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On metric Diophantine approximation in positive characteristic

by

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1. Introduction. Let $\mathbb{F}$ be a finite field with q elements. We denote by $\mathbb{F}[X]$ the set of polynomials with coefficients in $\mathbb{F}$, and by $\mathbb{F}(X)$ the quotient field of $\mathbb{F}[X]$. We also denote by $\mathbb{F}((X^{-1}))$ the set of formal Laurent power series:

$$\mathbb{F}((X^{-1})) = \{f = a_l X^l + a_{l-1}X^{l-1} + \ldots : l \in \mathbb{Z} \text{ and each } a_i \in \mathbb{F}\}.$$

For $f \in \mathbb{F}((X^{-1}))$, we denote by $[f]$ its polynomial part:

$$[f] = a_l X^l + a_{l-1}X^{l-1} + \ldots + a_0,$$

and define

$$|f| = \begin{cases} 0 & \text{if } f = 0, \\ q^l & \text{if } a_l \neq 0. \end{cases}$$

In this paper, we discuss the metric theory of Diophantine approximation of Laurent series on the analogy of the classical theory; here, $\mathbb{F}[X]$, $\mathbb{F}(X)$, and $\mathbb{F}((X^{-1}))$ play the role of integers, rational numbers, and real numbers, respectively. We put

$$\mathbb{L} = \{f = a_{-1}X^{-1} + a_{-2}X^{-2} + \ldots : a_i \in \mathbb{F} \text{ for } i \le -1\},$$

which plays the role of the unit interval $[0, 1)$. Then $\mathbb{L}$ is a compact Abelian group with the metric $d(f, g) = |f - g|$. We denote by m the normalized Haar measure on $\mathbb{L}$. Note that

$$m\{f = a_{-1}X^{-1} + a_{-2}X^{-2} + \ldots : a_{-1} = b_1, \ldots, a_{-l} = b_l\} = \frac{1}{q^l}$$

for any $b_1, \ldots, b_l \in \mathbb{F}$. Our aim is to study the following Diophantine inequality:

$$\left| f - \frac{P}{Q} \right| < \frac{\psi(Q)}{|Q|}, \qquad P, Q \text{ coprime},$$

where ψ is a non-negative function defined on $\mathbb{F}[X]$ and $\psi(Q) = \psi(Q')$ whenever $Q' = aQ$ for some non-zero $a \in \mathbb{F}$. The question is whether this

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inequality has an infinite number of solutions P/Q for m-a.e. $f \in \mathbb{L}$. In the case of real numbers:

$$(1) \qquad \left| x - \frac{p}{q} \right| < \frac{\psi(q)}{q}, \qquad (p,q) = 1,$$

a number of sufficient conditions are known for this question. For example, if $q\psi(q)$ is non-increasing, then (1) has infinitely many solutions for a.e. $x \in \mathbb{R}$ if and only if $\sum \psi(q)$ diverges. This can be proved by using the continued fraction expansion of x (see Billingsley [2], for example). We refer to [7] and [5] for the formal power series version of this theorem. In general, we cannot make use of continued fractions for this type of problem. We refer to [9] for general cases. In what follows, we first restrict to the case where $\psi(Q)$ depends only on the degree of Q. In this case, it is easy to give a necessary and sufficient condition on ψ for having infinitely many solutions for a.e. $f \in \mathbb{L}$. Indeed, we have the following:

THEOREM 1. *Let ψ be a non-negative function defined on $\mathbb{F}[X]$ such that $\psi(Q)$ depends only on the degree of $Q \in \mathbb{F}[X]$. For any set S of positive integers, the inequality*

$$\left| f - \frac{P}{Q} \right| < \frac{\psi(Q)}{|Q|}$$

with P, Q coprime and $\deg Q \in S$ has infinitely many solutions for almost every $f \in \mathbb{L}$ if and only if

$$\sum_{d \in S} q^d \psi(X^d) = \infty.$$

To prove this theorem, we use continued fractions over $\mathbb{F}(X)$ to compute the number of fractions P/Q with $\deg Q = n$ for $n \geq 1$ (see [3]). We discuss this in Section 2 and give the proof in Section 3. If $\psi(Q)$ does not depend only on the degree of Q, then it is not easy to give a necessary and sufficient condition for the existence of infinitely many solutions (a.e.). Let ψ be a $\{q^{-n} : n \geq 0\} \cup \{0\}$-valued function defined on the set of monic polynomials in $\mathbb{F}[X]$ of the form

$$X^l + a_{l-1}X^{l-1} + \ldots + a_1 X + a_0, \qquad a_i \in \mathbb{F},\ 0 \leq i \leq l-1.$$

