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On the number of representations of integers as the sum of k terms

by

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1. Introduction. Let $\mathbb{N}$ denote the set of positive integers. Let $k > 2$ be a fixed integer and let $\mathcal{A} = \{a_1, a_2, \dots\}$ $(a_1 < a_2 < \cdots)$ be an infinite sequence of positive integers. For $n = 1, 2, \dots$ let $R_k(n)$ denote the number of solutions of $a_{i_1} + \cdots + a_{i_k} = n$, $a_{i_1} \in \mathcal{A}, \dots, a_{i_k} \in \mathcal{A}$. For $k = 2$, P. Erdős and A. Sárközy studied how regular the behaviour of the function $R_2(n)$ can be. In [2] they proved the following theorem:

THEOREM 1. *If $F(n)$ is an arithmetic function such that*

$$F(n) \to \infty,$$
$$F(n+1) \geq F(n) \quad \text{for } n \geq n_0,$$
$$F(n) = o\left(\frac{n}{(\log n)^2}\right),$$

and we write

$$\Delta(N) = \sum_{n=1}^{N} (R_2(n) - F(n))^2,$$

then

$$\Delta(N) = o(NF(N))$$

cannot hold.

In [3] they showed that the above result is nearly best possible:

THEOREM 2. *If $F(n)$ is an arithmetic function satisfying*

$$F(n) > 36 \log n \quad \text{for } n > n_0,$$

and there exists a real function $g(x)$, defined for $0 < x < \infty$, and real numbers x_0, n_1 such that

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(i) *$g'(x)$ exists and it is continuous for* $0 < x < \infty$,
(ii) $g'(x) \le 0$ *for* $x \ge x_0$,
(iii) $0 < g(x) < 1$ *for* $x \ge x_0$,
(iv) $|F(n) - 2\int_0^{n/2} g(x)g(n-x)\,dx| < (F(n)\log n)^{1/2}$ *for* $n > n_1$,

then there exists a sequence $\mathcal{A}$ *such that*

$$|R_2(n) - F(n)| < 8(F(n)\log n)^{1/2} \quad \text{for } n > n_2.$$

In [6] G. Horváth extended Theorem 1 to any $k > 2$:

THEOREM 3. *If* $F(n)$ *is an arithmetic function such that*

$$F(n) \to \infty,$$
$$F(n+1) \ge F(n) \quad \text{for } n \ge n_0,$$
$$F(n) = o\left(\frac{n}{(\log n)^2}\right),$$

and we write

$$\Delta(N) = \sum_{n=1}^{N}(R_k(n) - F(n))^2,$$

then

$$\Delta(N) = o(NF(N))$$

cannot hold.

A. Sárközy proposed to prove an analogue of Theorem 2 for $k > 2$ [8, Problem 3]. In this paper my goal is to extend Theorem 2 to any $k > 2$, i.e., to show that Theorem 3 is nearly best possible. In fact, I will prove the following theorem:

THEOREM 4. *If* $k > 2$ *is a positive integer*, c_8 *is a constant large enough in terms of* k, $F(n)$ *is an arithmetic function satisfying*

$$F(n) > c_8 \log n \quad \text{for } n > n_0,$$

and there exists a real function $g(x)$, *defined for* $0 < x < \infty$, *and real numbers* x_0, n_1 *and constants* c_7, c_9 *such that*

(i) $0 < g(x) \le \dfrac{(\log x)^{1/k}}{x^{1-(k+1)/k^2}} < 1$ *for* $x \ge x_0$,

(ii) $\left|F(n) - k! \sum_{\substack{x_1+\dots+x_k=n \\ 1 \le x_1 < \dots < x_k < n}} g(x_1)\dots g(x_k)\right| < c_7(F(n)\log n)^{1/2}$ *for* $n > n_1$,

