_k}|M(1/2+\sigma+it;f)L(1/2+\sigma+it;f)|^2 \le 1 + O(k^{-\theta\sigma}),$$

and for all t,

$$M(3/2+it;f)L(3/2+it;f) = 1 + O(k^{-\theta}).$$

To deduce Theorem 1.1 from this proposition, apply the lemma to $M\cdot L(s;f)$ with box bounded by $1/2+1/\log k \pm 2iT$ and $3/2 \pm 2iT$. The special feature of the lemma, which permits uniformity even for small $T \asymp 1/\log k$, is that only the real part of the logarithm appears in the part of the integral contained in the critical strip, so that this part may be bounded using the second moment estimate of the proposition. The further details of the deduction of Theorem 1.1 are not difficult, and may be found both in Selberg's original argument, and in the treatments in [CS] and [KM]. In the remainder of the paper we are concerned with the proof of the proposition, which takes place in three stages: first we calculate the harmonic twisted moment, proving Theorem 1.2. Next we mollify the second moment with respect to the harmonic weights. Finally we remove the harmonic weights via the method of [KM].

3. Some lemmas

Lemma 3.1 (Hecke relations). *For each Hecke eigenform f, the Fourier coefficients of f satisfy the relation*

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right).$$

This is equivalent to the Euler product (1.1).

The basic orthogonality relation on H_k is the Petersson trace formula.

Lemma 3.2 (Petersson trace formula). *We have*

$$\sum_{f\in H_k}^{h} \lambda_f(m)\lambda_f(n) = \delta_{m=n} + 2\pi i^k \sum_{c=1}^{\infty} \frac{S(m,n;c)}{c} J_{k-1}\left(\frac{4\pi}{c}\sqrt{mn}\right).$$

Proof. See e.g. [IK, p. 360]. ∎

Recall that we denote by

$$\tau_\nu(n) = \sum_{n_1 n_2 = n} \left(\frac{n_1}{n_2}\right)^{\nu} \tag{3.1}$$

the generalized divisor function. We will use the following version of the Voronoi summation formula.

Lemma 3.3. *Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be smooth with compact support. Let $c \geq 1$ and $(a,c) = 1$ with $ad \equiv 1 \bmod c$. We have*

$$\sum_{m=1}^{\infty} \tau_{it}(m) e\left(\frac{am}{c}\right) g(m)$$
$$= c^{2it-1}\zeta(1-2it)\int_0^{\infty} g(x)x^{-it}\,dx + c^{-2it-1}\zeta(1+2it)\int_0^{\infty} g(x)x^{it}\,dx$$

$$+\frac{1}{c}\sum_{n=1}^{\infty}\tau_{it}(n)e\left(\frac{-dn}{c}\right)\int_0^{\infty}g(x)J^{+}_{2it}\left(\frac{4\pi}{c}\sqrt{nx}\right)dx$$
$$+\frac{1}{c}\sum_{n=1}^{\infty}\tau_{it}(n)e\left(\frac{dn}{c}\right)\int_0^{\infty}g(x)K^{+}_{2it}\left(\frac{4\pi}{c}\sqrt{nx}\right)dx$$

where

$$J^{+}_{\nu}(x)=\frac{-\pi}{\sin\frac{\pi\nu}{2}}(J_{\nu}(x)-J_{-\nu}(x)),\qquad K^{+}_{\nu}(x)=4\cos\frac{\pi\nu}{2}K_{\nu}(x).$$

Proof. This is a slight modification of [IK, Theorem 4.10]. ∎

In bounding oscillatory integrals we make use of the following simple estimate [T, Lemma 4.5].

LEMMA 3.4. *Let $F(x), G(x)$ be real-valued functions on $[a,b]$ such that $F'(x)/G(x)$ is monotonic and $F''(x) > r > 0$, $|G(x)| \le M$. Then*

$$\left|\int_a^b G(x)e^{iF(x)}\,dx\right| \le \frac{8M}{\sqrt{r}}.$$

3.1. Facts concerning Bessel functions. Bessel functions arise both in the Petersson trace formula and as transforms in the Voronoi summation formula; we record here the properties that we will need regarding these functions.

The Bessel function of the first kind, $J_\nu(x)$, has Taylor series about zero given by

$$J_\nu(x)=\sum_{m=0}^{\infty}\frac{(-1)^m(z/2)^{\nu+2m}}{m!\Gamma(\nu+1+m)}.$$

Differentiating, one obtains the relation

$$J'_\nu(x)=\frac{1}{2}(J_{\nu+1}(x)-J_{\nu-1}(x)). \tag{3.2}$$

Specializing to $\nu = k-1$, the Mellin transform is given by

$$\int_0^{\infty}J_{k-1}(x)x^{s-1}\,dx = 2^{s-1}\frac{\Gamma\left(\frac{k-1+s}{2}\right)}{\Gamma\left(\frac{k+1-s}{2}\right)}. \tag{3.3}$$

The behavior of all of the Bessel functions depends essentially on the relationship between the size of the order ν and the variable x. When x is large, $x > |\nu|^2$ (ν possibly complex), then J_ν is oscillatory of essentially fixed frequency, while the Bessel function of the third kind K_ν is exponentially small. Asymptotic evaluations are given by (cf. [EMOT, p. 85])

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}}\cos\Big(x - \frac{\pi\nu}{2} - \frac{\pi}{4}\Big)\Big[1 - \frac{P(\nu)}{128x^2}\Big] - \sin\Big(x - \frac{\pi\nu}{2} - \frac{\pi}{4}\Big)\frac{\nu^2 - 1/4}{2x} + O\Big(\frac{1+|\nu|^6}{x^3}\Big), \tag{3.4}$$

$$J_\nu^+(x) = -\sqrt{\frac{2\pi}{x}}\sin\Big(x - \frac{\pi}{4}\Big)\Big[1 - \frac{P(\nu)}{128x^2}\Big] - \pi\cos\Big(x - \frac{\pi}{4}\Big)\frac{\nu^2 - 1/4}{2x} + O\Big(\frac{1+|\nu|^6}{x^3}\Big), \tag{3.5}$$

where $P(\nu) = 16\nu^4 - 40\nu^2 + 9$. Also,

$$K_\nu(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x}\Big[1 + O\Big(\frac{1+|\nu|^2}{x}\Big)\Big]. \tag{3.6}$$

Since we regard t as small compared to k, $|t| < k^{1/4}$, these are the only evaluations we need regarding J_{it}, and K_{it}.

When x is small, $x \ll k$, then $J_{k-1}(x)$ is uniformly small. Taking absolute values in the Taylor expansion leads to the bound ([RS, p. 297])

$$|J_{k-1}(x)| \le \frac{(x/2)^{k-1}}{\Gamma(k-1)}e^{x/2}, \qquad x < 2k. \tag{3.7}$$

In particular, if $x < k/10$ then $J_{k-1}(x) < e^{-k}$.

In the transition region $k \ll x \ll k^2$, $J_k(x)$ increases to a global maximum of size $k^{-1/3}$ at a point near $x = k$, and thereafter oscillates with slowly increasing frequency and slowly decreasing amplitude. Langer's formulas [EMOT, p. 85, (32) and (34)] give an asymptotic evaluation:

$$J_k(x) = \frac{(\tanh^{-1} w - w)^{1/2}}{\pi w^{1/2}} K_{1/3}(z) + O(k^{-4/3}), \qquad x < k,\ w = (1 - x^2/k^2)^{1/2},\ z = k(\tanh^{-1} w - w), \tag{3.8}$$

$$J_k(x) = \frac{(w - \tan^{-1} w)^{1/2}}{w^{1/2}}\Big[J_{1/3}(z)\cos\frac{\pi}{6} - Y_{1/3}(z)\sin\frac{\pi}{6}\Big] + O(k^{-4/3}), \qquad x > k,\ w = (x^2/k^2 - 1)^{1/2},\ z = k(w - \tan^{-1} w). \tag{3.9}$$

Here $Y_\nu(x)$ is the Bessel function of the second kind, related to J_ν by

$$Y_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin \nu\pi}.$$

Since Langer's formulas depend on the functions $J_{1/3}$, $Y_{1/3}$ and $K_{1/3}$, we record their further asymptotic properties. For $x \gg 1$ the evaluations of $J_{1/3}$ and $K_{1/3}$ are given by (3.4) and (3.6), while the evaluation of $Y_{1/3}$ is the same as for $J_{1/3}$ except that the places of cos and sin are interchanged. When $x < 1$ we have the bounds

$$J_{1/3}(x) \ll x^{1/3}, \quad Y_{1/3}(x) \ll x^{-1/3}, \quad K_{1/3}(x) \ll x^{-1/3}. \tag{3.10}$$

We collect together these facts in the following lemma.

Lemma 3.5. *In the region $|x-k| < k^{1/3}$ we have the bound*

$$J_k(x) \ll k^{-1/3}. \tag{3.11}$$

For $0 < x < k - k^{1/3}$ we have

$$J_k(x) = \frac{e^{k(w-\tanh^{-1} w)}}{\sqrt{2\pi k w}}[1 + O(k^{-1}w^{-3})] + O(k^{-4/3}) \tag{3.12}$$

with $w = (1 - x^2/k^2)^{1/2}$. For $x > k + k^{1/3}$ we have

$$J_k(x) = \sqrt{\frac{2}{\pi k w}} \cos\biggl(k(w - \tan^{-1} w) - \frac{\pi}{4}\biggr) + O\biggl(k^{-4/3} + \frac{1+w^{-2}}{k^{3/2}w^{3/2}}\biggr) \tag{3.13}$$

with, now, $w = (x^2/k^2 - 1)^{1/2}$.

Proof. Note that for $x = k \pm k^{\Delta}$ and $\Delta < 1$, $w \asymp k^{(\Delta-1)/2}$. Thus for $|x-k| < k^{1/3}$ the bound follows from Langer's formulas and the bounds in (3.10). For $|x-k| > k^{1/3}$ we have $w \gg k^{-1/3}$ and, therefore, $z \gg 1$. The remaining formulas thus follow from the asymptotic evaluations of $J_{1/3}, Y_{1/3}, K_{1/3}$ at large argument, together with Langer's formulas. ∎

One further consequence is the following simple lemma.

