ACTA ARITHMETICA
117.2 (2005)

On the error term of an asymptotic formula of Ramanujan

by

H. Maier (Ulm) and A. Sankaranarayanan (Mumbai)

Dedicated to Professor Yoichi Motohashi
on his sixtieth birthday

1. Introduction. In [21], Ramanujan records (without proof) many curious asymptotic formulae. One of them is

$$d^2(1)+d^2(2)+\cdots+d^2(n)=An(\log n)^3+Bn(\log n)^2+Cn\log n+Dn+O(n^{3/5+\varepsilon}). \tag{1.1}$$

Also he records (without proof) the result that on the assumption of the Riemann hypothesis, the error term in (1.1) can be improved to $O(n^{1/2+\varepsilon})$. In view of a method due to H. L. Montgomery and R. C. Vaughan (see [17]), it is very likely that the error term is $O(n^{1/2})$. We propose this as a conjecture (see also [19], [22]). Unconditionally, the error term related to $d^2(j)$ is known to be $O(n^{1/2+\varepsilon})$ for any positive constant ε (see for example equation (14.30) of [10] and also [5] and [28]). Professor A. Schinzel has already considered some of the problems of Ramanujan (see [24]), namely for the arithmetic function $r^2(n)$, and he has proved that the corresponding error term is $\Omega(n^{3/8})$. Also the corresponding error term is $O(n^{1/2}(\log n)^{8/3}(\log\log n)^{1/3})$ which is due to M. Kühleitner and W. G. Nowak (see [15], [16]). Let

$$E(x)=\sum_{n\le x}d^2(n)-xP_3(\log x), \tag{1.2}$$

where $P_3(y)$ is a polynomial in y of degree 3. From a general theorem of M. Kühleitner and W. G. Nowak (see e.g. (5.4) of [15]), it follows that

$$E(x)=\Omega(x^{3/8}).$$

2000 *Mathematics Subject Classification*: Primary 11M; Secondary 11M06, 11N05.
Key words and phrases: error term, Riemann zeta-function, Möbius function, Minkowski's inequality.

Let

$$(1.3)\qquad \sum_{n\le x} d_4(n) = ax(\log x)^3 + bx(\log x)^2 + cx(\log x) + dx + O(x^{\alpha}).$$

Assuming that $\alpha < 1/2$ in (1.3), D. Suryanarayana and R. Sitaramachandra rao (see [25]) showed that (with some $A > 0$)

$$(1.4)\qquad E(x) \ll x^{1/2}\exp(-A(\log x)^{3/5}(\log\log x)^{-1/5}),$$

and assuming additionally the Riemann hypothesis, they established (see [25]) that

$$(1.5)\qquad E(x) \ll x^{\frac{2-\alpha}{5-4\alpha}}\exp(A(\log x)(\log\log x)^{-1}).$$

In [20], the second author jointly with K. Ramachandra proved that unconditionally, we have

$$(1.6)\qquad E(x) \ll x^{1/2}(\log x)^5(\log\log x).$$

It should be mentioned that recently M. Kühleitner and W. G. Nowak (see [16]) have given a precise upper bound for the error term related to the average number of solutions of the Diophantine equation $u^2 + v^2 = w^3$, and their arguments are in fact more general. For some more general interesting results, we refer to for example [1], [2], [3] and [23]; we also mention some related references [4], [14] and [27].

The main aim of this paper is to prove:

For $Y \ge Y_0$, we have (unconditionally)

$$(1.7)\qquad \frac{1}{Y}\int_{Y}^{2Y} (E(x))^2\,dx \ll Y\exp(-C(\log Y)^{3/5}(\log\log Y)^{-1/5})$$

for an effective positive constant C.

That is, the natural but unproven conjectural inequality (1.4) is true in mean-square. This is established in a more general frame involving the integers k, l in Theorem 2 below.