then there exists a sequence $\mathcal{A}$ *such that*

$$|R_k(n) - F(n)| < c_9(F(n)\log n)^{1/2} \quad \text{for } n > n_2.$$

It is easy to see that the following functions satisfy the conditions of Theorem 4: $g(x) = c_{10}((\log x)^{\beta}/x^{\alpha})$, where c_{10} is a positive constant, $\alpha > 1-(k+1)/k^2$, or $\alpha = 1-(k+1)/k^2$ and $\beta \le 1/k$. It follows that for $F(n) = n^{\delta}(\log n)^{\gamma}$ with $0 < \delta \le 1/k$, or $0 \le \gamma < 1$ there is a sequence $\mathcal{A}$ for which $R_k(n)$ satisfies the conclusion of the theorem. For $k = 2$ in [3] P. Erdős and A. Sárközy used the probabilistic method to construct $\mathcal{A}$. In the case $k = 2$, certain events in their paper were mutually independent. For $k > 2$ the independence fails, thus to prove Theorem 4 we need deeper probabilistic tools.

2. Probabilistic tools. The proof of Theorem 4 is based on the probabilistic method due to Erdős and Rényi. There is an excellent summary of this method in Halberstam and Roth's book [5]. We use the notation and terminology of that book. First we give a survey of the probabilistic tools and notation we use in the proof of Theorem 4. Let Ω denote the set of strictly increasing sequences of positive integers. In this paper we denote the probability of an event E by $P(E)$.

LEMMA 1. *Let*

$$\alpha_1, \alpha_2, \ldots \tag{1}$$

be real numbers satisfying

$$0 \le \alpha_n \le 1 \quad (n = 1, 2, \ldots). \tag{2}$$

Then there exists a probability space (Ω, S, P) *with the following two properties*:

(i) *For every natural number* n, *the event* $E^{(n)} = \{\mathcal{A} \in \Omega : n \in \mathcal{A}\}$ *is measurable*, *and* $P(E^{(n)}) = \alpha_n$.
(ii) *The events* $E^{(1)}, E^{(2)}, \ldots$ *are independent.*

See [5, Theorem 13, p. 142]. We denote the characteristic function of the event $E^{(n)}$ by $\varrho(\mathcal{A}, n)$:

$$\varrho(\mathcal{A}, n) = \begin{cases} 1 & \text{if } n \in \mathcal{A}, \\ 0 & \text{if } n \notin \mathcal{A}. \end{cases}$$

Furthermore, we denote the number of solutions of $a_{i_1} + \cdots + a_{i_k} = n$ by $r_k(n)$, where $a_{i_1} \in \mathcal{A}, \ldots, a_{i_k} \in \mathcal{A}$, $1 \le a_{i_1} < \cdots < a_{i_k} < n$. Thus

$$r_k(n) = \sum_{\substack{(a_1,\ldots,a_k)\in\mathbb{N}^k \\ 1\le a_1<\cdots<a_k<n \\ a_1+\cdots+a_k=n}} \varrho(\mathcal{A}, a_1)\ldots\varrho(\mathcal{A}, a_k). \tag{3}$$

Let $r_k^*(n)$ denote the number of those representations $n = a_{i_1} + \cdots + a_{i_k}$ in which there are at least two equal terms. Thus

$$R_k(n) = k!r_k(n) + r_k^*(n). \tag{4}$$

It is easy to see from (3) that $r_k(n)$ is a sum of random variables. However, for $k > 2$ these variables are not independent because the same $\varrho(\mathcal{A}, a_i)$ may appear in many terms; therefore we need deeper probabilistic tools.

Our proof is based on a method of J. H. Kim and V. H. Vu. In the next section we give a short survey of their method. The interested reader can find more details in [7], [9], [10]. Assume that $t_1, \ldots, t_n$ are independent binary (i.e., $\{0,1\}$-valued) random variables. Consider a polynomial Y in $t_1, \ldots, t_n$ of degree k. We say Y is *positive* if it can be written in the form $Y = \sum_i e_i \Gamma_i$, where the e_i's are positive and each Γ_i is a product of some t_j's. Given a (multi-) set A, $\partial_A(Y)$ denotes the partial derivative of Y with respect to the variables with indices in A. For instance, if $Y = t_1 t_2^2$ and $A_1 = \{1,2\}$ and $A_2 = \{2,2\}$ then $\partial_{A_1}(Y) = 2t_2$ and $\partial_A Y = 2t_1$. If A is empty then $\partial_A(Y) = Y$. Let $E_A(Y)$ denote the expectation of $\partial_A(Y)$. Furthermore, set

$$E_j(Y) = \max_{|A| \geq j} E_A(Y) \quad \text{for } j = 0, 1, \ldots, k,$$

so $E_0(Y) = E(Y)$.