Lemma 3.6. *For any integer $k > 0$ and any $A < k^2$,*

$$\int_0^A |J_k(x)|\, dx \ll \sqrt{A}.$$

Proof. In the range $A < k - k^{1/2}$ the formula (3.12) and $w \gg k^{-1/4}$ imply uniformly $J_k(x) \ll k^{-4/3} + e^{-\Omega(k^{1/4})}k^{O(1)}$. For $k - k^{1/2} < x < k + k^{1/2}$ bound simply $J_k(x) = O(1)$. In the range $k + k^{1/2} < A < 2k$ use (3.13) to bound

$$\int_{k+k^{1/2}}^{A} |J_k(x)|\, dx \ll \frac{A}{k^{4/3}} + \int_{k+k^{1/2}}^{A} \biggl(\frac{1}{\sqrt{kw}} + \frac{1}{k^{3/2}w^{7/2}}\biggr) dx.$$

For $k < x < 2k$, $w \gg \sqrt{(x-k)/k}$, so the last integral is

$$\ll \int_{k^{1/2}}^{A-k} \biggl(\frac{1}{(ky)^{1/4}} + \frac{k^{1/4}}{y^{7/4}}\biggr) dy \ll \frac{A^{3/4}}{k^{1/4}} + k^{-3/8} \ll \sqrt{A}.$$

Finally, for $x > 2k$, $w = \Omega(1)$ and so (3.13) says that $|J_k(x)| \ll \frac{1}{\sqrt{x}} + \frac{1}{k^{4/3}}$, which plainly suffices. ∎

With an eye toward applying Lemma 3.4 and with $x > k$ and $w = \sqrt{x^2/k^2 - 1}$ as above, we record

$$\begin{aligned} &\frac{\partial}{\partial x}(kw - k\tan^{-1} w) = \frac{kw}{x}, \\ &\frac{\partial^2}{\partial x^2}(kw - k\tan^{-1} w) = \frac{k^2}{x^2(x^2-k^2)^{1/2}}. \end{aligned} \tag{3.14}$$

3.2. Approximate functional equation. Fix, once and for all, a smooth function $H : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

(1) $H(x) \equiv 1$ for $x \in [0, 1/2]$,
(2) $H(x) + H(1/x) = 1$.

In particular, the Mellin transform $\hat{H}(s)$ has a single simple pole at 0 of residue 1, is odd, and satisfies the bounds

$$\hat{H}(s) \ll_A \frac{1}{s(s+1)\cdots(s+A-1)}, \qquad A = 1, 2, \ldots,$$

and $|\hat{H}(s)| \ll 2^{\Re(s)}$ for $\Re(s) > 1$.

We record an approximate formula for $|L(1/2 + \sigma + it; f)|^2$.

PROPOSITION 3.7 (Approximate functional equation). *We have*

$$\begin{aligned} |L(1/2+\sigma+it; f)|^2 = \sum_{d=1}^{\infty} \frac{1}{d^{1+2\sigma}} \sum_{m=1}^{\infty} \frac{\lambda_f(m)\tau_{it}(m)}{m^{1/2+\sigma}} \\ \times (W_{k,\sigma+it}(md^2) + (4\pi^2 md^2)^{2\sigma} \tilde{W}_{k,-\sigma+it}(md^2)) \end{aligned} \tag{3.15}$$

with

$$W_{k,\sigma+it}(\xi) = \frac{1}{2\pi i} \int_{(3)} \frac{\hat{H}(s)}{(4\pi^2\xi)^s} \frac{\Gamma(\sigma+k/2+it+s)\Gamma(\sigma+k/2-it+s)}{\Gamma(\sigma+k/2+it)\Gamma(\sigma+k/2-it)}\, ds,$$

$$\tilde{W}_{k,-\sigma+it}(\xi) = \frac{1}{2\pi i} \int_{(3)} \frac{\hat{H}(s)}{(4\pi^2\xi)^s} \frac{\Gamma(-\sigma+k/2+it+s)\Gamma(-\sigma+k/2-it+s)}{\Gamma(\sigma+k/2+it)\Gamma(\sigma+k/2-it)}\, ds.$$

Proof. See [IK, pp. 97–100] . ∎

The functions $W_{k,\sigma+it}$ and $\tilde{W}_{k,-\sigma+it}$ have the following properties.

LEMMA 3.8. *As functions of a real variable, both $W_{k,\sigma+it}$ and $\tilde{W}_{k,-\sigma+it}$ are real-valued. For $t < k^{1/4}$ and $|\sigma| < 2$ we have*

$$W_{k,\sigma+it}(\xi) = 1 + O\bigg(\bigg(\frac{400\xi}{k^2}\bigg)^{k^{1/4}}\bigg),$$

$$W_{k,\sigma+it}(\xi) = O\bigg(\bigg(\frac{k^2}{80\xi}\bigg)^{k^{1/4}}\bigg), \qquad \xi^j\bigg(\frac{\partial}{\partial\xi}\bigg)^j W_{k,\sigma+it}(\xi) \ll_j 1,$$
$$\tilde{W}_{k,-\sigma+it}(\xi) = \frac{\Gamma(-\sigma+k/2+it)\Gamma(-\sigma+k/2-it)}{\Gamma(\sigma+k/2+it)\Gamma(\sigma+k/2-it)} + O\bigg(\bigg(\frac{400\xi}{k^2}\bigg)^{k^{1/4}}\bigg),$$
$$\tilde{W}_{k,-\sigma+it}(\xi) = O\bigg(\bigg(\frac{k^2}{80\xi}\bigg)^{k^{1/4}}\bigg), \qquad \xi^j\bigg(\frac{\partial}{\partial\xi}\bigg)^j \tilde{W}_{k,\sigma+it}(\xi) \ll_j k^{-4\sigma}.$$

Proof. Pair s and $\overline{s}$ in the defining integrals to prove that W and $\tilde{W}$ are real.

For the bounds on the functions, shift the contour to $\Re(s) = \pm k^{1/4}$ and estimate the ratio of Gamma factors using Stirling's approximation. In particular, for $|\Delta| < \Re(z)^{1/2}$ we use the estimate

$$\frac{\Gamma(z+\Delta)}{\Gamma(z)} = \exp\bigg(\Delta\log z + \frac{\Delta^2}{2z} + O(|\Delta|\,|z|^{-1})\bigg). \tag{3.16}$$

The derivatives are bounded by estimating directly on the $\Re(s) = 0$ line. ∎

4. Twisted second moment, Proof of Theorem 1.2. From the approximate functional equation,

$$\begin{aligned}\sum_{f\in H_k}^h &\lambda_f(l)|L(1/2+\sigma+it;f)|^2 \\ &= \sum_{m,d=1}^{\infty} \frac{\tau_{it}(m)}{m^{1/2+\sigma}d^{1+2\sigma}}[W_{k,\sigma+it}(md^2) + (4\pi^2md^2)^{2\sigma}\tilde{W}_{k,-\sigma+it}(md^2)] \\ &\qquad\times \sum_{f\in H_k}^h \lambda_f(l)\lambda_f(m).\end{aligned}$$

Applying the Petersson inner product we obtain a diagonal term

$$(\mathrm{D}) := \frac{\tau_{it}(l)}{l^{1/2+\sigma}}\sum_{d=1}^{\infty}\frac{1}{d^{1+2\sigma}}[W_{k,\sigma+it}(ld^2) + (4\pi^2ld^2)^{2\sigma}\tilde{W}_{k,-\sigma+it}(ld^2)]$$

and an off-diagonal term

$$\begin{aligned}(\mathrm{OD}) := 2\pi i^k \sum_{m,d=1}^{\infty} &\frac{\tau_{it}(m)}{m^{1/2+\sigma}d^{1+2\sigma}}[W_{k,\sigma+it}(md^2) + (4\pi^2md^2)^{2\sigma}\tilde{W}_{k,-\sigma+it}(md^2)] \\ &\times \sum_{c=1}^{\infty}\frac{S(m,l;c)}{c}J_{k-1}\bigg(\frac{4\pi}{c}\sqrt{lm}\bigg).\end{aligned}$$

Introducing the integrals defining W and $\tilde{W}$, the diagonal terms are given by

$$(\mathrm{D}) = \frac{\tau_{it}(l)}{l^{1/2+\sigma}}\biggl\{\frac{1}{2\pi i}\int\limits_{(3)}\frac{\zeta(1+2s)}{(4\pi^2 l)^{s-\sigma}}\frac{\Gamma(s+k/2+it)\Gamma(s+k/2-it)}{\Gamma(\sigma+k/2+it)\Gamma(\sigma+k/2-it)} \times (\hat{H}(s-\sigma)+\hat{H}(s+\sigma))\,ds\biggr\},$$

and this evaluates to the first two main terms, with an error that is $O(1/k)$, by shifting the contour to the line $\Re(s) = -1/2+\sigma$ (2).

The two remaining main terms come from off the diagonal, so we now work to isolate these terms. The crucial fact in evaluating the off-diagonal terms is that the summations over c and d are very short. Exchanging the order of summation, we write

$$(\mathrm{OD}) = 2\pi i^k \sum_{cd<10000\sqrt{l}} \frac{\Sigma_{c,d}}{cd^{1+2\sigma}} + O\Bigl(\sum_{cd\ge 10000\sqrt{l}} \frac{|\Sigma_{c,d}|}{cd^{1+2\sigma}}\Bigr),$$

$$\Sigma_{c,d} = \sum_{m=1}^{\infty} \frac{\tau_{it}(m)S(m,l;c)}{m^{1/2+\sigma}} J_{k-1}\Bigl(\frac{4\pi}{c}\sqrt{lm}\Bigr) \times [W_{k,\sigma+it}(md^2) + (4\pi^2 md^2)^{2\sigma}\tilde{W}_{k,-\sigma+it}(md^2)].$$

We show that the sum over large cd is an error that is $o(1)$.