Let $k \ge 2$ and $l \ge 2$ be integers. We define the Dirichlet series (in $\sigma > 1$)

$$F(s) = \frac{\zeta^k(s)}{\zeta(ls)} = \sum_{n=1}^{\infty} b_n n^{-s}.$$

Then, from the Perron formula (see for example [18]), we obtain

$$(1.8)\qquad \begin{aligned}\sum_{n\le x} b_n &= A_{(k-1)}x(\log x)^{k-1} + A_{(k-2)}x(\log x)^{k-2}\\ &\quad + A_{(k-3)}x(\log x)^{k-3} + \cdots + A_{(0)}x + E_{k,l}(x)\\ &=: M_{k,l}(x) + E_{k,l}(x),\end{aligned}$$

$$\begin{aligned}(1.9)\qquad \sum_{n\le x} d_k(n) &= D_{(k-1)}x(\log x)^{k-1} + D_{(k-2)}x(\log x)^{k-2}\\ &\quad + D_{(k-3)}x(\log x)^{k-3} + \cdots + D_{(0)}x + \Delta_k(x)\\ &=: M_k(x) + \Delta_k(x).\end{aligned}$$

Note that the coefficients $A_{(j)}$ in (1.8) will depend on l whereas $D_{(j)}$ in (1.9) are independent of l.

We study the more general error term $E_{k,l}(x)$ of the Ramanujan type. We define

$$(1.10)\qquad \alpha_k := \inf\{\alpha : \Delta_k(x) \ll x^{\alpha}\}$$

and

$$(1.11)\qquad \beta_k := \inf\Big\{\beta : \int_2^Y (\Delta_k(x))^2\,dx \ll Y^{1+2\beta}\Big\}.$$

It is already known from the work of Kolesnik (see [13] and [12] respectively) that $\alpha_2 \le 139/429, \alpha_3 \le 43/96$ (a better value of $\alpha_2 \le 23/73$ is known from the work of M. N. Huxley (see [9]) and in fact $\alpha_2 \le 131/416$ from an unpublished work of M. N. Huxley), and from the work of D. R. Heath-Brown (see [7] and [8]) that

$$\alpha_k = \begin{cases} 3/4 - 1/k & \text{for } 4 \le k \le 8,\\ 1 - 3/k & \text{for } k \ge 8.\end{cases}$$

Better upper bounds are available for certain intermediate values of k (see Theorem 13.2 of [10]), namely $\alpha_9 \le 35/54, \alpha_{10} \le 41/60, \alpha_{11} \le 7/10$ and $\alpha_{12} \le 5/7$.

General Conjecture. *For every integer $k \ge 2$, we have*

$$\alpha_k = \frac{k-1}{2k}.$$

Regarding β_k, first of all we observe that $\beta_k \le \alpha_k$. It is already known that (see Theorem 12.6(A) of [26])

$$\beta_k \ge \frac{k-1}{2k}.$$

We also know (see Theorem 12.8 of [26] and also Theorems 13.9 and 13.10 of [10]) that $\beta_2 = 1/4, \beta_3 = 1/3$, and from the work of D. R. Heath-Brown (see [7] and [8]) that $\beta_4 = 3/8$. We should also mention a result of Jutila (see [11]) which states that if $\alpha_2 = 1/4$, then $\mu(1/2) \le 3/20$ and $E^*(T) \ll T^{5/16+\varepsilon}$, where $\mu(1/2) = \inf\{\xi : \zeta(1/2+it) \ll (|t|+10)^{\xi}\}$ and

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt = T\log\left(\frac{T}{2\pi}\right) + (2\gamma-1)T + E^*(T).$$

Throughout the paper, we write

$$(1.12)\qquad \delta(x) := \exp(-A(\log x)^{3/5}(\log\log x)^{-1/5})$$

with A being a positive constant, and we assume that $x \geq e^{le^l}$ and $Y \geq 100e^{le^l}$. We prove

THEOREM 1. *For every $\varepsilon > 0$ and for $x \geq x_0(l)$, we have*

$$E_{k,l}(x) \ll \begin{cases} x^{\alpha_k+2\varepsilon} & \text{if } l\alpha_k \geq 1 - l\varepsilon, \\ x^{1/l} & \text{if } l\alpha_k < 1 - l\varepsilon. \end{cases}$$

THEOREM 2. *For every $\varepsilon > 0$ and $Y \geq Y_0(l)$, we have*

$$\begin{aligned} I &= \frac{1}{Y}\int_Y^{2Y} (E_{k,l}(x))^2\,dx \\ &\ll \begin{cases} Y^{2\beta_k+2\varepsilon} & \text{if } l\beta_k \geq 1 - l\varepsilon/2, \\ Y^{2/l}\exp(-C(\log Y)^{3/5}(\log\log Y)^{-1/5}) & \text{if } l\beta_k < 1 - l\varepsilon/2, \end{cases} \end{aligned}$$

where C is an effective positive constant depending only on k, l and ε.