THEOREM 5 (Kim–Vu). *For every positive integer k there are positive constants d_k and b_k depending only on k such that for any positive polynomial $Y = Y(t_1, \ldots, t_n)$ of degree k, where the t_i's are independent binary random variables*,

$$P\big(|Y - E(Y)| \geq d_k \lambda^k \sqrt{E_0(Y) E_1(Y)}\big) \leq b_k e^{-\lambda/4 + (k-1)\log n}.$$

See [7] for the proof. Finally, we need the Borel–Cantelli lemma (see [5]):

LEMMA 2. *Let $\{B_i\}$ be a sequence of events in a probability space. If*

$$\sum_{j=1}^{\infty} P(B_j) < \infty,$$

then with probability 1, *at most a finite number of the events B_j can occur.*

3. Proof of Theorem 4. Fix a number n and write

$$S_n = \{(a_1, \ldots, a_k) \in \mathbb{N}^k : 0 < a_1 < \cdots < a_k < n,\ a_1 + \cdots + a_k = n\}.$$

Define a sequence (1) of real numbers by

$$\alpha_n = \begin{cases} g(n) & \text{if } n \geq x_0, \\ 0 & \text{otherwise,} \end{cases}$$

and let (Ω, S, P) be the probability space as in Lemma 1. Clearly the sequence α_n satisfies (2). Thus

$$r_k(n) = \sum_{(a_1, \ldots, a_k) \in S_n} t_{a_1} \ldots t_{a_k},$$

where

$$t_{a_i} = \begin{cases} 1 & \text{if } a_i \in \mathcal{A}, \\ 0 & \text{if } a_i \notin \mathcal{A}. \end{cases}$$

Then we have

$$\lambda_n = E(r_k(n)) = \sum_{(a_1,\ldots,a_k)\in S_n} P(a_1 \in \mathcal{A}) \ldots P(a_k \in \mathcal{A}),$$

where $E(\zeta)$ denotes the expectation of the random variable ζ. To prove Theorem 4 we will give an upper estimate for $|R_k(n) - k!\lambda_n|$. As Vu in [10], we split $r_k(n)$ into two parts, as follows. Let a be a small positive constant, say $a < 1/2(k+1)$, and let $S_n^{[1]}$ be the subset of all $(a_1,\ldots,a_k) \in S_n$ with $a_1 \geq n^a$, and $S_n^{[2]} = S_n \setminus S_n^{[1]}$. We split $r_k(n)$ into the sum of two terms corresponding to $S_n^{[1]}$ and $S_n^{[2]}$, respectively:

$$r_k(n) = r_k^{[1]}(n) + r_k^{[2]}(n),$$

where

$$r_k^{[j]}(n) = \sum_{(a_1,\ldots,a_k)\in S_n^{[j]}} t_{a_1} \ldots t_{a_k}, \tag{5}$$

and set

$$\lambda_n^{[j]} = E(r_k^{[j]}(n)).$$

Clearly

$$\begin{aligned} |R_k(n) - k!\lambda_n| &\leq |R_k(n) - k!r_k(n)| + k!|r_k(n) - \lambda_n| \\ &= r_k^*(n) + k!|r_k^{[1]}(n) + r_k^{[2]}(n) - \lambda_n^{[1]} - \lambda_n^{[2]}| \\ &\leq r_k^*(n) + k!|r_k^{[1]}(n) - \lambda_n^{[1]}| + k!|r_k^{[2]}(n) - \lambda_n^{[2]}| \\ &= r_k^*(n) + I_1 + I_2. \end{aligned} \tag{6}$$

The rest of the proof of Theorem 4 has four parts. In the first part we estimate I_1, in the second I_2, in the third $r_k^*(n)$, and in the last part we complete the proof.

Estimating I_1. We will apply Theorem 5 so we need an upper bound for $E_1(r_k^{[1]}(n))$. To do this, it is clear from the definition of E_1 that we need the following lemma, which guarantees that every partial derivative of $r_k^{[1]}(n)$ has small expectation.