We denote by E the set of $f \in \mathbb{L}$ such that the inequality

$$(2) \qquad \left| f - \frac{P}{Q} \right| < \frac{\psi(Q)}{|Q|}, \qquad P, Q \text{ coprime}, \ Q \text{ monic},$$

has infinitely many solutions. In Section 4, we prove that $m(E) = 0$ or 1 (Theorem 4), which is an analogue of Gallagher's theorem (see [6]). Moreover we show that the Duffin–Schaeffer type theorem (see [4] and [9] for the classical case) holds.

THEOREM 2. *Let ψ be a $\{q^{-n} : n \geq 0\} \cup \{0\}$-valued function which satisfies*

$$\sum_{n=1}^{\infty} \sum_{\substack{\deg Q=n \\ Q \text{ monic}}} \psi(Q) = \infty.$$

Suppose there are infinitely many positive integers n such that

$$\sum_{\substack{\deg Q\leq n \\ Q \text{ monic}}} \psi(Q) < C \sum_{\substack{\deg Q\leq n \\ Q \text{ monic}}} \psi(Q) \frac{\Phi(Q)}{|Q|} \tag{3}$$

for a constant C. Then the inequality

$$\left| f - \frac{P}{Q} \right| < \frac{\psi(Q)}{|Q|}, \quad (P, Q) = 1,$$

has infinitely many solutions P/Q for a.e. $f \in \mathbb{L}$.

Here, $(P, Q) = 1$ means that P and Q are coprime polynomials, and $\Phi(Q)$ is the number of monic polynomials Q' such that

$$\deg Q' < \deg Q, \quad (Q, Q') = 1.$$

2. Continued fractions. We refer to Berthé and Nakada [1] for the details of the continued fraction expansions of power series.

Let T be the map of $\mathbb{L}$ onto itself defined by

$$T(f) = f^{-1} - [f^{-1}], \quad f \in \mathbb{L}.$$

Henceforth, we denote by 1 the unity of multiplication of $\mathbb{F}$, and by 0 the unity of addition. Then we have

$$f = \cfrac{1}{p_1 + \cfrac{1}{p_2 + \ddots}} =: [0; p_1, p_2, \ldots] \quad \text{with} \quad p_n = [(T^{n-1} f)^{-1}].$$

As in the classical case, we define

$$\begin{aligned} P_n &= p_n P_{n-1} + P_{n-2}, \quad & P_0 &= 0, \quad & P_1 &= 1, \\ Q_n &= p_n Q_{n-1} + Q_{n-2}, \quad & Q_0 &= 1, \quad & Q_1 &= p_1, \end{aligned} \tag{4}$$

and we have

$$P_n Q_{n-1} - Q_n P_{n-1} = \pm 1,$$

$$\frac{P_n}{Q_n} = \cfrac{1}{p_1 + \cfrac{1}{p_2 + \ddots + \cfrac{1}{p_n}}} =: [0; p_1, \ldots, p_n] \quad \text{for } n \geq 1.$$

We call P_n/Q_n the nth *convergent fraction* of f. Since

$$f = \frac{P_n + T^n f \cdot P_{n-1}}{Q_n + T^n f \cdot Q_{n-1}},$$

it is easy to see that

$$\left| f - \frac{P_n}{Q_n} \right| < \frac{1}{|Q_n|^2} \quad \text{for } n \geq 1.$$

Moreover, we have the following:

LEMMA 1. *If two relatively prime non-zero polynomials P, Q satisfy*

$$\left| f - \frac{P}{Q} \right| < \frac{1}{|Q|^2},$$

then

$$\frac{P}{Q} = \frac{P_n}{Q_n} \quad \textit{for some } n \geq 1.$$

We put

$$W_n = \left\{ \frac{P}{Q} \in \mathbb{L} : \deg Q = n,\ (P, Q) = 1 \right\} \quad \text{for } n \geq 1.$$

The following lemma, shown in [3], is essential in the next section. Here we prove it by the use of continued fractions.