2. Notation and preliminaries. C and A (with or without suffixes) denote effective positive constants unless otherwise specified, which need not be the same at each occurrence. We write $f(x) \ll g(x)$ to mean $|f(x)| < C_1 g(x)$ (sometimes we use the O notation also). The notation $[x]$ denotes the integral part of x. The implied constants are all effective. We assume that $x \geq x_0(l)$ and $Y \geq Y_0(l)$ where $x_0(l)$ and $Y_0(l)$ are positive constants depending only on l.

3. Some lemmas

LEMMA 3.1. *We have the relation*

$$b_n = \sum_{j^l \mid n} \mu(j) d_k\left(\frac{n}{j^l}\right).$$

Proof. The proof is obvious. ■

LEMMA 3.2. *For $s > 1$ and $r \geq 0$, we have*

$$\sum_{n \leq x} n^{-s}\mu(n)(\log n)^r = (-1)^r \eta^{(r)}(s) + O(x^{-(s-1)}\delta(x)(\log x)^r), \tag{3.2.1}$$

where $\eta^{(0)}(s) = \eta(s) = (\zeta(s))^{-1}$ and $\eta^{(r)}(s)$ for $r \geq 1$ denotes the rth derivative of $\eta(s) = (\zeta(s))^{-1}$.

Proof. This is Lemma 2.2 of [25]. ■

LEMMA 3.3. *For $x \geq x_0(l)$, we have*

$$\begin{aligned} &E_{k,l}(x) \\ &\ll \Big|\sum_{n \leq \varrho x^{1/l}} \mu(n)\Delta_k\left(\frac{x}{n^l}\right)\Big| + x^{1/l}\varrho^{1-l}\delta(\varrho x^{1/l})\left(\log\left(\max\left(\frac{1}{\varrho}, x\right)\right)\right)^{C_k}, \end{aligned} \tag{3.3.1}$$

where $0 < \varrho\ (= \varrho(x)) < 1$ *and* C_k *is an effective positive constant depending only on* k.

Proof. We fix $z = x^{1/l}$ and let $\varrho\ (= \varrho(x))$ be a number (or a function of x) which satisfies $0 < \varrho < 1$. We will choose ϱ appropriately later. We notice that if $n^l r \le x$, then both $n > \varrho z$ and $r > \varrho^{-l}$ cannot hold simultaneously, and hence

$$(3.3.2)\qquad \sum_{n\le x} b_n = \sum_{\substack{n^l r\le x\\ n\le \varrho z}} \mu(n)d_k(r) + \sum_{\substack{n^l r\le x\\ r\le \varrho^{-l}}} \mu(n)d_k(r) - \sum_{\substack{n\le \varrho z\\ r\le \varrho^{-l}}} \mu(n)d_k(r) =: S_1 + S_2 - S_3.$$

From (1.9), we have

$$(3.3.3)\qquad S_1 = \sum_{\substack{n^l r\le x\\ n\le \varrho z}} \mu(n)d_k(r) = \sum_{n\le \varrho z} \mu(n) \sum_{r\le xn^{-l}} d_k(r)$$
$$= \sum_{n\le \varrho z} \mu(n)\{M_k(xn^{-l}) + \Delta_k(xn^{-l})\}$$
$$= \{D_{(k-1)}x(\log x)^{k-1} + D_{(k-2)}x(\log x)^{k-2} + \cdots + D_{(0)}x\} \times \Big(\sum_{n\le \varrho z} \mu(n)n^{-l}\Big)$$
$$- lx\bigg(D_{(k-1)}\binom{k-1}{1}(\log x)^{k-2} + D_{(k-2)}\binom{k-2}{1}(\log x)^{k-3} + \cdots\bigg) \times \Big(\sum_{n\le \varrho z} \mu(n)n^{-l}(\log n)\Big)$$
$$+ l^2x\bigg(D_{(k-1)}\binom{k-1}{2}(\log x)^{k-3} + D_{(k-2)}\binom{k-2}{2}(\log x)^{k-4} + \cdots\bigg) \times \Big(\sum_{n\le \varrho z} \mu(n)n^{-l}(\log n)^2\Big)$$
$$- \cdots + (-1)^{k-1}D_{(k-1)}l^{k-1}x\Big(\sum_{n\le \varrho z} \mu(n)n^{-l}(\log n)^{k-1}\Big)$$
$$+ \sum_{n\le \varrho z} \mu(n)\Delta_k(xn^{-l}).$$