Lemma 4.1. *When $cd \ge 10000\sqrt{l}$ we have the bound*

$$\Sigma_{c,d} \ll \frac{(cdkl)^{O(1)}}{\Gamma(k-1)} e^{4\pi k\sqrt{\frac{l^{1/2}}{cd}}}\Bigl(2\pi k\sqrt{\frac{l^{1/2}}{cd}}\Bigr)^{k-1} + cd^{4\sigma}\Bigl(\frac{l^{1/2}}{80cd}\Bigr)^{k^{1/4}}.$$

This suffices, since when summed over $cd > 10000\sqrt{l}$, the bound of the lemma yields

$$\Sigma_{c,d} \ll l^{O(1)} e^{-k^{1/4}} = o(1).$$

Proof of Lemma 4.1. Split the sum over m according as $m \le k^2 c/l^{1/2} d$ or not. For small m, each term in the sum is bounded by applying the bound (3.7) for the Bessel function, bounding the Kloosterman sum trivially by c and bounding W and $\tilde{W}$ by $O(1)$. This yields

$$\ll cd^{4\sigma}\frac{1}{\Gamma(k-1)} \sum_{m<k^2 c/dl^{1/2}} e^{\frac{4\pi}{c}\sqrt{lm}}\Bigl(\frac{2\pi}{c}\sqrt{lm}\Bigr)^{k-1}$$

$$\le \frac{(cdkl)^{O(1)}}{\Gamma(k-1)} e^{4\pi k\sqrt{l^{1/2}/cd}}\Bigl(2\pi k\sqrt{\frac{l^{1/2}}{cd}}\Bigr)^{k-1},$$

(2) The pole of ζ does not contribute since $\hat{H}(-\sigma)+\hat{H}(\sigma) = 0$.

by bounding each term in the sum by the largest term. In the part of the sum with $m > k^2c/l^{1/2}d$ we bound the Bessel function by $O(1)$, the Kloosterman sum by c and $W, \tilde{W}$ by $\ll (k^2/80d^2m)^{k^{1/4}}$, which gives

$$\ll cd^{4\sigma} \sum_{m>k^2c/l^{1/2}d} \frac{1}{m^{1/2+\sigma}} \left(\frac{k^2}{80md^2}\right)^{k^{1/4}} \ll cd^{4\sigma} \left(\frac{l^{1/2}}{80cd}\right)^{k^{1/4}}. \blacksquare$$

In order to apply the Voronoi summation formula to the sum over m we open the Kloosterman sum and introduce a function of compact support. Let $F \in C_c^\infty(\mathbb{R}^+)$ satisfy

(1) $F(x) \equiv 1$ for $k/1000 < x < 1000k\sqrt{l}$,
(2) $\operatorname{supp}(F) \subset [k/2000, 2000k\sqrt{l}]$,
(3) for each $j = 0, 1, \ldots$ and all x, $x^j \frac{d^j}{dx^j} F(x) \ll_j 1$,

and consider the perturbed sum

$$\tilde{\Sigma}_{c,d} = \sum_{a \bmod c}^{*} e\left(\frac{\overline{a}l}{c}\right) \sum_{m=1}^{\infty} \frac{\tau_{it}(m)e(\frac{am}{c})}{m^{1/2+\sigma}} J_{k-1}\left(\frac{4\pi}{c}\sqrt{lm}\right) F\left(\frac{4\pi}{c}\sqrt{lm}\right) \times [W_{k,\sigma+it}(md^2) + (4\pi^2 md^2)^{2\sigma} \tilde{W}_{k,-\sigma+it}(md^2)].$$

This negligibly changes the sum, since for those c, m for which $F(\frac{4\pi}{c}\sqrt{lm})$ is not identically 1, either J_k or W or $\tilde{W}$ is extremely small: there are $O(l^{O(1)}k^{O(1)})$ terms with $\frac{4\pi}{c}\sqrt{lm} < k/1000$ and for these terms, the Bessel function is bounded by e^{-k}. Meanwhile, if $\frac{4\pi}{c}\sqrt{lm} > 1000kl$ then $m > (1000/4\pi)^2 k^2 c^2$ so that the sum is bounded by

$$\ll l^{O(1)}k^{O(1)} \sum_{cd<1000\sqrt{l}} \sum_{m>(1000/4\pi)^2k^2c^2} \left(\frac{k^2}{80md^2}\right)^{k^{1/4}} \ll l^{O(1)}k^{O(1)}e^{-k^{1/4}}.$$

Introduce functions

$$g_{c,d}(x) = \frac{1}{x^{1/2+\sigma}} W_{k,\sigma+it}(d^2x) J_{k-1}\left(\frac{4\pi}{c}\sqrt{lx}\right) F\left(\frac{4\pi}{c}\sqrt{lx}\right),$$
$$\tilde{g}_{c,d}(x) = \frac{1}{x^{1/2-\sigma}} \tilde{W}_{k,-\sigma+it}(d^2x) J_{k-1}\left(\frac{4\pi}{c}\sqrt{lx}\right) F\left(\frac{4\pi}{c}\sqrt{lx}\right)$$

so that

$$\tilde{\Sigma}_{c,d} = \sum_{a \bmod c}^{*} e\left(\frac{\overline{a}l}{c}\right) \sum_{m} \tau_{it}(m) e\left(\frac{am}{c}\right) \{g_{c,d}(m) + (4\pi^2 d^2)^{2\sigma} \tilde{g}_{c,d}(m)\}.$$

Applying, for each c, d, Voronoi summation in the sum over m, we express the off-diagonal terms as ([3])

$$\frac{(\mathrm{OD})}{2\pi i^k}+o(1)=\zeta(1-2it)\sum_{cd<10000l^{1/2}}\frac{S(0,l;c)}{c^{2-2it}d^{1+2\sigma}}\int_0^\infty g_{c,d}(x)x^{-it}\,dx$$
$$+\zeta(1+2it)\sum_{cd<10000l^{1/2}}\frac{S(0,l;c)}{c^{2+2it}d^{1+2\sigma}}\int_0^\infty g_{c,d}(x)x^{it}\,dx$$
$$(\mathrm{J})\qquad +\sum_{cd<10000l^{1/2}}\sum_{n=1}^{\infty}\frac{\tau_{it}(n)S(0,l-n;c)}{c^2d^{1+2\sigma}}\int_0^\infty g_{c,d}(x)J_{2it}^{+}\Big(\frac{4\pi}{c}\sqrt{nx}\Big)\,dx$$
$$(\mathrm{K})\qquad +\sum_{cd<10000l^{1/2}}\sum_{n=1}^{\infty}\frac{\tau_{it}(n)S(0,l+n;c)}{c^2d^{1+2\sigma}}\int_0^\infty g_{c,d}(x)K_{2it}^{+}\Big(\frac{4\pi}{c}\sqrt{nx}\Big)\,dx$$
$$+\text{ analogous terms coming from }\tilde g.$$

We are going to show that the first two terms combine with the corresponding terms from $\tilde g$ to yield the remaining two main terms of the theorem, and that (J) and (K) are error terms.

4.1. The off-diagonal main terms. Expanding the definition of $g_{c,d}(x)$, the first two terms above are equal to

$$(4.1)\qquad 2i^k\Re\bigg\{2\pi\zeta(1-2it)\sum_{cd<10000l^{1/2}}\frac{S(0,l;c)}{c^{2-2it}d^{1+2\sigma}}$$
$$\times\int_0^\infty W_{k,\sigma+it}(d^2x)J_{k-1}\Big(\frac{4\pi}{c}\sqrt{lx}\Big)F\Big(\frac{4\pi}{c}\sqrt{lx}\Big)x^{-1/2-\sigma-it}\,dx\bigg\}.$$

With negligible error the function F may be removed from the integrand, and then the sums extended to all c and d, this justified by the continuous analog of the arguments given above involving summations over m ([4]). Inserting the definition of $W_{k,\sigma+it}$ we obtain for the integral in (4.1) with F removed

$$\int_0^\infty\bigg\{\bigg[\frac{1}{2\pi i}\int_{(3)}\frac{\hat H(s)}{(4\pi^2xd^2)^s}\,\frac{\Gamma(s+\sigma+k/2+it)\Gamma(s+\sigma+k/2-it)}{\Gamma(\sigma+k/2+it)\Gamma(\sigma+k/2-it)}\,ds\bigg]$$
$$\times J_{k-1}\Big(\frac{4\pi}{c}\sqrt{lx}\Big)x^{1/2-\sigma-it}\bigg\}\frac{dx}{x}.$$

([3]) Note that summation over a mod c^* has been replaced by Ramanujan sums.

([4]) We bound only the real part of the error. Recall that W and $\tilde W$ are real, so that the imaginary parts of c^{it} and x^{it} are $O(t\log l)$ and $O(t\log x)$.

In view of the bound (3.7), both integrals are absolutely convergent. Put $w = \frac{4\pi}{c}\sqrt{lx}$ and exchange the order of the integration to rewrite this as

$$2\left(\frac{c}{4\pi\sqrt{l}}\right)^{1-2\sigma-2it}\frac{1}{2\pi i}\int_{(3)}\left(\frac{4l}{c^2d^2}\right)^s\frac{\Gamma(s+\sigma+k/2+it)\Gamma(s+\sigma+k/2-it)}{\Gamma(\sigma+k/2+it)\Gamma(\sigma+k/2-it)} \times\left[\int_0^\infty J_{k-1}(w)w^{1-\sigma-2it-2s}\,\frac{dw}{w}\right]\hat{H}(s)\,ds.$$

The bracketed integral is the Mellin transform of J_{k-1}, given by (3.3).