LEMMA 2.

$$\# W_n = q^{2n} - q^{2n-1} \quad \textit{for } n \geq 1.$$

Proof. If $n = 1$, all elements in W_1 are of the form

$$\frac{P}{Q} = \frac{a}{X + b} \quad \text{with } a, b \in \mathbb{F},\ a \neq 0.$$

This implies the assertion. Now suppose

$$\# W_i = q^{2i} - q^{2i-1} \quad \text{for } 1 \leq i \leq n.$$

Fix $P/Q \in W_{n+1}$. Then we have a unique continued fraction expansion

$$\frac{P}{Q} = [0; p_1, \ldots, p_m].$$

So we can define a unique element $P'/Q' \in W_j$ for some j, $1 \leq j \leq n$, by

$$\frac{P'}{Q'} = [0; p_1, \ldots, p_{m-1}]$$

unless $m = 1$. On the other hand, for any $P'/Q' \in W_j$, $1 \leq j \leq n$, we have $q^{n+1-j}(q-1)$ fractions $P/Q \in W_{n+1}$ by (4). The number of P/Q with

$\deg Q = n+1$ and $\deg P = 0$ is $q^{n+1}(q-1)$. Thus

$$\begin{aligned}\#W_{n+1} &= \sum_{j=1}^{n} q^{n+1-j}(q-1)(q^{2j}-q^{2j-1}) + q^{n+1}(q-1)\\ &= q^{2n+2}-q^{2n+1}. \ \blacksquare\end{aligned}$$

3. Proof of Theorem 1. In what follows, we always assume that P and Q are coprime non-zero polynomials whenever P/Q denotes a rational function.

For P/Q with $\deg Q = n$, we put

$$E_n\left(\frac{P}{Q}\right) = \left\{f \in \mathbb{L} : \left|f - \frac{P}{Q}\right| < \frac{1}{q^{2n}}\right\}$$

and also put

$$E_n = \left\{f \in \mathbb{L} : \text{there exists } \frac{P}{Q} \text{ such that } \deg Q = n,\ \left|f - \frac{P}{Q}\right| < \frac{1}{q^{2n}}\right\}.$$

LEMMA 3. *For a fixed integer $n \geq 1$, if $P/Q \neq P'/Q'$ with* $\deg Q = \deg Q' = n$, *then*

$$E_n\left(\frac{P}{Q}\right) \cap E_n\left(\frac{P'}{Q'}\right) = \emptyset.$$

Proof. Since $|\cdot|$ is ultrametric, we see that if the intersection were non-empty, then

$$\left|\frac{P}{Q} - \frac{P'}{Q'}\right| < \frac{1}{q^{2n}}.$$

However,

$$\left|\frac{P}{Q} - \frac{P'}{Q'}\right| \geq \frac{1}{|Q|\,|Q'|} = \frac{1}{q^{2n}},$$

which gives a contradiction. $\blacksquare$

LEMMA 4. *For any $n \geq 1$,*

$$m(E_n) = 1 - \frac{1}{q}.$$

Proof. Since $m\{f \in \mathbb{L} : |f - P/Q| < 1/q^{2n}\} = 1/q^{2n}$ for a fixed P/Q with $\deg Q = n$, and the number of P/Q is $q^{2n} - q^{2n-1}$ from Lemma 2, we have the assertion. $\blacksquare$

LEMMA 5. *For any $n \geq 1$ and $k \geq 1$, we have*

$$m(E_n \cap E_{n+k}) = m(E_n)m(E_{n+k}) = \left(1 - \frac{1}{q}\right)^2.$$

Proof. If $f \in E_n \cap E_{n+k}$, say

$$\left|f - \frac{P}{Q}\right| < \frac{1}{q^{2n}}, \qquad \left|f - \frac{P'}{Q'}\right| < \frac{1}{q^{2n+2k}}$$

with $\deg Q = n$, $\deg Q' = n+k$, then $|P'/Q' - P/Q| < 1/q^{2n}$, so that by Lemma 1, P/Q is a convergent of the continued fraction of P'/Q'. Conversely, when this is the case, and $|f - P'/Q'| < 1/q^{2n+2k}$, then $f \in E_n \cap E_{n+k}$. Therefore

$$m(E_n \cap E_{n+k}) = Z(n, n+k)\,\frac{1}{q^{2n+2k}}, \tag{5}$$

where $Z(n, n+k)$ is the number of pairs P/Q, P'/Q' with P/Q a convergent to P'/Q', and $\deg Q = n$, $\deg Q' = n+k$. The number of choices for P/Q is $\#W_n = q^{2n}(1 - 1/q)$. For given P/Q, we will find the number of choices for P'/Q'. Suppose that P'/Q' satisfies

$$\left|f - \frac{P'}{Q'}\right| < \frac{1}{q^{2n+2k}}, \qquad \deg Q' = n+k \quad \text{for } f \in E_n\left(\frac{P}{Q}\right).$$