LEMMA 3. *For all non-empty multi-sets A of size at most $k-1$,*

$$E(\partial_A(r_k^{[1]}(n))) = O(n^{-a/2k^2}).$$

Proof. This can be proved similarly to Lemma 5.3 in [10]. For completeness I will present the proof. Consider a multi-set A of $k-l$ elements and

$\sum_{x\in A} x = n - m$. There exists a constant $c(k)$ such that

$$\partial_A(r_k^{[1]}(n)) \le c(k) \sum_{\substack{n^a<a_1<\dots<a_l\\ a_1+\dots+a_l=m}} t_{a_1}\dots t_{a_l}.$$

As $a_l \ge m/l$ and $\sum_{x=1}^m x^{1/k-1} \approx \int_1^m z^{1/k-1}\,dz \approx m^{1/k}$, and using assumption (i) of Theorem 4, we have

$$\begin{aligned}
&E(\partial_A(r_k^{[1]}(n)))\\
&= O\Big(\sum_{\substack{n^a<a_1<\dots<a_l\\ a_1+\dots+a_l=m}} P(a_1\in\mathcal{A})\dots P(a_l\in\mathcal{A})\Big) = O\Big(\sum_{\substack{n^a<a_1<\dots<a_l\\ a_1+\dots+a_l=m}} g(a_1)\dots g(a_l)\Big)\\
&= O(\log n)\sum_{\substack{n^a<a_1<\dots<a_l\\ a_1+\dots+a_l=m}} a_1^{(k+1)/k^2-1}\dots a_l^{(k+1)/k^2-1}\\
&= O(\log n)O\Big(\Big(\sum_{x=1}^m x^{(k+1)/k^2-1}\Big)^{l-1}(m/l)^{(k+1)/k^2-1}\Big)\\
&= O(\log n)O(m^{(l-1)(k+1)/k^2}(m/l)^{(k+1)/k^2-1})\\
&= O(\log n)O(m^{(l(k+1)-k^2)/k^2}) = O(n^{-a/2k^2}),
\end{aligned}$$

since $k-1\ge l$ and $m \ge n^a$. The proof of Lemma 3 is complete.

By the definition of $E_1(r_k^{[1]}(n))$, and from Lemma 3, it is clear that $E_1(r_k^{[1]}(n)) = \max_{|A|\ge 1} E_A(r_k^{[1]}(n)) \le cn^{-a/2k^2}$, where c is a constant. It is clear from (5) that $r_k^{[1]}(n)$ is a positive polynomial of degree k. Now we apply Theorem 5 with $\lambda = (\log n/E_1(r_k^{[1]}(n)))^{1/2k}$. If n is large enough we have

$$\begin{aligned}
P\Bigg(|r_k^{[1]}(n)-\lambda_n^{[1]}| \ge d_k\sqrt{\frac{\log n}{E_1(r_k^{[1]}(n))}}\sqrt{\lambda_n^{[1]}E_1(r_k^{[1]}(n))}\Bigg)&\\
&\le b_k\exp\Bigg(-\frac{1}{4}\sqrt[2k]{\frac{\log n}{E_1(r_k^{[1]}(n))}}+(k-1)\log n\Bigg)\\
&\le b_k\exp\Bigg(-\frac{1}{4}\sqrt[2k]{\frac{\log n}{n^{-a/2k^2}}}+(k-1)\log n\Bigg)\\
&< \exp(-2\log n) = \frac{1}{n^2}.
\end{aligned}$$

Applying the above result we obtain

$$\sum_{n=1}^{\infty} P\Big(|r_k^{[1]}(n)-\lambda_n^{[1]}| \ge d_k\sqrt{\lambda_n^{[1]}\log n}\Big) < \sum_{n=1}^{\infty}\frac{1}{n^2} < \infty.$$

By the Borel–Cantelli lemma, with probability 1, there exists n_0 such that

$$|r_k^{[1]}(n) - \lambda_n^{[1]}| < d_k\sqrt{\lambda_n^{[1]}\log n} \quad \text{for } n > n_0. \tag{7}$$