We now pass the summations over c and d under the integral. Recall that the Ramanujan sum evaluates to

$$\begin{aligned}
&S(0,a;p) = -1,\\
&S(0,a;p^e) = 0,\\
&S(0,p;p) = p-1, \qquad (a,p)=1,\ e\ge 1,\\
&S(0,ap,p^2) = -p,\\
&S(0,ap,p^{e+1}) = 0.
\end{aligned}$$

Thus the resulting Dirichlet series $\sum_{c,d} S(0,l;c)/(cd)^{1+2\sigma+2s}$ collapses to the finite product

$$\prod_{p|l}\left(1-\frac{1}{p^{1+2\sigma+2s}}\right)^{-1}\left(1+\frac{p-1}{p^{1+2\sigma+2s}}-\frac{p}{p^{2+4\sigma+4s}}\right) = \prod_{p|l}\left(1+\frac{1}{p^{2\sigma+2s}}\right) = l^{-\sigma-s}\tau_{s+\sigma}(l).$$

Combining these steps we arrive at

$$(4.1) = o(1) + 2i^k\Re\left\{\frac{\zeta(1-2it)(2\pi)^{2\sigma+2it}}{l^{1/2-it}} \times\frac{1}{2\pi i}\int_{(3)}\tau_{s+\sigma}(l)\frac{\Gamma(\sigma+k/2-it+s)\Gamma(-\sigma+k/2-it-s)}{\Gamma(\sigma+k/2+it)\Gamma(\sigma+k/2-it)}\hat{H}(s)\,ds\right\}.$$

Repeating these steps, one proves that the main terms coming from $\tilde{g}_{c,d}$ are (again with error $o(1)$)

$$2i^k\Re\left\{\frac{\zeta(1-2it)(2\pi)^{2\sigma+2it}}{l^{1/2-it}} \times\frac{1}{2\pi i}\int_{(3)}\tau_{-s+\sigma}(l)\frac{\Gamma(\sigma+k/2-it-s)\Gamma(-\sigma+k/2-it+s)}{\Gamma(\sigma+k/2+it)\Gamma(\sigma+k/2-it)}\hat{H}(s)\,ds\right\}.$$

In this integral we change s to $-s$. Recall that $\hat{H}(-s) = -\hat{H}(s)$, so that the combined contribution from the $g_{c,d}$ and $\tilde{g}_{c,d}$ main terms is equal to

$$\frac{1}{2\pi i}\Big\{\int_{(3)} - \int_{(-3)}\Big\}\Big[\tau_{s+\sigma}(l)\frac{\Gamma(\sigma+k/2-it+s)\Gamma(-\sigma+k/2-it-s)}{\Gamma(\sigma+k/2+it)\Gamma(\sigma+k/2-it)}\hat{H}(s)ds\Big].$$

Thus the two terms together are just equal to the residue of the integrand at the pole at 0, that is,

$$2i^k\Re\bigg\{\zeta(1-2it)(2\pi)^{2\sigma+2it}\frac{\tau_\sigma(l)}{l^{1/2-it}}\frac{\Gamma(-\sigma+k/2-it)}{\Gamma(\sigma+k/2+it)}\bigg\}$$
$$= 2\Re\bigg\{\zeta(1-2it)\Big(\frac{k}{4\pi}\Big)^{-2\sigma-2it}\frac{\tau_\sigma(l)}{l^{1/2-it}}\bigg\} + O((1+t^2)k^{-1}).$$

4.2. The terms containing Bessel integrals. The term (K) is extremely small, since the K-Bessel function is exponentially small for large variable and the support of F in the function $g_{c,d}$ localizes the variable to be of size at least $k\sqrt{n/l}$. The term (J) requires some more care, and we get cancellation from the changing rate of oscillation of the J-Bessel function in its transition region.

The integral in the term (J) is equal to

$$\int_0^\infty W_{k,\sigma+it}(d^2x)J_{k-1}\Big(\frac{4\pi}{c}\sqrt{lx}\Big)F\Big(\frac{4\pi}{c}\sqrt{lx}\Big)J^+_{2it}\Big(\frac{4\pi}{c}\sqrt{nx}\Big)\frac{dx}{x^{1/2+\sigma}}.$$

Substituting $y = \frac{4\pi}{c}\sqrt{lx}$ we obtain

$$2\pi(J) = \frac{(4\pi)^{1+2\sigma}}{l^{1/2-\sigma}}\sum_{cd<10000\sqrt{l}}\frac{1}{(cd)^{1+2\sigma}}\sum_{n=1}^\infty \tau_{it}(n)S(0,l-n;c)$$
$$\times\int_0^\infty W_{k,\sigma+it}\Big(\frac{c^2d^2y^2}{(4\pi)^2l}\Big)J_{k-1}(y)J^+_{2it}\Big(y\sqrt{\frac{n}{l}}\Big)F(y)y^{-2\sigma}\,dy.$$

Now replace J^+_{2it} with its asymptotic expansion

$$J^+_{2it}\Big(y\sqrt{\frac{n}{l}}\Big) = -\sqrt{\frac{2\pi}{y}\sqrt{\frac{l}{n}}}\sin\Big(y\sqrt{\frac{n}{l}}-\frac{\pi}{4}\Big)\Big[1-\frac{P(2it)l}{128y^2n}\Big]$$
$$-\pi\cos\Big(y\sqrt{\frac{n}{l}}-\frac{\pi}{4}\Big)\frac{-4t^2-1/4}{2y} + O\Big(\frac{(1+t^6)l^{3/2}}{y^3n^{3/2}}\Big).$$

By the integral bound in Lemma 3.6, the error contributes $O\big(\frac{l^{2+\sigma}(1+t^6)}{k^{5/2+2\sigma-\epsilon}}\big)$. In the remaining terms we can integrate by parts several times to truncate the sum over n at $n < lk^\epsilon$, with negligible error. We only show how to bound the contribution from integrating against the main term

$$-\sqrt{\frac{2\pi}{y}\sqrt{\frac{l}{n}}}\sin\left(y\sqrt{\frac{n}{l}}-\frac{\pi}{4}\right); \tag{4.2}$$

the rest of the main term can be handled in exactly the same way, and it produces an error of smaller size.

We will prove the following lemma.

Lemma 4.2. *We have the bound*

$$\int_0^\infty W_{k,\sigma+it}\left(\frac{c^2d^2y^2}{(4\pi)^2l}\right)J_{k-1}(y)\sin\left(y\sqrt{\frac{n}{l}}-\frac{\pi}{4}\right)F(y)y^{-1/2-2\sigma}\,dy \ll l^{1/4-\sigma}k^{-1/2-2\sigma+\epsilon}. \tag{B}$$

Assuming this bound for the moment we find that the contribution to (J) from integration against (4.2) is

$$\ll \frac{1}{k^{1/2+2\sigma-\epsilon}}\sum_{cd<10000\sqrt{l}}\frac{1}{(cd)^{1+2\sigma}}\sum_{n\ll lk^\epsilon}\frac{|S(0,l-n;c)|}{n^{1/4-\epsilon}}.$$

Here the $n=l$ term contributes $\ll l^{1/4-\sigma}k^{-1/2-2\sigma+\epsilon}$ while the $n\neq l$ terms give

$$\ll \frac{1}{k^{1/2+2\sigma-\epsilon}}\sum_{\substack{n\ll lk^\epsilon\\ n\neq l}}\frac{1}{n^{1/4-\epsilon}}\sum_{\substack{c_1\mid n-l\\ c_1c_2d\leq 10000\sqrt{l}}}\frac{1}{c_1^{2\sigma}c_2^{1+2\sigma}d^{1+2\sigma}}\ll l^{3/4}k^{-1/2-2\sigma+\epsilon},$$

and both of these bounds suffice for the theorem. The term corresponding to (J) coming from $\tilde{g}$ is handled in an analogous way, so it only remains to prove the bound (B).

Proof of Lemma 4.2. We split the integral into the ranges $y<k-k^{1/3}$, $k-k^{1/3}<y<k+k^{1/3}$, and $k+k^{1/3}<y<2000k\sqrt{l}$.

For $y<k-k^{1/3}$ we set $y=k-k^\Delta$ so that $w=\sqrt{k^2/x^2-1}$ satisfies $w\asymp k^{(\Delta-1)/2}$. Then the bound from (3.12),

$$J_{k-1}(y)\ll\frac{e^{kw-k\tanh^{-1}w}}{\sqrt{kw}}+O(k^{-4/3}),$$

easily suffices for the result, since for small w,

$$kw-k\tanh^{-1}w\sim-\frac{kw^3}{3}\asymp -k^{(3\Delta-1)/2}.$$

For $k-k^{1/3}<y<k+k^{1/3}$ we bound simply $J_{k-1}(y)\ll k^{-1/3}$, so that this part also contributes $\ll k^{-1/2-2\sigma}$.

In the remaining part of the integral we have from (3.13) (set $k' = k-1$)

$$J_{k'}(y) = \sqrt{\frac{2}{\pi k' w}}\cos\left(k'w - k'\tan^{-1} w - \frac{\pi}{4}\right) + O\left(k^{-4/3} + \frac{1+w^{-2}}{k^{3/2}w^{3/2}}\right)$$

with $w = \sqrt{y^2/k'^2 - 1}$. For $y > 2k$ we have $w \gg 1$, while for $k + k^{1/3} < y < 2k$ we have $w \asymp \left(\frac{y-k}{k}\right)^{1/2}$. Therefore, integration of the error term produces

$$\ll l^{1/4-\sigma}k^{-5/6-2\sigma} + \int_{k+k^{1/3}}^{2k} \frac{k^{1/4}}{y^{1/2+2\sigma}(y-k)^{7/4}} \ll l^{1/4-\sigma}k^{-5/6-2\sigma} + k^{-1/2-2\sigma}.$$

Now consider a dyadic interval $[k+A, k+2A]$ with $A > k^{1/3}$. On such an interval we find that w is fixed to within a constant. Moreover,

$$\sqrt{\frac{2}{\pi k' w}}\cos\left(k'w - k'\tan^{-1} w - \frac{\pi}{4}\right)\sin\left(y\sqrt{\frac{n}{l}} - \frac{\pi}{4}\right)$$

may be written as a linear combination of exponentials of the form

$$\sqrt{\frac{2}{\pi k' w}}\, e^{i[\pm(k'w - k'\tan^{-1} w - \pi/4)\pm(y\sqrt{n/l}-\pi/4)]} = G(y)e^{iF(y)}.$$