Applying Lemma 3.2 for $r = 0, 1, \dots, k-1$ and $s = l$, we obtain

$$(3.3.4)\qquad S_1 = \{D_{(k-1)}x(\log x)^{k-1} + D_{(k-2)}x(\log x)^{k-2} + \cdots + D_{(0)}x\} \times ((\zeta(l))^{-1} + O((\varrho z)^{1-l}\delta(\varrho z)))$$

$$
\begin{aligned}
&- lx\left(D_{(k-1)}\binom{k-1}{1}(\log x)^{k-2} + D_{(k-2)}\binom{k-2}{1}(\log x)^{k-3} + \cdots \right)\\
&\times (-\eta^{(1)}(l) + O((\varrho z)^{1-l}\delta(\varrho z)\log(\varrho z)))\\
&+ l^2 x\left(D_{(k-1)}\binom{k-1}{2}(\log x)^{k-3} + D_{(k-2)}\binom{k-2}{2}(\log x)^{k-4} + \cdots \right)\\
&\times (\eta^{(2)}(l) + O((\varrho z)^{1-l}\delta(\varrho z)(\log(\varrho z))^2))\\
&- \cdots + (-1)^{k-1} D_{(k-1)} l^{k-1} x((-1)^{k-1}\eta^{(k-1)}(l)\\
&+ O((\varrho z)^{1-l}\delta(\varrho z)(\log(\varrho z))^{k-1})) + \sum_{n\le \varrho z}\mu(n)\Delta_k(xn^{-l})\\
=\ & M_{k,l}(x) + O(x(\varrho z)^{1-l}\delta(\varrho z)(\log x)^{k-1}) + \sum_{n\le \varrho z}\mu(n)\Delta_k(xn^{-l}).
\end{aligned}
$$

We find that

$$
\begin{aligned}
(3.3.5)\qquad S_2 &= \sum_{\substack{n^l r\le x\\ r\le \varrho^{-l}}} \mu(n) d_k(r) = \sum_{r\le \varrho^{-l}} d_k(r) \sum_{n\le (x/r)^{1/l}} \mu(n)\\
&= \sum_{r\le \varrho^{-l}} d_k(r) M((x/r)^{1/l})\\
&\ll x^{1/l} \sum_{r\le \varrho^{-l}} d_k(r) r^{-1/l}\left(\delta\left(\left(\frac{x}{r}\right)^{1/l}\right)\right)\\
&\ll x^{1/l}\varrho^{1-l}\delta(\varrho z)(\log(\varrho^{-l}))^{C_k},
\end{aligned}
$$

since $\left(\frac{x}{r}\right)^{1/l} > \varrho z$, δ is decreasing, $\delta\left(\left(\frac{x}{r}\right)^{1/l}\right) \le \delta(\varrho z)$, and

$$
\sum_{r\le \varrho^{-l}} d_k(r) r^{-1/l} = \sum_{r\le \varrho^{-l}} \frac{d_k(r)}{r} r^{1-1/l} \ll \varrho^{1-l}(\log(\varrho^{-l}))^{C_k}.
$$

We also notice that

$$
\begin{aligned}
(3.3.6)\qquad S_3 &= \sum_{\substack{n\le \varrho z\\ r\le \varrho^{-l}}} \mu(n) d_k(r) = \sum_{r\le \varrho^{-l}} d_k(r) M(\varrho z)\\
&\ll \varrho^{-l}(\log(\varrho^{-l}))^{C_k'}(\varrho z)\delta(\varrho z)\\
&\ll x^{1/l}\varrho^{1-l}\delta(\varrho z)(\log(\varrho^{-l}))^{C_k'}
\end{aligned}
$$

for $z = x^{1/l}$. Now the lemma follows from (3.3.2) and (3.3.4)–(3.3.6). ∎

4. Proof of the theorems

Proof of Theorem 1. We choose $\varrho = 1/10$ and note that $z = x^{1/l}$. Therefore (from Lemma 3.3 and from the definition (1.10)), we obtain