Estimating I_2. We will prove similarly to the proof in [10] that for almost every sequence $\mathcal{A}$, there is a finite number $c_{11}(\mathcal{A})$ such that $r_k^{[2]}(n) \le c_{11}(\mathcal{A})$ for all sufficiently large n. Let $r_l(n)$ denote the number of representations of n as the sum of l distinct numbers from $\mathcal{A}$. First we estimate $E(r_l(n))$ similarly to [4]. Fix $2 \le l \le k-1$. As $n/l < a_l$, by assumption (i) of Theorem 4, we have

$$\begin{aligned}
E(r_l(n)) &\le \sum_{\substack{a_1+\dots+a_l=n\\ 1\le a_1<\dots<a_l<n}} P(a_1\in\mathcal{A})\dots P(a_l\in\mathcal{A})\\
&< \sum_{\substack{a_1+\dots+a_l=n\\ 1\le a_1<\dots<a_l<n}} g(a_1)\dots g(a_l)\\
&\le \sum_{\substack{a_1+\dots+a_l=n\\ 1\le a_1<\dots<a_l<n}} \frac{(\log a_1)^{1/k}}{a_1^{1-(k+1)/k^2}}\dots\frac{(\log a_l)^{1/k}}{a_l^{1-(k+1)/k^2}}\\
&= n^{o(1)} \sum_{\substack{a_1+\dots+a_l=n\\ 1\le a_1<\dots<a_l<n}} \frac{1}{(a_1\dots a_l)^{1-(k+1)/k^2}}\\
&\le n^{o(1)}\Bigg(n^{(k+1)/k^2-1+o(1)} \sum_{\substack{a_1+\dots+a_l=n\\ 1\le a_1<\dots<a_l<n}} \frac{1}{(a_1\dots a_{l-1})^{1-(k+1)/k^2}}\Bigg)\\
&\le n^{(k+1)/k^2-1+o(1)} \sum_{\substack{1\le a_i\le n\\ i=1,\dots,l-1}} \frac{1}{(a_1\dots a_{l-1})^{1-(k+1)/k^2}}\\
&= n^{(k+1)/k^2-1+o(1)}\Bigg(\sum_{1\le a_1\le n}\frac{1}{a_1^{1-(k+1)/k^2}}\Bigg)^{l-1}\\
&= n^{(k+1)/k^2-1+o(1)}\big(n^{(k+1)/k^2+o(1)}\big)^{l-1} = n^{-1+l(k+1)/k^2+o(1)}.
\end{aligned} \tag{8}$$

Let $T_1 = \{a_1,\dots,a_k\}$, $T_2 = \{b_1,\dots,b_k\}$, $T_1 \ne T_2$, $T_1, T_2 \subset \mathcal{A}$ and

$$a_1 + \dots + a_k = b_1 + \dots + b_k = n.$$

We say these representations are *disjoint* if they share no element in common. Let $f_l(n)$ denote the maximum number of pairwise disjoint representations of n as the sum of l distinct numbers from $\mathcal{A}$. We show that with probability 1, $f_l(n)$ is bounded. We will apply the following result due to Erdős and Tetali which is called the *disjointness lemma*. We say events $G_1,\dots,G_n$ are *independent* if for all subsets $I \subseteq \{1,\dots,n\}$, $P(\bigcap_{i\in I} G_i) = \prod_{i\in I} P(G_i)$.

Lemma 4. *If* $\sum_i P(B_i) \le \mu$, *then*

$$\sum_{\substack{(B_1,\dots,B_l)\\ \text{independent}}} P(B_1 \cap \dots \cap B_l) \le \mu^l / l!.$$

Proof. This is Lemma 1 in [4]. Let

$$\mathcal{B} = \{(a_1,\dots,a_l) \in \mathcal{A}^n : a_1 + \dots + a_l = n, 1 \le a_1 < \dots < a_l < n\}.$$