By further subdividing $[k+A, k+2A]$ into $O(1)$ subintervals we may assume that $F'(y)/G(y)$ is monotonic. Recalling (3.14), we have

$$\frac{d^2}{dy^2}(k'w - k'\tan^{-1} w) = \frac{k'}{y^2 w},$$

and we deduce from Lemma 3.4 that for each $B \in [k+A, k+2A]$,

$$\int_{k+A}^{B} \sqrt{\frac{2}{\pi k' w}}\cos\left(k'w - k'\tan^{-1} w - \frac{\pi}{4}\right)\sin\left(y\sqrt{\frac{n}{l}} - \frac{\pi}{4}\right)dy$$
$$\ll \frac{1}{\sqrt{kw}}\sqrt{\frac{B^2 w}{k}} \ll 1 + \frac{A}{k}.$$

Thus summing dyadically we conclude that for all $z \in [k + k^{1/3}, 2000k\sqrt{l}]$ we have

$$I_z = \int_{k+k^{1/3}}^{z} \sqrt{\frac{2}{\pi k' w}}\cos\left(k'w - k'\tan^{-1} w - \frac{\pi}{4}\right)\sin\left(y\sqrt{\frac{n}{l}} - \frac{\pi}{4}\right)dy$$

is bounded by $\ll \frac{z\log l}{k}$. Write

$$\int_{k+k^{1/3}}^{2000k\sqrt{l}} W_{k,\sigma+it}\left(\frac{c^2d^2y^2}{(4\pi)^2 l}\right)\sqrt{\frac{2}{\pi k' w}}\cos\left(k'w - k'\tan^{-1} w - \frac{\pi}{4}\right) \times \sin\left(y\sqrt{\frac{n}{l}} - \frac{\pi}{4}\right)F(y)\frac{dy}{y^{1/2+2\sigma}}$$
$$= \int_{k+k^{1/3}}^{2000k\sqrt{l}} W_{k,\sigma+it}\left(\frac{c^2d^2y^2}{(4\pi)^2 l}\right)F(y)y^{-1/2-2\sigma}\,dI_y$$

and integrate by parts. Substituting our absolute bound for I_y and the bounds

$$\frac{\partial}{\partial y}W_{k,\sigma+it}\left(\frac{c^2d^2y^2}{(4\pi)^2 l}\right) \ll \frac{1}{y}, \qquad F'(y) \ll \frac{1}{y}$$

gives the result $\ll k^{-1/2-2\sigma} l^{1/4-\sigma}\log l$. This completes the proof of (B). ■

5. Mollification. Write the inverse of the L-function $L(s;f)$ as

$$L(s;f)^{-1} = \sum_{n=1}^{\infty}\frac{a_f(n)}{n^s} = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{1}{p^{2s}}\right), \qquad \Re(s) > 1.$$

The coefficients $a_f(n)$ are supported on cube-free numbers, and for m,n square-free, $(m,n)=1$, we have $a_f(mn^2) = \mu(m)\lambda_f(m)$. We define a mollifier for $L(s;f)$ by

$$M(s;f) = \sum_{n=1}^{\infty}\frac{a_f(n)F(s(n))}{n^s}. \tag{5.1}$$

Here $s(n) = \prod_{p\mid n} p$ denotes the square-free kernel of n and $F(n)$ is a cut-off function to be given explicitly later, but for which we stipulate $F(n) \ll n^{\epsilon}$ and $F(n) = 0$ for $n > M = k^{\theta}$ for some $\theta < 1/5$. In particular, we have the representation

$$|M(1/2+\sigma+it;f)|^2 = \left|\sideset{}{^\flat}\sum_{(m,n)=1}\frac{\mu(m)\lambda_f(m)F(mn)}{m^{1/2+\sigma+it}n^{1+2\sigma+2it}}\right|^2 \tag{5.2}$$
$$= \sideset{}{^\flat}\sum_{d}\frac{1}{d^{1+2\sigma}} \sideset{}{^\flat}\sum_{\substack{(m_1,n_1)=1\\(m_2,n_2)=1\\(m_1n_1m_2n_2,d)=1}} \frac{\mu(m_1)\mu(m_2)\lambda_f(m_1m_2)F(dm_1n_1)F(dm_2n_2)}{m_1^{1/2+\sigma+it}m_2^{1/2+\sigma-it}n_1^{1+2\sigma+2it}n_2^{1+2\sigma-2it}}.$$

From this representation, we find

$$(5.3)\qquad \sum_{f\in H_k}^{h} |ML(1/2+\sigma+it;f)|^2$$
$$= \sum_d^{\flat} \frac{1}{d^{1+2\sigma}} \sum^{\flat}_{\substack{(m_1,n_1)=1\\(m_2,n_2)=1\\(m_1n_1m_2n_2,d)=1}} \frac{\mu(m_1)\mu(m_2)F(m_1n_1d)F(m_2n_2d)}{m_1^{1/2+\sigma+it}m_2^{1/2+\sigma-it}n_1^{1+2\sigma+2it}n_2^{1+2\sigma-2it}}$$
$$\times \sum_{f\in H_k}^{h} \lambda_f(m_1m_2)|L(1/2+\sigma+it;f)|^2.$$

Substituting our expression for the twisted second moment, we find that

$$\text{expr. (5.3)} + O(k^{5\theta/2-2\sigma-1/2+\epsilon})$$
$$= \sum_d^{\flat} \frac{1}{d^{1+2\sigma}} \sum^{\flat}_{\substack{(m_1,n_1)=1\\(m_2,n_2)=1\\(m_1n_1m_2n_2,d)=1}} \frac{\mu(m_1)\mu(m_2)F(m_1n_1d)F(m_2n_2d)}{m_1^{1/2+\sigma+it}m_2^{1/2+\sigma-it}n_1^{1+2\sigma+2it}n_2^{1+2\sigma-2it}}$$
$$\times \left\{\zeta(1+2\sigma)\frac{\tau_{it}(m_1m_2)}{(m_1m_2)^{1/2+\sigma}} + \zeta(1-2\sigma)\left(\frac{k}{4\pi}\right)^{-4\sigma}\frac{\tau_{it}(m_1m_2)}{(m_1m_2)^{1/2-\sigma}}\right.$$
$$\left. + i^k 2\Re\left[\zeta(1+2it)\left(\frac{k}{4\pi}\right)^{-2\sigma+2it}\frac{\tau_\sigma(m_1m_2)}{(m_1m_2)^{1/2+it}}\right]\right\}$$
$$= \mathcal{S}_1 + \mathcal{S}_2 + 2i^k\Re\mathcal{S}_3.$$

We may rewrite the divisor sums:

$$\tau_s(m_1m_2) = \sum_{l_1l_2=m_1m_2}\left(\frac{l_1}{l_2}\right)^s = \sum_{g|(m_1,m_2)}\mu(g)\tau_s\left(\frac{m_1}{g}\right)\tau_s\left(\frac{m_2}{g}\right).$$

Doing so and shifting the sum over g to the front we separate the variables m_1 and m_2. Thus we find

$$\mathcal{S}_1 = \zeta(1+2\sigma)\sum_d^{\flat}\frac{1}{d^{1+2\sigma}}\sum_{(g,d)=1}\frac{\mu(g)}{g^{2+4\sigma}}\left|\sum^{\flat}_{\substack{(m,n)=1\\(mn,gd)=1}}\frac{\mu(m)\tau_{it}(m)F(mngd)}{m^{1+2\sigma+it}n^{1+2\sigma+2it}}\right|^2$$

and similar expressions for $\mathcal{S}_2$, and $\mathcal{S}_3$, although the inner sum in $\mathcal{S}_3$ is not a square. In fact, there is substantial cancellation in the inner summation for $\mathcal{S}_1$ above coming from the Möbius function. The sum is in fact equal to

$$\mathcal{S}_1 = \zeta(1+2\sigma)\sum_d^{\flat}\frac{1}{d^{1+2\sigma}}\sum_{(g,d)=1}\frac{\mu(g)}{g^{2+4\sigma}}\left|\sum^{\flat}_{(m,gd)=1}\frac{\mu(m)F(mgd)}{m^{1+2\sigma}}\right|^2.$$

We also find

$$\mathcal{S}_2 = \zeta(1-2\sigma)\left(\frac{k}{4\pi}\right)^{-4\sigma} \sideset{}{^\flat}\sum_{d} \frac{1}{d^{1+2\sigma}} \sum_{(g,d)=1} \frac{\mu(g)}{g^2} \times \left| \sideset{}{^\flat}\sum_{\substack{(m,n)=1\\(mn,gd)=1}} \frac{\mu(m)\tau_{it}(m)F(mngd)}{m^{1+it}n^{1+2\sigma+2it}}\right|^2,$$

$$\mathcal{S}_3 = \zeta(1+2it)\left(\frac{k}{4\pi}\right)^{-2\sigma+2it} \sideset{}{^\flat}\sum_{d} \frac{1}{d^{1+2\sigma}} \sum_{(g,d)=1} \frac{\mu(g)}{g^{2+2\sigma+2it}} \times \sideset{}{^\flat}\sum_{(m_1,gd)=1} \frac{\mu(m_1)F(m_1gd)}{m_1^{1+2it}} \sideset{}{^\flat}\sum_{\substack{(m_2,gd)=1\\ m_2^1m_2^2m_2^3=m_2}} \frac{\mu(m_2^1)\mu(m_2^2)F(m_2gd)}{(m_2^1)(m_2^2)^{1+2\sigma}(m_2^3)^{1+2\sigma-2it}}.$$

5.1. Upper bound for the harmonic mollified second moment. We now fix the cut-off function F and prove an upper bound for the mollified second moment. Let

$$F(x) = \begin{cases} 1, & 0 \le x \le \sqrt{M}, \\ P\left(\dfrac{\log(M/x)}{\log M}\right), & \sqrt{M} \le x \le M, \\ 0, & x \ge M, \end{cases} \tag{5.4}$$

where $P(t) = 12t^2 - 16t^3$ satisfies $P(1/2) = 1$ and $P'(1/2) = P(0) = P'(0) = 0$. The function F is continuously differentiable. Its Mellin transform is equal to

$$\hat{F}(s) = \frac{24(M^s + M^{s/2})}{s^3(\log M)^2} - \frac{96(M^s - M^{s/2})}{s^4(\log M)^3}. \tag{5.5}$$

It has a simple pole at $s = 0$ with residue 1. Also, expanding $\hat{F}(s)$ in its Laurent series about 0,

$$\hat{F}(s) = \frac{1}{s} + \sum_{n=0}^{\infty} c_n s^n, \tag{5.6}$$

we see that the coefficients c_n satisfy the bound

$$c_n \ll \frac{(\log M)^{n+1}}{(n+3)!}.$$

For this choice of cut-off function we prove

Proposition 5.1. *Let $M = k^\theta$ with $\theta < 1/5$ and suppose $1/\log k < \sigma$ and $|t| < k^{1/4}$. For $M(1/2 + \sigma + it; f)$ defined by (5.1) and cut-off function*

F as in (5.4) *we have*

$$\sum_{f\in H_k}^{h} |ML(1/2+\sigma+it;f)|^2 \le 1 + O(k^{5\theta/2-2\sigma-1/2+\epsilon}) + O(k^{-\theta\sigma}).$$

Proof. We prove $\mathcal{S}_1 = 1+O(K^{-\theta\sigma})$ and $\Re(\mathcal{S}_3) = O(K^{-\theta\sigma})$. This suffices because $\mathcal{S}_2 \le 0$ since $\zeta(1-2\sigma) < 0$.