$$(4.1)\quad E_{k,l}(x)$$
$$\ll_l \sum_{n\le z/10}\left|\Delta_k\left(\frac{x}{n^l}\right)\right| + x^{1/l}\delta\left(\frac{z}{10}\right)(\log x)^{C_k}$$
$$\ll_l \sum_{n\le z/10}\left(\frac{x}{n^l}\right)^{\alpha_k+\varepsilon} + x^{1/l}\delta\left(\frac{z}{10}\right)(\log x)^{C_k}$$
$$\ll_l \begin{cases} x^{\alpha_k+2\varepsilon} + x^{1/l}\delta(z/10)(\log x)^{C_k} & \text{if } l\alpha_k \ge 1-l\varepsilon,\\ x^{\alpha_k+\varepsilon}(z/10)^{1-l\alpha_k-l\varepsilon} + x^{1/l}\delta(z/10)(\log x)^{C_k} & \text{if } l\alpha_k < 1-l\varepsilon\end{cases}$$
$$\ll_{l,\varepsilon} \begin{cases} x^{\alpha_k+2\varepsilon} & \text{if } l\alpha_k \ge 1-l\varepsilon,\\ x^{1/l} & \text{if } l\alpha_k < 1-l\varepsilon,\end{cases}$$

since, for $x \ge e^{le^l}$, we note that

$$x^{1/l}\delta\left(\frac{z}{10}\right)(\log x)^{C_k} = x^{1/l}\delta\left(\frac{x^{1/l}}{10}\right)(\log x)^{C_k}$$
$$\ll x^{1/l}\exp(-C(\log x)^{3/5}(\log\log x)^{-1/5}).$$

This proves Theorem 1. ∎

Proof of Theorem 2. We choose here $\varrho = (\delta(x^{1/l}))^{1/10}$ and note that $z = x^{1/l}$. Set

$$f(x) := \varrho z = x^{1/l}((\delta(x^{1/l}))^{1/10}).$$

From Lemma 3.3, we have

$$(4.2)\quad E_{k,l}(x) \ll \sum_{n\le \varrho x^{1/l}}\left|\Delta_k\left(\frac{x}{n^l}\right)\right| + x^{1/l}\varrho^{1-l}\delta(\varrho x^{1/l})\left(\log\left(\max\left(\frac{1}{\varrho},x\right)\right)\right)^{C_k}$$
$$\ll E_1 + E_2.$$

Without loss of generality the constant A in (1.12) can be taken to be < 1. Note that $x \ge e^{le^l}$. Now, we observe that

$$(4.3)\quad f(x) := \varrho z = x^{1/l}((\delta(x^{1/l}))^{1/10}) \ge x^{1/2l}$$

if $x \ge e^{l^{1/2}}$; but we have already assumed that $x \ge e^{le^l}$. Since the function δ is decreasing, we find that

$$\delta(\varrho z) \le \delta(x^{1/2l}).$$

Note that

$$(4.4)\quad \varrho^{1-l}\delta(\varrho z) \le \varrho^{-1}\delta(\varrho z) \le \delta(x^{1/2l})(\delta(x^{1/l}))^{-1/10}$$

$$\begin{aligned}&= \exp(-A(\log(x^{1/2l}))^{3/5}(\log\log(x^{1/2l}))^{-1/5})\\ &\quad\times \exp\left(\frac{A}{10}(\log(x^{1/l}))^{3/5}(\log\log(x^{1/l}))^{-1/5}\right)\\ &\le \exp(-C(\log x)^{3/5}(\log\log x)^{-1/5}),\end{aligned}$$

provided $x \ge e^{l^2}$. Hence, clearly,

$$\frac{1}{Y}\int_Y^{2Y} E_2^2\,dx \ll Y^{2/l}\exp(-C(\log Y)^{3/5}(\log\log Y)^{-1/5}). \tag{4.5}$$

We note that for $Y \le x \le 2Y$, we have $f(x) \le f(2Y)$. Now,

$$\begin{aligned} I_1 &:= \int_Y^{2Y} E_1^2\,dx = \int_Y^{2Y}\Bigg(\sum_{n\le \varrho x^{1/l}}\left|\Delta_k\left(\frac{x}{n^l}\right)\right|\Bigg)^2 dx \\ &\ll \int_Y^{2Y}\left(|\Delta_k(x)| + \left|\Delta_k\left(\frac{x}{2^l}\right)\right| + \cdots + \left|\Delta_k\left(\frac{x}{[f(x)]^l}\right)\right|\right)^2 dx \\ &\quad + \int_Y^{2Y}\left(\frac{x}{[f(x)]^l}\right)^{2(\alpha_k+\varepsilon)} dx \\ &\ll \int_Y^{2Y}\left(|\Delta_k(x)| + \left|\Delta_k\left(\frac{x}{2^l}\right)\right| + \cdots + \left|\Delta_k\left(\frac{x}{[f(x)]^l}\right)\right|\right)^2 dx + Y^{1+10\varepsilon}. \end{aligned} \tag{4.6}$$