Let $H(\mathcal{B}) = \{\mathcal{T} \subset \mathcal{B}$: all the $K \in \mathcal{T}$ are pairwise disjoint$\}$ and c_1 be a constant. It is clear that the pairwise disjointness of the sets implies the independence of the associated events, i.e., if K_1 and K_2 are pairwise disjoint representations, then the events $K_1 \subset \mathcal{A}$, $K_2 \subset \mathcal{A}$ are independent. Thus by (8) and Lemma 4 we have

$$(9)\qquad P(f_l(n) > c_1) \le P\Big(\bigcup_{\substack{\mathcal{T}\subset H(\mathcal{B})\\ |\mathcal{T}|=c_1+1}} \bigcap_{K\in\mathcal{T}} K\Big) \le \sum_{\substack{\mathcal{T}\subset H(\mathcal{B})\\ |\mathcal{T}|=c_1+1}} P\Big(\bigcap_{K\in\mathcal{T}} K\Big)$$

$$= \sum_{\substack{(K_1,\dots,K_{c_1+1})\\ \text{pairwise}\\ \text{disjoint}}} P(K_1 \cap \dots \cap K_{c_1+1}) \le \frac{1}{(c_1+1)!}\,(E(f_l(n)))^{c_1+1}$$

$$\le \frac{1}{(c_1+1)!}\,(E(r_l(n)))^{c_1+1} \le \frac{1}{(c_1+1)!}\, n^{-2+o(1)}$$

if c_1 large enough. By the Borel–Cantelli lemma, with probability 1 for almost every random sequence $\mathcal{A}$ there is a finite number $c_1(\mathcal{A})$ such that for any $l < k$ and all n, the maximal number of disjoint l-representations of n from $\mathcal{A}$ is at most $c_1(\mathcal{A})$.

In the next step we estimate $E(r_k^{[2]}(n))$ as in Lemma 3. Using also the fact that $\sum_{x=1}^m x^{1/k-1} \approx \int_1^m z^{1/k-1}\,dz \approx m^{1/k}$, and $a_k \ge n/k$, $a < 1/(2(k+1))$, and (i) of Theorem 4, we have

$$E(r_k^{[2]}(n)) = E\Big(\sum_{(a_1,\dots,a_k)\in S_n^{[2]}} t_{a_1}\dots t_{a_k}\Big)$$

$$= O\Big(\sum_{(a_1,\dots,a_k)\in S_n^{[2]}} P(a_1\in\mathcal{A})\dots P(a_k\in\mathcal{A})\Big)$$

$$= O(\log n) \sum_{\substack{a_1+\dots+a_k=n\\ a_1\le n^a}} a_1^{(k+1)/k^2-1} \dots a_k^{(k+1)/k^2-1}$$

$$= O(\log n)O\Big(\sum_{x=1}^{n^a} x^{(k+1)/k^2-1}\Big(\sum_{x=1}^{n} x^{(k+1)/k^2-1}\Big)^{k-2}(n/k)^{(k+1)/k^2-1}\Big)$$
$$= O(n^{(a(k+1)-1)/k^2}\log n) = O(n^{-1/2k^2}).$$

Thus by Lemma 4 and the Borel–Cantelli lemma, with probability 1, there is a constant c_2 such that almost surely the maximum number of disjoint representations of n in $r_k^{[2]}(n)$ is at most c_2 for all large n.

To finish the proof it suffices to show that $r_k^{[2]}(n)$ is bounded by a constant. The proof is purely combinatorial. We need the following well-known result due to Erdős and Rado [1]. Let r be a positive integer, $r \geq 3$. A collection of sets $D_1, \ldots, D_r$ forms a Δ-*system* if the sets have pairwise the same intersection.

Lemma 5. *If H is a collection of sets of size at most k and $|H| > (r-1)^k k!$ then H contains r sets forming a Δ-system.*

Set $C(\mathcal{A}) = (\max(c_1(\mathcal{A}), c_2))^k k!$ and assume that n is sufficiently large. To each representation of n counted in $r_k^{[2]}(n)$ we assign the set formed by the k terms occurring in this representation. We will apply Lemma 5 with H being the collection of these sets. It is clear that if $r_k^{[2]}(n) > C(\mathcal{A})$, then by Lemma 5, $r_k^{[2]}(n)$ contains a Δ-system with $c_3 = \max(c_1(\mathcal{A}), c_2) + 1$ sets. If the intersection of these sets is empty, then they form a family of c_3 disjoint k-representations of n, which contradicts the definition of c_3. Otherwise, assume that the intersection of these sets is $\{y_1, \ldots, y_j\}$, where $1 \leq j \leq k-1$ and $\sum_{i=1}^{j} y_i = m$. Removing the common intersection of these sets we can find $c_1(\mathcal{A})+1$ $(k-j)$-representations of $n-m = n-\sum_{i=1}^{j} y_i$. These $c_1(\mathcal{A})+1$ sets are disjoint due to the definition of the Δ-system. Therefore in both cases we obtain a contradiction.