There exist $n = j_0 < j_1 < \ldots < j_{l-1} < j_l = n+k$ (uniquely) such that

$$\frac{P'}{Q'} = \frac{P_{m+l}}{Q_{m+l}} = [0; p_1, p_2, \ldots, p_m, \ldots, p_{m+l}]$$

with

$$\deg p_{m+i} = j_i - j_{i-1}, \qquad 1 \le i \le l.$$

Since $\#\{p \in \mathbb{F}[X] : \deg p = u\} = q^u(q-1)$, we have

$$\#\left\{\frac{P'}{Q'} : \deg p_{m+i} = j_i - j_{i-1},\ 1 \le i \le l\right\}$$
$$= q^{j_1 - j_0}(q-1)q^{j_2 - j_1}(q-1)\ldots q^{j_l - j_{l-1}}(q-1) = q^k(q-1)^l$$

for each fixed $(j_1, \ldots, j_l)$. The number of choices for $n < j_1 < \ldots < j_{l-1} < n+k$ is $\binom{k-1}{l-1}$ and l runs from 1 to k. Hence

$$\#\left\{\frac{P'}{Q'} : \left|f - \frac{P'}{Q'}\right| < \frac{1}{q^{2n+2k}} \text{ for some } f \in E_n\left(\frac{P}{Q}\right)\right\}$$
$$= \sum_{l=1}^{k} \binom{k-1}{l-1} q^k (q-1)^l = q^{2k}\left(1 - \frac{1}{q}\right).$$

Consequently,

$$Z(n, n+k) = q^{2n+2k}\left(1 - \frac{1}{q}\right)^2,$$

and by (5), we get

$$m(E_n \cap E_{n+k}) = \left(1 - \frac{1}{q}\right)^2 = m(E_n)m(E_{n+k}). \ \blacksquare$$

By the Borel–Cantelli lemma, this implies the following:

PROPOSITION 1. *For any sequence $n_1 < n_2 < \ldots$ of positive integers the inequality*

$$\left| f - \frac{P}{Q} \right| < \frac{1}{|Q|^2}, \qquad \deg Q = n_i,$$

has infinitely many solutions for m-a.e. $f \in \mathbb{L}$.

According to this proposition, we can assume that $\psi(Q) < 1/q^n$ for any $n \geq 1$. Then we rewrite Theorem 1 as follows:

THEOREM 3. *For any sequences $n_1 < n_2 < \ldots$ and $l_1, l_2, \ldots$ of positive integers, the inequality*

$$\left| f - \frac{P}{Q} \right| < \frac{1}{q^{2n_i + l_i}}, \qquad \deg Q = n_i,$$

has infinitely many solutions for m-a.e. $f \in \mathbb{L}$ if and only if

$$\sum_{i=1}^{\infty} q^{-l_i} = \infty.$$

Proof. Put

$$F_i = \left\{ f \in \mathbb{L} : \text{there exists } \frac{P}{Q} \text{ such that } \left| f - \frac{P}{Q} \right| < \frac{1}{q^{2n_i + l_i}}, \deg Q = n_i \right\}.$$

Given P/Q, the measure of $f \in \mathbb{L}$ with $|f - P/Q| < 1/q^{2n_i + l_i}$ is $1/q^{2n_i + l_i}$. The number of P/Q in W_{n_i} is $q^{2n_i} - q^{2n_i - 1}$, therefore

$$m(F_i) = \frac{q-1}{q} \cdot \frac{1}{q^{l_i}}. \tag{6}$$

Now the assertion follows from the next lemma together with (6) by Theorem 3 of [8].