By Mellin inversion,

$$\mathcal{S}_1 = \zeta(1+2\sigma)\left(\frac{1}{2\pi i}\right)^2 \int_{(2)}\int_{(2)} \hat{F}(\alpha)\hat{F}(\beta)G(\alpha,\beta;\sigma)\,d\alpha\,d\beta \tag{5.7}$$

where

$$\begin{aligned} G(\alpha,\beta;\sigma) &= \sum_{d}^{\flat} \frac{1}{d^{1+2\sigma+\alpha+\beta}} \sum_{(g,d)=1} \frac{\mu(g)}{g^{2+4\sigma+\alpha+\beta}} \sum_{\substack{(m_1,gd)=1\\(m_2,gd)=1}}^{\flat} \frac{\mu(m_1)\mu(m_2)}{m_1^{1+2\sigma+\alpha}m_2^{1+2\sigma+\beta}} \\ &= \prod_p (1-p^{-1-2\sigma-\alpha}-p^{-1-2\sigma-\beta}+p^{-1-2\sigma-\alpha-\beta}) \\ &= \frac{\zeta(1+2\sigma+\alpha+\beta)}{\zeta(1+2\sigma+\alpha)\zeta(1+2\sigma+\beta)} H(\alpha,\beta;\sigma). \end{aligned}$$

The Euler product defining H converges absolutely in the region

$$\alpha+2\sigma > -1/2, \qquad \beta+2\sigma > -1/2, \qquad \alpha+\beta+2\sigma > -1/2.$$

To evaluate the integral, shift both contours to the line $\Re(\alpha) = \Re(\beta) = 1/\log k$ and truncate the β integral at $|\Im(\beta)| \le k$ with error $O(k^{-2+\epsilon})$. Then shift the α integral to the contour $\mathcal{C}$ given by

$$\mathcal{C} := \{\alpha : \Re(\alpha) = -2\sigma - \log^{3/4}(2+|\Im(\alpha)|)\}.$$

In shifting the α contour to $\mathcal{C}$ we encounter poles at $\alpha = 0$ and $\alpha = -2\sigma-\beta$. This first pole yields a residue

$$\frac{1}{\zeta(1+2\sigma)}\frac{1}{2\pi i}\int_{1/\log k-ik}^{1/\log k+ik} \hat{F}(\beta)\,d\beta = \frac{1+O(k^{-2})}{\zeta(1+2\sigma)}. \tag{5.8}$$

The second pole has residue

$$\frac{1}{2\pi i}\int_{1+1/\log k-ik}^{1+1/\log k+ik} \hat{F}(\beta)\hat{F}(-2\sigma-\beta)\frac{H(-2\sigma-\beta,\beta;\sigma)}{\zeta(1-\beta)\zeta(1+2\sigma+\beta)}\,d\beta$$

Here we can extend the integration to the full line, and shift the contour to $\Re(\beta) = -\sigma$. On this line, $H(-\sigma+is,-\sigma-is;\sigma)$ is uniformly bounded, and so the integral is bounded by

$$\int_{-\infty}^{\infty} \left| \frac{\hat{F}(-\sigma+is)}{\zeta(1+\sigma-is)} \right|^2 ds \tag{5.9}$$
$$\ll M^{-\sigma}\left[\int_{-1}^{1} \left| \frac{\log^{-2} M}{|\sigma+is|^2} + \frac{\log^{-3} M}{|\sigma+is|^3} \right|^2 d|s| + O((\log M)^{-4}) \right].$$

Now using $(a+b)^2 \le 2(a^2+b^2)$, we find that the right hand side is bounded by

$$k^{-\theta\sigma}\left[O((\log k)^{-4}) + \frac{1}{(\log M)^4} \int_{-\infty}^{\infty} \frac{ds}{(s^2+\sigma^2)^2} + \frac{1}{(\log M)^6} \int_{-\infty}^{\infty} \frac{ds}{(s^2+\sigma^2)^3} \right].$$

Since $\sigma \ge 1/\log k$ we deduce that the second residue is $\ll k^{-\theta\sigma}/\log k$. The remaining integral, for α on $\mathcal{C}$, is bounded in view of standard bounds for ζ in the zero-free region and is quite small. Since $\zeta(1+2\sigma) \ll \log k$ we have the claimed evaluation of $\mathcal{S}_1$.

In bounding $2\Re(\mathcal{S}_3)$ we handle separately the cases $t \le 1/4\log k$ and $t > 1/4\log k$.

When $t > 1/4\log k$ we bound $\mathcal{S}_3$ in magnitude as we did $\mathcal{S}_1$. By Mellin inversion,

$$\mathcal{S}_3 = \zeta(1+2it)\left(\frac{k}{4\pi}\right)^{-2\sigma+2it}\left(\frac{1}{2\pi i}\right)^2 \int_{(2)}\int_{(2)} \hat{F}(\alpha)\hat{F}(\beta) G(\alpha,\beta;\sigma,t)\, d\alpha\, d\beta \tag{5.10}$$

where now $G(\alpha,\beta;\sigma,t)$ is given by

$$\begin{aligned} G(\alpha,\beta;\sigma,t) &= \sum_d{}^{\flat} \frac{1}{d^{1+2\sigma+\alpha+\beta}} \sum_{(g,d)=1} \frac{\mu(g)}{g^{2+2\sigma+2it+\alpha+\beta}} \sum_{(m_1m_2,gd)=1}{}^{\flat} \frac{\mu(m_1)}{m_1^{1+2it+\alpha}} \\ &\quad \times \sum_{m_2^1m_2^2m_2^3=m_2} \frac{\mu(m_2^1)\mu(m_2^2)}{(m_2^1)^{1+\beta}(m_2^2)^{1+2\sigma+\beta}(m_2^3)^{1+2\sigma-2it+\beta}} \\ &= \prod_p \left[1 - \frac{1}{p^{1+\beta}} - \frac{1}{p^{1+2\sigma+\beta}} + \frac{1}{p^{1+2\sigma-2it+\beta}} - \frac{1}{p^{1+2it+\alpha}} \right. \\ &\quad \left. + \frac{1}{p^{1+2\sigma+\alpha+\beta}} + \frac{1}{p^{2+2it+\alpha+\beta}} - \frac{1}{p^{2+2\sigma+\alpha+\beta}}\right] \\ &= \frac{\zeta(1+2\sigma-2it+\beta)\zeta(1+2\sigma+\alpha+\beta)}{\zeta(1+\beta)\zeta(1+2\sigma+\beta)\zeta(1+2it+\alpha)} H(\alpha,\beta;\sigma,t). \end{aligned}$$

Here the Euler product defining $H(\alpha,\beta;\sigma,t)$ converges absolutely for

$$\min(\Re(\alpha),\Re(\beta),\Re(\alpha+\beta)) > -1/2.$$

To evaluate the integral in (5.10), shift the α and β contours to the lines $\Re(\alpha)=\Re(\beta)=1/\log k$. We may assume that $\sigma<\frac{100\log\log k}{\log k}$ since otherwise $k^{-2\sigma}<k^{-\sigma}/(\log k)^{100}$ and the integral may be bounded directly using standard bounds for ζ and ζ^{-1} to the right of the 1-line. Now truncate the β contour at $|\Im(\beta)|<k$ and shift the α contour to $\mathcal{C}'$ given by

$$\mathcal{C}' := \{\alpha : \Re(\alpha) = -\log^{3/4}(2+|\Im(\alpha+2it)|)\}.$$

In doing so we pass two poles, at $\alpha=0$ and at $\alpha=-\beta-2\sigma$. The first pole has residue

$$\zeta(1+2it)^{-1}\frac{1}{2\pi i}\int_{1/\log k-ik}^{1/\log k+ik}\frac{\hat{F}(\beta)\zeta(1+2\sigma-2it+\beta)}{\zeta(1+\beta)}H(0,\beta;\sigma,t)\,d\beta$$
$$=\zeta(1+2it)^{-1}\left(\frac{\hat{F}(-2\sigma+2it)}{\zeta(1-2\sigma+2it)}H(0,-2\sigma+2it;\sigma,t)+O(1)\right).$$

Expressing $\hat{F}(-2\sigma+2it)$ using either the Laurent expansion (5.6) for $|t|<1/\log k$ or the direct definition (5.4) for $|t|>1/\log k$, together with the bound $1/\zeta(1-s)\ll s$ valid in the standard zero-free region, we deduce that this residue is $O(\zeta(1+2it)^{-1})$.