We note that (for $Y \ge 100e^{le^l}$),

$$(\delta((2Y)^{1/l}))^{(1/10)(1-l\beta_k-l\varepsilon/2)} \le \exp(-C(\log Y)^{3/5}(\log\log Y)^{-1/5}), \tag{4.7}$$

provided $1 - l\beta_k - l\varepsilon/2 > 0$.

Therefore, from (4.6) and the Minkowski inequality (see item 200 of [6]), we get (using the inequality $(a+b)^{1/2} \le a^{1/2} + b^{1/2}$ for $a \ge 0$ and $b \ge 0$ and the definition (1.11))

$$\begin{aligned} I_1^{1/2} &\ll \sum_{n\le f(2Y)}\left\{\int_Y^{2Y}\left(\Delta_k\left(\frac{x}{n^l}\right)\right)^2 dx\right\}^{1/2} + Y^{1/2+5\varepsilon} \\ &\ll \sum_{n\le f(2Y)} n^{l/2}\left(\frac{Y}{n^l}\right)^{(1/2)(1+2\beta_k+\varepsilon)} + Y^{1/2+5\varepsilon} \\ &\ll Y^{1/2+\beta_k+\varepsilon/2}\sum_{n\le f(2Y)} n^{-l\beta_k - l\varepsilon/2} + Y^{1/2+5\varepsilon} \\ &\ll \begin{cases} Y^{1/2+\beta_k+\varepsilon} + Y^{1/2+5\varepsilon} & \text{if } l\beta_k \ge 1 - l\varepsilon/2, \\ Y^{1/2+\beta_k+\varepsilon/2}(f(2Y))^{1-l\beta_k-l\varepsilon/2} + Y^{1/2+5\varepsilon} & \text{if } l\beta_k < 1 - l\varepsilon/2 \end{cases} \\ &\ll \begin{cases} Y^{1/2+\beta_k+\varepsilon} & \text{if } l\beta_k \ge 1 - l\varepsilon/2, \\ Y^{1/2+1/l}\exp(-C(\log Y)^{3/5}(\log\log Y)^{-1/5}) & \text{if } l\beta_k < 1 - l\varepsilon/2. \end{cases} \end{aligned} \tag{4.8}$$

Hence, we obtain

$$\frac{I_1}{Y} \ll \begin{cases} Y^{2\beta_k+2\varepsilon} & \text{if } l\beta_k \geq 1 - l\varepsilon/2, \\ Y^{2/l}\exp(-C(\log Y)^{3/5}(\log\log Y)^{-1/5}) & \text{if } l\beta_k < 1 - l\varepsilon/2. \end{cases} \tag{4.9}$$

This proves Theorem 2. ∎

Remark. From the work of Heath-Brown (see [7] and [8]), we know that $\beta_4 = 3/8$. If we fix $k = 4$ and $l = 2$ in Theorem 2, then we find that $l\beta_4 = 3/4 < 1$, and hence the inequality (1.7) follows from Theorem 2.

Acknowledgements. This project was carried out and completed when the second author visited the Department for Number Theory and Probability Theory, University of Ulm, Germany in 2004; he wishes to thank the University of Ulm for its warm hospitality and generous support. The authors are grateful to the referee for some useful comments.

References

[1] R. Balasubramanian and K. Ramachandra, *Some problems of analytic number theory, II*, Studia Sci. Math. Hungar. 14 (1979), 193–202.

[2] —, —, *Some problems of analytic number theory, III*, Hardy-Ramanujan J. 4 (1981), 13–40.

[3] R. Balasubramanian, K. Ramachandra and M. V. Subbarao, *On the error function in the asymptotic formula for the counting function of k-full numbers*, Acta Arith. 50 (1988), 107–118.

[4] K. Chandrasekharan, *Introduction to Analytic Number Theory*, Grundlehren Math. Wiss. 148, Springer, New York, 1968.