LEMMA 6. (a) *$F_i \cap F_{i+j} = \emptyset$ if $n_i + l_i \geq n_{i+j}$.*
(b) *$m(F_i \cap F_{i+j}) = m(F_i) m(F_{i+j})$ if $n_i + l_i < n_{i+j}$.*

Proof. If $f \in F_i \cap F_{i+j}$, say

$$\left| f - \frac{P}{Q} \right| < \frac{1}{q^{2n_i + l_i}}, \qquad \left| f - \frac{P'}{Q'} \right| < \frac{1}{q^{2n_{i+j} + l_{i+j}}}$$

with $\deg Q = n_i$, $\deg Q' = n_{i+j}$, then

$$\left| \frac{P}{Q} - \frac{P'}{Q'} \right| < \frac{1}{q^{2n_i + l_i}}, \tag{7}$$

and on the other hand

$$\left| \frac{P}{Q} - \frac{P'}{Q'} \right| \geq \frac{1}{|Q|\,|Q'|} = \frac{1}{q^{n_i + n_{i+j}}}.$$

When $n_i + l_i \geq n_{i+j}$ these inequalities contradict each other, so $F_i \cap F_{i+j} = \emptyset$.

Suppose, then, that $n_i + l_i < n_{i+j}$. It follows from (7) that P/Q is a convergent to P'/Q'. Write again

$$\frac{P}{Q} = [0; p_1, \ldots, p_m], \qquad \frac{P'}{Q'} = [0; p_1, \ldots, p_m, p_{m+1}, \ldots, p_{m+l}].$$

Then by a well-known formula,

$$\left|\frac{P}{Q} - \frac{P'}{Q'}\right| = \frac{1}{|Q|^2 |p_{m+1}|} = \frac{1}{q^{2n_i + \deg p_{m+1}}},$$

yielding $\deg p_{m+1} > l_i$. In analogy to (5) we obtain

$$m(F_i \cap F_{i+j}) = Z(n_i, n_{i+j}, l_i)\,\frac{1}{q^{2n_{i+j}+l_{i+j}}}, \tag{8}$$

where $Z(n_i, n_{i+j}, l_i)$ is the number of pairs P/Q, P'/Q' as above with $\deg p_{m+1} > l_i$. Now, the number of choices for $p_{m+1}, \ldots, p_{m+l}$ is

$$q^{\deg p_{m+1}}(q-1)q^{\deg p_{m+2}}(q-1)\ldots q^{\deg p_{m+l}}(q-1) = q^{n_{i+j}-n_i}(q-1)^l.$$

Thus

$$\begin{aligned} &Z(n_i, n_{i+j}, l_i) \\ &\quad = (q^{2n_i} - q^{2n_i - 1}) \sum_{l=1}^{n_{i+j}-n_i-l_i} \binom{n_{i+j} - n_i - l_i - 1}{l-1} q^{n_{i+j}-n_i}(q-1)^l \\ &\quad = (q^{2n_i} - q^{2n_i-1}) q^{n_{i+j}-n_i}(q-1) q^{n_{i+j}-n_i-l_i-1} \\ &\quad = q^{2n_{i+j}-l_i}\left(1 - \frac{1}{q}\right)^2, \end{aligned}$$

which together with (8) yields the lemma. ∎

Example 1. Put

$$\psi(Q) = \begin{cases} 1/|Q| & \text{if } \deg Q \text{ is prime,} \\ 0 & \text{otherwise.} \end{cases}$$

Then we see that there are infinitely many solutions of

$$\left| f - \frac{P}{Q} \right| < \frac{1}{|Q|^2}, \qquad \deg Q \text{ prime},$$

for a.e. $f \in \mathbb{L}$.

4. General case. For a given polynomial

$$h = a_l X^l + a_{l-1} X^{l-1} + \ldots + a_1 X + a_0, \qquad a_i \in \mathbb{F},\ 0 \le i \le l,\ a_l \ne 0,$$

we denote by $\langle h \rangle$ the cylinder set defined by

$$\{ f \in \mathbb{L} : [X^{l+1} \cdot f] = h \}.$$

LEMMA 7. *Let h_k, $k \geq 1$, be a sequence of polynomials with*

$$\lim_{k\to\infty} \deg h_k = \infty,$$

and E_k be a sequence of measurable subsets of $\mathbb{L}$ for which $E_k \subset \langle h_k\rangle$. Suppose that $m(E_k) \geq \delta m(\langle h_k\rangle)$ for some $\delta > 0$. Then

$$m\Big(\bigcap_{l=1}^{\infty}\bigcup_{k=l}^{\infty} E_k\Big) = m\Big(\bigcap_{l=1}^{\infty}\bigcup_{k=l}^{\infty}\langle h_k\rangle\Big).$$