The second residue is equal to

$$\frac{1}{2\pi i}\int_{1/\log k-ik}^{1/\log k+ik}\frac{\hat{F}(-\beta-2\sigma)\hat{F}(\beta)\zeta(1+2\sigma-2it+\beta)}{\zeta(1+\beta)\zeta(1+2\sigma+\beta)\zeta(1+2it-2\sigma-\beta)}H(-2\sigma-\beta,\beta)\,d\beta.$$

Shifting this integral to the line $\Re(\beta)=-2\sigma$ (the horizontal integrals are very small), and taking absolute values, we obtain a bound

$$\ll\int_{-k}^{k}\frac{|\hat{F}(-2\sigma+is)|}{|\zeta(1-2\sigma+is)|}\,\frac{|\hat{F}(-is)|}{|\zeta(1-is)|}\,ds.$$

Arguing as above we have

$$\frac{|\hat{F}(-is)|}{|\zeta(1-is)|}=O(1)$$

for all real s, while for $|s|\le k$,

$$\frac{|\hat{F}(-2\sigma+is)|}{|\zeta(1-2\sigma+is)|}\ll M^{-\sigma}\left[\frac{1}{(\log M)^2|\sigma+is|^2}+\frac{1}{(\log M)^3|\sigma+is|^3}\right],$$

so that the integral is $O(k^{-\theta\sigma}/\log k)$ as in the second residue calculation for $\mathcal{S}_1$. The remaining double integral with α on the contour $\mathcal{C}-2it$ is again small. Thus for $1/4\log k < t$, we have

$$O(\zeta(1+it)^{-1}) + O\left(\frac{k^{-\theta\sigma}}{\log k}\right)$$

for the integral in (5.10), which suffices since in this range, $\zeta(1+2it) = O(\log k)$.

When $|t| < 1/4\log k$, we bound $2\Re\mathcal{S}_3$ to balance the fact that $\zeta(1+2it)$ can be quite large (but mostly imaginary). Following the method of [CS], let $\mathcal{O}$ be the circle $|w| = 1/2\log k$. By Cauchy's residue theorem,

$$2\Re\mathcal{S}_3 = \frac{1}{2\pi i}\int_{\mathcal{O}}\left(\frac{k}{4\pi}\right)^{-2\sigma+w}\zeta(1+w)\eta(w;\sigma)\left[\frac{1}{w+2it}+\frac{1}{w-2it}\right]dw$$

with

$$\eta(w;\sigma) = \sum_d^{\flat}\frac{1}{d^{1+2\sigma}}\sum_{(g,d)=1}\frac{\mu(g)}{g^{2+2\sigma+w}}\sum_{(m_1m_2,gd)=1}^{\flat}\frac{\mu(m_1)F(m_1gd)F(m_2gd)}{m_1^{1+w}}$$
$$\times\sum_{m_2^1m_2^2m_2^3=m_2}\frac{\mu(m_2^1)\mu(m_2^2)}{(m_2^1)(m_2^2)^{1+2\sigma}(m_2^3)^{1+2\sigma-w}}.$$

As before, we may assume that $\sigma < \frac{100\log\log k}{\log k}$. The evaluation of $\eta(w;\sigma)$ by Mellin inversion is exactly analogous to the integral performed in calculating $\mathcal{S}_3$ when $1/4\log k < |t|$: there is a main term equal to $\zeta(1+w)^{-1}O(1)$, a secondary residue term of size $M^{-\sigma}/\log k$ and a smaller error integral. Thus $\eta(w;\sigma) = O(1/\log k)$. Thus the integrand in the integral over $\mathcal{O}$ is $O(k^{-\theta\sigma}\log k)$. Since the length of $\mathcal{O}$ is $O(1/\log k)$ the integral itself is $O(k^{-\theta\sigma})$. ∎

6. Removing the harmonic weights. The starting point for the Kowalski–Michel [KM] method for removing harmonic weights is the formula ([ILS])

$$w_f^{-1} = \frac{L(1,\mathrm{sym}^2 f)}{\zeta(2)}|H_k| + O(\log^3 k),$$

where $L(s,\mathrm{sym}^2 f)$ is the symmetric square L-function associated to f, defined by

$$L(s,\mathrm{sym}^2 f) = \sum_{n=1}^{\infty}\frac{\rho_f(n)}{n^s} = \zeta(2s)\sum_n\frac{\lambda_f(n^2)}{n^s}.$$

Thus the natural average is expressed as

$$\frac{1}{|H_k|}\sum_{f\in H_k}|ML(1/2+\sigma+it;f)|^2 = \frac{1}{|H_k|}\sum_{f\in H_k}^{h} w_f^{-1}|ML(1/2+\sigma+it;f)|^2$$
$$= \frac{1}{\zeta(2)}\sum_{f\in H_k}^{h}L(1,\mathrm{sym}^2 f)|ML(1/2+\sigma+it;f)|^2 + O(k^{-1+\epsilon}).$$

The method replaces $L(1,\mathrm{sym}^2 f)$ with a short Dirichlet polynomial approximation

$$w_f(x) = \sum_{n\le x} \frac{\rho_f(n)}{n}, \qquad x = k^\kappa.$$

A minor modification to the proof of Proposition 2 of [KM] yields the following result.

PROPOSITION 6.1. *Assume that the mollifier $M(1/2+\sigma+it;f)$ is such that*

$$\sup_{f\in H_k} w_f |ML(1/2+\sigma+it;f)|^2 < k^{-\delta}, \qquad \delta > 0, \tag{6.1}$$

and

$$\sum_{f\in H_k}^{h} |ML(1/2+\sigma+it;f)|^2 < (\log k)^A. \tag{6.2}$$

Let $x = k^\kappa$ for some $\kappa > 0$. Then there is a $\gamma = \gamma(\delta,\kappa,A) > 0$ such that

$$\frac{1}{|H_k|}\sum_{f\in H_k} |ML(1/2+\sigma+it;f)|^2 = \sum_{f\in H_k}^{h} w_f(x)|ML(1/2+\sigma+it;f)|^2 + O(k^{-\gamma}).$$

The result of the previous section guarantees condition (6.2) so long as the mollifier is a Dirichlet polynomial of length less than k^θ with $\theta < 1/5$. Trivially $|M(1/2+\sigma+it)| < k^{\theta/2+\epsilon}$ and the best known subconvex bound (see [JM]) implies that $L(1/2+\sigma+it) \ll (k+|t|)^{1/3-2\sigma/3+\epsilon}$. Thus condition (6.1) holds uniformly in $|t| < k$ for $\theta < 1/3$. Therefore, we complete the proof of Proposition 2.2 by proving the following uniform bound.

PROPOSITION 6.2. *For sufficiently small $\kappa,\delta,\theta > 0$ there exists $\gamma(\kappa,\delta,\theta) > 0$ such that, uniformly in $1/\log k < \sigma \le 1$ and $|t| < k^\delta$,*

$$\frac{1}{\zeta(2)}\sum_{f\in H_k}^{h}\Big(\sum_{n\le x=k^\kappa}\frac{\rho_f(n)}{n}\Big)|ML(1/2+\sigma+it;f)|^2 \le 1 + O(k^{-\theta\sigma} + k^{-\kappa/2+\epsilon}),$$

where M is the mollifier from the previous section, having length $M = k^\theta$.

Proof. Combining expression (5.2) for $|M(1/2+\sigma+it;f)|^2$ with $\sum_{n\le x}\rho_f(n)/n = \sum_{l^2 d<x}\lambda_f(d^2)/l^2 d$ and the Hecke relations, we obtain

$$\sum_{n\le x}\frac{\rho_f(n)}{n}|M(1/2+\sigma+it;f)|^2=\sum_{l^2d<x}\frac{1}{l^2d}\sideset{}{^\flat}\sum_{g}\frac{1}{g^{1+2\sigma}}$$
$$\times\sideset{}{^\flat}\sum_{\substack{(m_1,n_1)=1\\(m_2,n_2)=1\\(m_in_i,g)=1}}\frac{\mu(m_1)\mu(m_2)F(m_1n_1g)F(m_2n_2g)}{m_1^{1/2+\sigma+it}m_2^{1/2+\sigma-it}n_1^{1+2\sigma+2it}n_2^{1+2\sigma-2it}}\sum_{h|(d^2,m_1m_2)}\lambda_f\Big(\frac{m_1m_2d^2}{h^2}\Big).$$

Write $h=h_1h_2^2$ where h_1 and h_2 are square-free. Clearly $h_2\,|\,(m_1,m_2)$ and $h_2\,|\,d$. Also, $h_1\,|\,(d,m_1m_2)$. Shifting the orders of summation, and then introducing our expression for the twisted second moment, we obtain

$$\sideset{}{^h}\sum_{f\in H_k}\Big(\sum_{n\le x=k^\kappa}\frac{\rho_f(n)}{n}\Big)|ML(1/2+\sigma+it;f)|^2=\sum_{l}\frac{1}{l^2}\sideset{}{^\flat}\sum_{(g,h_2)=1}\frac{1}{g^{1+2\sigma}h_2^{2+2\sigma}}$$
$$\times\sideset{}{^\flat}\sum_{\substack{(m_1,n_1)=1\\(m_2,n_2)=1\\(m_1m_2n_1n_2,gh_2)=1}}\frac{\mu(m_1)\mu(m_2)F(m_1n_1gh_2)F(m_2n_2gh_2)}{m_1^{1/2+\sigma+it}m_2^{1/2+\sigma-it}n_1^{1+2\sigma+2it}n_2^{1+2\sigma-2it}}\sideset{}{^\flat}\sum_{h_1|m_1m_2}\frac{1}{h_1}$$
$$\times\sum_{d<x/l^2h_1h_2}\Big\{\frac{\zeta(1+2\sigma)\tau_{it}(m_1m_2d^2)}{d(m_1m_2d^2)^{1/2+\sigma}}+\Big(\frac{k}{4\pi}\Big)^{-4\sigma}\frac{\zeta(1-2\sigma)\tau_{it}(m_1m_2d^2)}{d(m_1m_2d^2)^{1/2-\sigma}}$$
$$+2i^k\Re\Big(\Big(\frac{k}{4\pi}\Big)^{-2\sigma+2it}\frac{\zeta(1+2it)\tau_\sigma(m_1m_2d^2)}{d(m_1m_2d^2)^{1/2+it}}\Big)+O\Big(\frac{d^{3/2}(m_1m_2)^{3/4}}{k^{1/2+\sigma-\epsilon}}\Big)\Big\}.$$

The error term contributes $\ll k^{-1/2+5\theta/2-2\sigma+\epsilon}x^{2+2\sigma}\ll k^{-1/2+5\theta/2+\epsilon}x^2$.