Estimating $r_k^*(n)$. If we collect the equal terms, we have

$$u_1a_1 + u_2a_2 + \cdots + u_ha_h = n, \tag{10}$$

where the u_i's are positive integers, and

$$u_1 + u_2 + \cdots + u_h = k. \tag{11}$$

Thus $r_k^*(n)$ denotes the number of representations (10) of n, where the a_i's are different. It can be proved similarly to the estimate of $r_k^{[2]}(n)$ that $r_k^*(n)$ is also bounded by a constant. For completeness we sketch the proof leaving the details to the reader. Fix $2 \leq h \leq k-1$. For fixed $u_1, \ldots, u_h$ let $s_h(n)$ denote the number of representations (10) of n. We show that $s_h(n)$ is bounded by a constant. (Note that we have already proved this when all u_i's are equal to one, and $h = k$.) First we estimate $E(s_h(n))$, with a calculation similar to (8). Using the definition of $s_h(n)$, and $n/h < a_h$, we have

$$E(s_h(n)) \le \sum_{\substack{u_1a_1+\dots+u_ha_h=n\\ 1\le a_1<\dots<a_h<n}} P(a_1\in\mathcal{A})\dots P(a_h\in\mathcal{A}) \tag{12}$$
$$= \sum_{\substack{u_1a_1+\dots+u_ha_h=n\\ 1\le a_1<\dots<a_h<n}} g(a_1)\dots g(a_h)$$
$$\le \sum_{\substack{u_1a_1+\dots+u_ha_h=n\\ 1\le a_1<\dots<a_h<n}} \frac{(\log a_1)^{1/k}}{a_1^{1-(k+1)/k^2}}\dots\frac{(\log a_h)^{1/k}}{a_h^{1-(k+1)/k^2}}$$
$$= n^{-1+h(k+1)/k^2+o(1)}.$$

Let $s_h^*(n)$ denote the size of a maximal collection of pairwise disjoint representations (10). The same argument as in (9) shows that almost always there exists a constant v_h such that $s_h^*(n) < v_h$ for n large enough. In view of (12), and applying Lemma 4, we have

$$P(s_h^*(n) > v_h) < n^{-2+o(1)}$$

if v_h is large enough. Thus by the Borel–Cantelli lemma, with probability 1, $s_h^*(n) < v_h$ for every large enough n. We say that an m-tuple $(a_1,\dots,a_m)$ $(m\le h)$ is an *m-representation* of n in the form (10) if there is a permutation π of $\{1,\dots,h\}$ such that $\sum_{i=1}^m u_{\pi(i)}a_i = n$. For all $m<h$, let $s_m^*(n)$ denote the size of a maximal collection of pairwise disjoint such representations of n. The same argument as above shows that almost always there exists a constant p_m such that $s_m^*(n) < p_m$ for every large enough n.

In the last step we apply Lemma 5 to prove that $s_h(n)$ is bounded by a constant. Let $C = (\max(p_m h!, v_h))^h h!$. Let H in Lemma 5 be the collection of representations (10) of n. Clearly $|H| = s_h(n)$. If $s_h(n) > C$, and n is sufficiently large then by Lemma 5, H contains a Δ-system with $C+1$ sets. If the intersection of these sets is empty, then they form a family of disjoint h-representations (10). Otherwise, let the common intersection of the sets be $\{y_1,\dots,y_s\}$, where $1\le s\le h-1$. By the pigeon-hole principle there exists a permutation π of $\{1,\dots,h\}$ such that we can find p_m+1 $(k-s)$-representations of $n'' = n-\sum_{i=1}^s u_{\pi(i)}y_s$. These p_m+1 sets are disjoint, thus in both cases we obtain a contradiction. Since there are only a finite number of partitions of k in the form (11), we conclude that $r_k^*(n)$ is bounded by a constant, i.e., there exists a constant C_3 such that $r_k^*(n) < C_3$. Let c_4, c_5, c_6 be constants. Thus by (6) and (7) we have