Proof. Let

$$H := \bigcap_{l=1}^{\infty}\bigcup_{k=l}^{\infty}\langle h_k\rangle, \quad E_l^* = \bigcup_{k=l}^{\infty} E_k, \quad H_l^* := H \setminus E_l^*.$$

We show that $m(H_l^*) = 0$ for any $l \geq 1$, which implies the assertion of this lemma. Suppose that $m(H_k^*) > 0$. For almost all $f_0 \in H_l^*$, there are infinitely many k such that $f_0 \in \langle h_k\rangle$. For $f = \sum_{i<0} a_i X^i \in \mathbb{L}$, we put $\iota(f) = \sum_{i<0} a_i q^i \in (0,1]$. The map ι is a measure isomorphism of $(\mathbb{L}, m)$ to $(0,1]$ with the Lebesgue measure. By this isomorphism, the cylinder sets $\langle h_k\rangle$ are mapped to q-adic rational intervals. So we can apply Lebesgue's density theorem to get

$$\frac{m(H_k^* \cap \langle h_k\rangle)}{m(\langle h_k\rangle)} > 1 - \frac{\delta}{2}$$

for some k. On the other hand, $H_k^* \cap E_k^* = \emptyset$. So

$$m(\langle h_k\rangle) \geq m(E_k) + m(H_k^* \cap \langle h_k\rangle) \geq \delta m(\langle h_k\rangle) + m(H_k^* \cap \langle h_k\rangle),$$

which says that $m(H_k^* \cap \langle h_k\rangle) \leq (1-\delta) m(\langle h_k\rangle)$. This is impossible. ■

LEMMA 8. *For any polynomial $h \in \mathbb{F}[X]$ and $g \in \mathbb{L}$, the map T of $\mathbb{L}$ onto itself defined by*

$$T(f) = hf + g - [hf + g] \quad \textit{for } f \in \mathbb{L}$$

is ergodic.

Proof. It is easy to see that both $f \mapsto h \cdot f$ and $f \mapsto f + g$ for $f \in \mathbb{L}$ are m-preserving. Then $\omega_i(f) = [h \cdot T^{i-1}]$, $1 \leq i < \infty$, is an independent and identically distributed sequence of random variables defined on $(\mathbb{L}, m)$. This implies the assertion of the lemma. ■

Let ψ be a $\{q^{-n} : n \geq 0\} \cup \{0\}$-valued function defined on the set of monic polynomials, that is, of the form

$$X^l + a_{l-1}X^{l-1} + \ldots + a_1 X + a_0, \quad a_i \in \mathbb{F},\ 0 \leq i \leq l-1.$$

Here $\psi(Q)$ depends on Q itself, and we put

$$E_Q = \{f \in \mathbb{L} : |f - P/Q| < \psi(Q)/|Q| \text{ for some polynomial } P \\ \text{with } \deg P < \deg Q \text{ and } (P,Q) = 1\}$$

for a monic polynomial Q. The following theorem is a formal power series version of [6].

THEOREM 4. *For any* ψ, $m(\bigcap_{n=1}^{\infty}\bigcup_{\deg Q \geq n} E_Q) = 0$ *or* 1.

Proof. If

$$\limsup_{\deg Q \to \infty} \frac{\psi(Q)}{q^{\deg Q}} > 0,$$

then we can find a sequence $Q_1, Q_2, \dots$ of monic polynomials and a positive integer l such that $\psi(Q_k)/q^{\deg Q} > q^{-l}$ for any $k \geq 1$. For any $f \in \mathbb{L}$ and sufficiently large k, we can find P ($\deg P < \deg Q_k$) such that

$$\left| f - \frac{P}{Q_k} \right| < \frac{1}{q^l} \qquad \left(< \frac{\psi(Q_k)}{q^{\deg Q_k}} \right)$$

and P and Q_k are coprime. Otherwise, Q_k has more than $q^{\deg Q_k - l}$ factors, which is impossible. This implies