[5] K. Chandrasekharan and A. Good, *On the number of integral ideals in Galois extensions*, Monatsh. Math. 95 (1983), 99–109.

[6] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, 2nd ed., Cambridge Univ. Press, 1952.

[7] D. R. Heath-Brown, *The twelfth power moment of the Riemann zeta-function*, Quart. J. Math. Oxford Ser. (2) 29 (1978), 443–462.

[8] —, *Mean values of the zeta function and divisor problems*, in: Recent Progress in Analytic Number Theory (Durham, 1979), Vol. I, Academic Press, London, 1981, 115–119.

[9] M. N. Huxley, *Exponential sums and lattice points, II*, Proc. London Math. Soc. (3) 66 (1993), 279–301.

[10] A. Ivić, *The Riemann Zeta-Function. The Theory of the Riemann Zeta-Function with Applications*, Wiley, New York, 1985.

[11] M. Jutila, *Riemann's zeta-function and the divisor problem*, Ark. Mat. 21 (1983), 75–96.

[12] G. Kolesnik, *On the estimation of multiple exponential sums*, in: Recent Progress in Analytic Number Theory (Durham, 1979), Vol. I, Academic Press, London, 1981, 231–246.

[13] —, *On the method of exponent pairs*, Acta Arith. 45 (1985), 115–143.

[14] M. Kühleitner, *On a question of A. Schinzel concerning the sum* $\sum_{n\le x}(r(n))^2$, Grazer Math. Ber. 318 (1993), 63–67.

[15] M. Kühleitner and W. G. Nowak, *An omega theorem for a class of arithmetic functions*, Math. Nachr. 165 (1994), 79–98.

[16] —, —, *The average number of solutions of the Diophantine equation* $u^2+v^2=w^3$ *and related arithmetic functions*, Acta Math. Hungar. 104 (2004), 225–240.

[17] H. L. Montgomery and R. C. Vaughan, *The distribution of squarefree numbers*, in: Recent Progress in Analytic Number Theory (Durham, 1979), Vol. I, Academic Press, London, 1981, 247–256.

[18] K. Ramachandra, *A remark on Perron's formula*, J. Indian Math. Soc. 65 (1998), 145–151.

[19] —, *Notes on prime number theorem*, *I*, in: Number Theory, R. P. Bambah, V. C. Dumir, R. J. Hans-Gill (eds.), Hindustan book Agency and Indian National Science Academy, 1999, 351–370.

[20] K. Ramachandra and A. Sankaranarayanan, *On an asymptotic formula of Srinivasa Ramanujan*, Acta Arith. 109 (2003), 349–357.

[21] S. Ramanujan, *Some formulae in the analytic theory of numbers*, Messenger of Math. 45 (1916), 81–84.

[22] —, *Ramanujan's Papers*, B. J. Venkatachala *et al.* (eds.), Prism Books, Bangalore, 2000, 169–173.

[23] A. Sankaranarayanan and K. Srinivas, *On a method of Balasubramanian and Ramachandra* (*on the abelian group problem*), Rend. Sem. Mat. Univ. Padova 97 (1997), 135–161.

[24] A. Schinzel, *On an analytic problem considered by Sierpiński and Ramanujan*, in: New Trends in Probability and Statistics (Palanga, 1991), Vol. 2, VSP, Utrecht, 1992, 165–171.

[25] D. Suryanarayana and R. Sitaramachandra rao, *On an asymptotic formula of Ramanujan*, Math. Scand. 32 (1973), 258–264.

[26] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed. (edited by D. R. Heath-Brown), Clarendon Press, Oxford, 1986.

[27] A. Walfisz, *Weylsche Exponentialsummen in der neueren Zahlentheorie*, Mathematische Forschungsberichte 15, Berlin, 1963.

[28] B. M. Wilson, *Proofs of some formulae enunciated by Ramanujan*, Proc. London Math. Soc. (2) 21 (1922), 235–255.

Department for Number Theory and Probability Theory
University of Ulm
D-89069 Ulm, Germany
E-mail: hamaier@mathematik.uni-ulm.de

School of Mathematics
TIFR
Homi Bhabha Road
Mumbai, 400 005, India
E-mail: sank@math.tifr.res.in

Received on 10.7.2004
and in revised form on 27.10.2004 (4804)