For $\sigma>1/4$, the terms involving $\zeta(1-2\sigma)$ and $\zeta(1+2it)$ are negligibly small and so we are left to consider only the $\zeta(1+2\sigma)$ term; otherwise, for $1/\log k<\sigma<1/4$ we consider all three terms. In either case, we may remove the restriction on the sum over d with error $\ll x^{-1/2+\epsilon}$. Thus

$$\frac{1}{\zeta(2)}\sideset{}{^h}\sum_{f\in H_k}\Big(\sum_{n\le x=k^\kappa}\frac{\rho_f(n)}{n}\Big)|ML(1/2+\sigma+it;f)|^2=\mathcal{S}_1+\mathcal{S}_2+2\Re\mathcal{S}_3$$

with the stipulation that $\mathcal{S}_2=\mathcal{S}_3=0$ if $\sigma>1/4$.

We use the following lemma.

Lemma 6.3. *Let m_1 and m_2 be square-free. For $\Re(s\pm\gamma)>1$ we have*

$$\sum_d\frac{\tau_\gamma(m_1m_2d^2)}{d^s}=\frac{\zeta(s)}{\zeta(2s)}\zeta(s+2\gamma)\zeta(s-2\gamma)$$
$$\times\prod_{p|\frac{m_1m_2}{(m_1,m_2)^2}}\frac{p^\gamma+p^{-\gamma}}{1+p^{-s}}\prod_{p|(m_1,m_2)}\frac{1+p^{2\gamma}+p^{-2\gamma}-p^{-s}}{1+p^{-s}}.$$

We first prove that for $\sigma < 1/4$, $\mathcal{S}_2 < 0$, so that it may be completely discarded. Since we assume $\sigma < 1/4$, we have

$$\begin{aligned}\mathcal{S}_2 = \zeta(1-2\sigma)\frac{\zeta(2-2\sigma)}{\zeta(4-4\sigma)}|\zeta(2-2\sigma+2it)|^2 \sideset{}{^\flat}\sum_{k=gh} \frac{1}{g^{1-2\sigma}h^{2-2\sigma}} \\ \times \sideset{}{^\flat}\sum_{\substack{(m_1,n_1)=1\\(m_2,n_2)=1\\(m_1n_1m_2n_2,k)=1}} \frac{\mu(m_1)\mu(m_2)F(m_1n_1k)F(m_2n_2k)}{m_1^{1+it}m_2^{1-it}n_1^{1+2\sigma+2it}n_2^{1+2\sigma-2it}} \\ \times \prod_{p|m_1}\left(\frac{p+1}{p}\,\frac{p^{it}+p^{-it}}{1+p^{-2+2\sigma}}\right)\prod_{p|m_2}\left(\frac{p+1}{p}\,\frac{p^{it}+p^{-it}}{1+p^{-2+2\sigma}}\right) \\ \times \prod_{p|(m_1,m_2)}\left(\frac{p}{p+1}\,\frac{(1+p^{2it}+p^{-2it}-p^{-2+2\sigma})(1+p^{-2+2\sigma})}{(p^{it}+p^{-it})^2}\right).\end{aligned}$$

This may be rearranged as

$$\begin{aligned}\zeta(1-2\sigma)\frac{\zeta(2-2\sigma)}{\zeta(4-4\sigma)}|\zeta(2-2\sigma+2it)|^2 \sideset{}{^\flat}\sum_{k=ghr} \frac{a(r)}{g^{1-2\sigma}h^{2-2\sigma}r^2} \\ \times \left|\sideset{}{^\flat}\sum_{\substack{(m_1,n_1)=1\\(m_1n_1,k)=1}} \frac{\mu(m_1)F(m_1n_1k)}{m_1^{1+it}n_1^{1+2\sigma+2it}}\prod_{p|m_1}\left(\frac{p+1}{p}\,\frac{p^{it}+p^{-it}}{1+p^{-2+2\sigma}}\right)\right|^2,\end{aligned}$$

where $a(r)$ is the multiplicative function, supported on square-free integers, and given on primes by

$$\begin{aligned}a(p) &= \frac{p+1}{p}\,\frac{1+p^{2it}+p^{-2it}-p^{-2+2\sigma}}{1+p^{-2+2\sigma}} - \left(\frac{p+1}{p}\,\frac{p^{it}+p^{-it}}{1+p^{-2+2\sigma}}\right)^2 \\ &= -\frac{p+1}{p} - (p^{it}+p^{-it})^2\left[\left(\frac{p+1}{p+p^{-1+2\sigma}}\right)^2 - \frac{p+1}{p+p^{-1+2\sigma}}\right].\end{aligned}$$

Now observe

$$\sum_{ghr=k}\frac{a(r)}{g^{1-2\sigma}h^{1-2\sigma}r^2} = \prod_{p|k} b(p), \qquad b(p) = \frac{1}{p^{1-2\sigma}} + \frac{1}{p^{2-2\sigma}} + \frac{a(p)}{p^2}.$$

We have $b(p) \geq 0$; indeed, it suffices to check this under the conditions $|p^{it}+p^{-it}| = 2$, $\sigma = 0$ and $p = 2$, and in this case we find a value of 0.135. In particular, since $\zeta(1-2\sigma) < 0$ this proves that $\mathcal{S}_2 \leq 0$.

Next we turn to $\mathcal{S}_1$. We have

$$\begin{aligned}\mathcal{S}_1 &= \zeta(1+2\sigma)\frac{\zeta(2+2\sigma)}{\zeta(4+4\sigma)}|\zeta(2+2\sigma+2it)|^2 \sideset{}{^\flat}\sum_{k=gh} \frac{1}{g^{1+2\sigma}h^{2+2\sigma}} \\ &\quad \times \sideset{}{^\flat}\sum_{\substack{(m_1,n_1)=1\\(m_2,n_2)=1\\(m_1n_1m_2n_2,k)=1}} \frac{\mu(m_1)\mu(m_2)F(m_1n_1k)F(m_2n_2k)}{m_1^{1+2\sigma+it}m_2^{1+2\sigma-it}n_1^{1+2\sigma+2it}n_2^{1+2\sigma-2it}} \\ &\quad \times \prod_{p|m_1}\left(\frac{p+1}{p}\,\frac{p^{it}+p^{-it}}{1+p^{-2-2\sigma}}\right)\prod_{p|m_2}\left(\frac{p+1}{p}\,\frac{p^{it}+p^{-it}}{1+p^{-2-2\sigma}}\right) \\ &\quad \times \prod_{p|(m_1,m_2)}\left(\frac{p}{p+1}\,\frac{(1+p^{2it}+p^{-2it}-p^{-2-2\sigma})(1+p^{-2-2\sigma})}{(p^{it}+p^{-it})^2}\right) \\ &= \zeta(1+2\sigma)\frac{\zeta(2+2\sigma)}{\zeta(4+4\sigma)}|\zeta(2+2\sigma+2it)|^2 \\ &\quad \times \left(\frac{1}{2\pi i}\right)^2 \int_{(2)}\int_{(2)} \hat{F}(\alpha)\hat{F}(\beta)G(\alpha,\beta;\sigma,t)\,d\alpha\,d\beta\end{aligned}$$

where G is given by

$$\begin{aligned}&G(\alpha,\beta;\sigma,t)\\ &\quad= \frac{\zeta(4+4\sigma)}{\zeta(2+2\sigma)}\prod_p\Bigg[1+\frac{1}{p^{2+2\sigma}}+\frac{1}{p^{1+2\sigma+\alpha+\beta}}+\frac{1}{p^{2+2\sigma+\alpha+\beta}}+\frac{1}{p^{4+4\sigma+\alpha+\beta}} \\ &\qquad -\frac{1}{p^{5+6\sigma+\alpha+\beta}}-\frac{1}{p^{1+2\sigma+\alpha}}-\frac{1}{p^{2+2\sigma+\alpha}}-\frac{1}{p^{2+2\sigma+2it+\alpha}} \\ &\qquad +\frac{1}{p^{3+4\sigma+2it+\alpha}}-\frac{1}{p^{1+2\sigma+\beta}}-\frac{1}{p^{2+\sigma+\beta}}-\frac{1}{p^{2+2\sigma-2it+\beta}}+\frac{1}{p^{3+4\sigma-2it+\beta}}\Bigg].\end{aligned}$$

Here

$$G(\alpha,\beta;\sigma,t) = \frac{\zeta(1+2\sigma+\alpha+\beta)}{\zeta(1+2\sigma+\alpha)\zeta(1+2\sigma+\beta)}\tilde{H}(\alpha,\beta;\sigma,t),$$

where $\tilde{H}$ is given by an absolutely convergent Euler product for

$$\min(\Re(\alpha),\Re(\beta),\Re(\alpha+\beta)) > -2\sigma - c$$

for some $c>0$. This is to say that the contour giving $\mathcal{S}_1$ under the natural average is the same as for the harmonic average up to a change in the absolutely convergent Euler product. Thus the analysis of $\mathcal{S}_1$ from the previous section goes through without change to give

$$\mathcal{S}_1 = \zeta(1+2\sigma)\frac{\zeta(2+2\sigma)}{\zeta(4+4\sigma)}|\zeta(2+2\sigma+2it)|^2 G(0,0;\sigma,t) + O(k^{-\theta\sigma})$$
$$= 1 + O(k^{-\theta\sigma}).$$

The reader may check that the contour integral giving $\mathcal{S}_3$ is the same for the natural average as for the harmonic average, up to an absolutely convergent Euler product. Thus the analysis of the previous section yields the bound $\Re(\mathcal{S}_3) = O(k^{-\theta\sigma})$, which completes the proof of Proposition 6.2. ■

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