$$m\Big(\bigcap_{l=1}^{\infty} \bigcup_{k=l}^{\infty} E_{Q_k} \Big) = 1.$$

Now we show the assertion of the theorem when

$$\limsup_{\deg Q \to \infty} \frac{\psi(Q)}{q^{\deg Q}} = 0.$$

This means we can apply Lemma 7. We put

$$E = \bigcap_{n=1}^{\infty} \bigcup_{\deg Q \geq n} E_Q.$$

Let R be an irreducible polynomial and consider

$$\left| f - \frac{P}{Q} \right| < \frac{\psi(Q)|R|^{n-1}}{|Q|}, \qquad (P,Q) = 1, \tag{9}$$

for $n \geq 1$. We put

$$E_0(n:R) = \{f \in \mathbb{L} : (9) \text{ has infinitely many solutions } P,Q \text{ with } R \nmid Q\},$$
$$E_1(n:R) = \{f \in \mathbb{L} : (9) \text{ has infinitely many solutions } P,Q \text{ with } R \,\|\, Q\}.$$

Then

$$E_i(1:R) \subset E_i(2:R) \subset \dots, \qquad E_i(1:R) \subset E \quad \text{for } i = 0,1.$$

From Lemma 7, we find that $m(E_i(n:R)) = m(E_i(1:R))$ for $n \geq 1$. Thus

$$m\Big(\bigcup_{n\geq 1} E_i(n:R)\Big) = m(E_i(1:R)).$$

Let

$$T_1(f) = R\cdot f - [R\cdot f] \quad \text{for } f \in \mathbb{L}.$$

Then

$$T_1\Big(\bigcup_{n\geq 1} E_0(n:R)\Big) = \bigcup_{n\geq 2} E_0(n:R).$$

From Lemma 8, we have

$$m\Big(\bigcup_{n\geq 1} E_0(n:R)\Big) = 0 \text{ or } 1.$$

Next we let

$$T_2(f) = R\cdot f + \frac{1}{R} - \left[R\cdot f + \frac{1}{R}\right] \quad \text{for } f \in \mathbb{L}.$$

Suppose (9) holds. We have

$$\left|\left(R\cdot f + \frac{1}{R}\right) - \frac{R\cdot P + Q/R}{Q}\right| < \frac{\psi(Q)|R|^n}{|Q|}, \qquad \left(R\cdot P + \frac{Q}{R}, Q\right) = 1,$$

and so

$$T_2\Big(\bigcup_{n\geq 1} E_1(n:R)\Big) = \bigcup_{n\geq 2} E_1(n:R).$$

Thus we have, again by Lemma 8,

$$m\Big(\bigcup_{n\geq 1} E_1(n:R)\Big) = 0 \text{ or } 1.$$

Hence, if either $m(E_0(1:R))$ or $m(E_1(1:R))$ is positive for some irreducible polynomial R, then $m(E) = 1$. Assume that $m(E_0(1:R)) = m(E_1(1:R)) = 0$ for any irreducible polynomial R. We put

$$F(R) = \{f \in \mathbb{L} : (2) \text{ has infinitely many solutions } P, Q \text{ with } R^2 \mid Q\}.$$

If $f \in F(R)$, then

$$\left|\left(f + \frac{U}{R}\right) - \frac{P + QU/R}{Q}\right| < \frac{\psi(Q)}{|Q|}, \qquad \left(P + \frac{QU}{R}, Q\right) = 1,$$

for any polynomial U with $0 \leq \deg U < \deg R$. This means that $f \in F(R)$ implies $f + U/R \in F(R)$. If we put $S(U;R) = \{f \in \mathbb{L} : [Rf] = U\}$, then

$$\bigcup_{U:\, 0\leq \deg U \leq \deg R} S(U;R) \cup \{f \in \mathbb{L} : \deg f < -\deg R\} = \mathbb{L}$$

and each measure is equal to $1/q^{\deg R}$. Since $F(R)$ is $(\cdot + U/R)$-invariant,

$$m(F(R) \cap S(U;R)) = \frac{m(F(R))}{q^{\deg R}}.$$

This implies

$$\frac{m(F(R) \cap S(U;R))}{m(S(U;R))} = m(F(R)).$$

By the density theorem, we have $m(E) = m(F(R)) = 1$ whenever $m(F(R)) > 0$ for some irreducible polynomial R; otherwise $m(E) = 0$, since $E = F(R) \cup E_0(1,R) \cup E_1(1,R)$. This concludes the proof of the theorem. ∎

REMARK. Note that the set E is the same as the one in the introduction.