$$|R_k(n) - k!\lambda_n| \le |R_k(n) - k!r_k(n)| + k!|r_k(n) - \lambda_n|$$
$$< C_3 + k!|r_n^{[1]} + r_n^{[2]} - \lambda_n^{[1]} - \lambda_n^{[2]}|$$

$$\le C_3 + k!|r_n^{[1]} - \lambda_n^{[1]}| + k!|r_n^{[2]} - \lambda_n^{[2]}|$$
$$\le C_3 + d_k k!\sqrt{\lambda_n^{[1]}\log n} + 2k!c_4 \le c_5 + d_k k!\sqrt{\lambda_n \log n}.$$

End of proof. We argue as in [3]. In view of the estimate above and assumption (ii), for large n we have

$$
\begin{aligned}
|R_k(n) - F(n)| &\le |R_k(n) - k!\lambda_n| + |k!\lambda_n - F(n)| \\
&< c_5 + d_k k!(\lambda_n \log n)^{1/2} + |k!\lambda_n - F(n)| \\
&\le c_5 + c_6\left(\left(\frac{1}{k!}\,F(n) + \frac{1}{k!}\,|k!\lambda_n - F(n)|\right)\log n\right)^{1/2} + |k!\lambda_n - F(n)| \\
&< c_5 + c_6\left(\left(\frac{1}{k!}\,F(n) + \frac{c_7}{k!}\,(F(n)\log n)^{1/2}\right)\log n\right)^{1/2} + c_7(F(n)\log n)^{1/2} \\
&< c_5 + c_6\left(\left(\frac{1}{k!}\,F(n) + \frac{c_7}{k!}\left(F(n)\,\frac{F(n)}{c_8}\right)^{1/2}\right)\log n\right)^{1/2} + c_7(F(n)\log n)^{1/2} \\
&= c_5 + c_6\left(\left(\frac{1}{k!} + \frac{c_7}{\sqrt{c_8 k!}}\right)F(n)\log n\right)^{1/2} + c_7(F(n)\log n)^{1/2} \\
&< c_9(F(n)\log n)^{1/2}.
\end{aligned}
$$

The proof of Theorem 4 is complete.

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References

[1] P. Erdős and R. Rado, *Intersection theorems for system of sets*, J. London Math. Soc. 35 (1960), 85–90.

[2] P. Erdős and A. Sárközy, *Problems and results on additive properties of general sequences I*, Pacific J. Math. 118 (1985), 347–357.

[3] —, —, *Problems and results on additive properties of general sequences II*, Acta Math. Hungar. 48 (1986), 201–211.

[4] P. Erdős and P. Tetali, *Representations of integers as the sum of k terms*, Random Structures Algorithms 1 (1990), 245–261.

[5] H. Halberstam and K. F. Roth, *Sequences*, Springer, New York, 1983.

[6] G. Horváth, *On an additive property of sequences of nonnegative integers*, Period. Math. Hungar. 45 (2002), 73–80.

[7] J. H. Kim and V. H. Vu, *Concentration of multivariate polynomials and its applications*, Combinatorica 20 (2000), 417–434.

[8] A. Sárközy, *On the number of additive representations of integers*, in: More Sets, Graphs and Numbers, A Salute to Vera T. Sós and András Hajnal, eds. E. Győri et al., Bolyai Soc. Math. Stud. 15, J. Bolyai Math. Soc. and Springer, 2006, 329–339.

[9] V. H. Vu, *Chernoff type bounds for sums of dependent random variables and applications in additive number theory*, in: Number Theory for the Millennium, III (Urbana, IL, 2000), A K Peters, Natick, MA, 2002, 341–356.
[10] —, *On the concentration of multi-variate polynomials with small expectation*, Random Structures Algorithms 16 (2000), 344–363.

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