Proof of Theorem 2. In what follows, we always assume that Q, Q_1, Q' and Q'_1 are monic. By the definition of E_Q,

$$m(E_Q) = \psi(Q)\,\frac{\Phi(Q)}{|Q|}. \tag{10}$$

Now consider the measure of the intersection of E_{Q_1} and E_Q ($\deg Q_1 \le \deg Q$). We let $N(Q_1, Q)$ be the number of pairs of polynomials P and P_1. For these polynomials, the conditions

$$\left|\frac{P}{Q} - \frac{P_1}{Q_1}\right| < \frac{\psi(Q)}{|Q|} + \frac{\psi(Q_1)}{|Q_1|}, \tag{11}$$

$$(P,Q) = (P_1,Q_1) = 1, \qquad \deg P < \deg Q, \qquad \deg P_1 < \deg Q_1,$$

hold for given Q and Q_1. Then

$$m(E_{Q_1} \cap E_Q) \le \min\left(\frac{\psi(Q_1)}{|Q_1|}, \frac{\psi(Q)}{|Q|}\right) N(Q_1,Q).$$

If

$$PQ_1 - P_1 Q = R \tag{12}$$

for some polynomial R, then $D = (Q_1, Q)$ divides R. Setting $Q_1 = DQ'_1, Q = DQ', R = DR'$, we have

$$PQ'_1 - P_1 Q' = R', \qquad (Q'_1, Q') = 1. \tag{13}$$

If P' and P'_1 also satisfy (12), then

$$P'Q'_1 - P'_1 Q' = R'. \tag{14}$$

From (13) and (14),

$$P = P' + KQ', \qquad K \text{ a polynomial.} \tag{15}$$

From (12), we see that

$$|P - P'| = |K|\,|Q'| < |Q| = |D|\,|Q|,$$

which implies $|K| < |D|$. The number of possible polynomials P satisfying (12) for a given R is no greater than $q^{\deg D}$. (11) implies

$$0 \neq |R| < |Q_1|\psi(Q) + |Q|\psi(Q_1)$$

and we can only take polynomials R divisible by D. We find that

$$N(Q_1, Q) \leq \frac{|Q_1|\psi(Q) + |Q|\psi(Q_1)}{|D|}\,|D| = |Q_1|\psi(Q) + |Q|\psi(Q_1).$$

Then

$$m(E_{Q_1} \cap E_Q) \leq 2\psi(Q_1)\psi(Q).$$

Since $\sum_{\deg Q \leq n} \psi(Q)$ diverges,

$$\sum_{\deg Q \leq n} \psi(Q) \leq \Big(\sum_{\deg Q \leq n} \psi(Q)\Big)^2$$

for sufficiently large n. Therefore

$$\sum_{\deg Q_1, \deg Q \leq n} m(E_{Q_1} \cap E_Q) \leq 2 \sum_{\substack{\deg Q_1, \deg Q \leq n \\ Q \neq Q_1}} \psi(Q_1)\psi(Q) + \sum_{\deg Q \leq n} \psi(Q)$$
$$< 3\Big(\sum_{\deg Q \leq n} \psi(Q)\Big)^2$$

for all sufficiently large $\deg Q$. From (3) and (10), we have

$$\sum_{\deg Q_1, \deg Q \leq n} m(E_{Q_1} \cap E_Q) < 3C^2\Big(\sum_{\deg Q \leq n} m(E_Q)\Big)^2$$

for infinitely many Q. Hence $m(E) > (3C^2)^{-1}$, by [9, Lemma 5, pp. 17–18]. Finally, applying Theorem 4, we have the assertion of the theorem. ∎

EXAMPLE 2. Put

$$\psi(Q) = \begin{cases} 1/|Q| & \text{if } Q \text{ is irreducible,} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\sum_{n=1}^{\infty} \sum_{Q : \deg Q = n} \psi(Q) > \sum_{k=1}^{\infty} \frac{1}{q^k} \cdot \frac{1}{k} \cdot q^k = \infty$$

and it is easy to see that

$$\sum_{\deg Q \leq n} \psi(Q) \leq C \sum_{\deg Q \leq n} \psi(Q)\,\frac{\Phi(Q)}{|Q|}.$$

Thus there are infinitely many solutions P/Q of

$$\left| f - \frac{P}{Q} \right| < \frac{1}{|Q|^2}, \qquad Q \text{ is irreducible,}$$

for a.e. $f \in \mathbb{L}$.

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