ACTA ARITHMETICA
110.3 (2003)

The integral mean of the sum-of-digits function of the Ostrowski expansion

by

Luís Roçadas (Vila Real)

1. Introduction and statement of results. Let $G = (G_j)_{j\ge 0}$ be a strictly increasing sequence of integers with $G_0 = 1$. Then every non-negative integer N has a unique G-ary expansion $N = \sum_{j\ge 0} b_j(N)G_j$ with integer digits $b_j(N)$ provided that $\sum_{j<k} b_j(N)G_j < G_k$. The *sum-of-digits function* $t_G(N)$ is given by

$$t_G(N) = \sum_{j\ge 0} b_j(N).$$

There are several papers concerning the distribution of $t_G(N)$ for fixed G, e.g. [1] and [2]. In [3] this function is studied by fixing N and considering the average values of $H^{-1}\sum_{g=2}^{H} t_G(N)$ with $G = (g^j)_{j\ge 0}$.

Let Ω be the set of all irrational numbers in the interval $[0,1]$. Then every $\alpha \in \Omega$ has a unique continued fraction expansion $\alpha = [0, a_1(\alpha), a_2(\alpha), \ldots]$ with convergents $p_n(\alpha)/q_n(\alpha)$. Given N, using the sequence $G = (q_j(\alpha))_{j\ge 0}$, we can obtain uniquely determined integers $m(N,\alpha)$ and $b_i(N,\alpha)$ (if it is clear from the context we omit the dependence on α in q_i, m and b_i and the dependence on N in m and b_i), $0 \le i \le m$, with the following properties:

(1) $N = \sum_{i=0}^{m} b_i q_i$.
(2) $b_m > 0$ and $0 \le b_i \le a_{i+1}$ for $0 \le i \le m$.
(3) If $0 < i \le m$ and $b_i = a_{i+1}$ then $b_{i-1} = 0$. Furthermore $b_0 < a_1$.

The expansion $N = \sum_{i=0}^{m} b_i q_i$ is called the *Ostrowski expansion* of N to base α. In analogy with the classical situation in [3] we study, for fixed N, the sum-of-digits function $s_N(\alpha)$ defined by $s_N(\alpha) = \sum_{i=0}^{m} b_i$ and the corresponding mean $\int_0^1 s_N(\alpha)\,d\alpha$. More precisely, our main goal is to prove, using some techniques and ideas from [5], the following theorem.

2000 *Mathematics Subject Classification*: Primary 11K16, 11K50, 11K55, 11A55.
Key words and phrases: Ostrowski expansion, sum-of-digits function, metrical results.

THEOREM. *For $N \in \mathbb{N}$,*

$$\int_0^1 s_N(\alpha)\,d\alpha = \frac{3}{\pi^2}\log^2 N + O((\log N)^{3/2}\log\log N).$$

For the proof of the Theorem we proceed as follows: We start by proving, for $b_i > 0$, the formulas $b_i = a_{i+1}\{Nq_i\alpha\} + O(1)$ for $2\,|\,i$, and $b_i = a_{i+1}(1 - \{Nq_i\alpha\}) + O(1)$ for $2\nmid i$. As $b_i = 0$ only for α in a small set $A_{N,i}$, by integrating these formulas we get the relation

$$\int_0^1 b_i(\alpha)\,d\alpha = \frac{1}{2}\int_0^1 \left(1 + \frac{q_{i-1}(\alpha)}{q_i(\alpha)}\right)\log^+\left(\frac{N}{q_i(\alpha)}\right)d\alpha + O(1).$$

Note that in contrast to $\int_0^1 a_{i+1}(\alpha)\,d\alpha = \infty$ the corresponding integral over $b_i(\alpha)$ is finite. Finally we calculate $\int_0^1 s_N(\alpha)\,d\alpha$ using the asymptotic relation

$$\sum_{i\ge 0}\int_0^1 \left(1 + \frac{q_{i-1}(\alpha)}{q_i(\alpha)}\right)\log^+\left(\frac{N}{q_i(\alpha)}\right)d\alpha = \frac{6}{\pi^2}\log^2 N + O((\log N)^{3/2}\log\log N).$$

2. Definitions and notations. For $\mathbf{a} = (a_n)_{n\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ define sequences $(P_n(\mathbf{a}))_{n\in\mathbb{Z}_+}$ and $(Q_n(\mathbf{a}))_{n\in\mathbb{Z}_+}$ by $P_0(\mathbf{a}) = 0$, $Q_0(\mathbf{a}) = 1$, $P_1(\mathbf{a}) = 1$, $Q_1(\mathbf{a}) = a_1$, and

$$P_{k+1}(\mathbf{a}) = a_{k+1}P_k(\mathbf{a}) + P_{k-1}(\mathbf{a}), \quad Q_{k+1}(\mathbf{a}) = a_{k+1}Q_k(\mathbf{a}) + Q_{k-1}(\mathbf{a}), \quad k \ge 1.$$

Then $P_k(\mathbf{a})$ and $Q_k(\mathbf{a})$ depend at most on $a_1, \ldots, a_k$; hence we may write $P_k(a_1, \ldots, a_k)$ for $P_k(\mathbf{a})$ and $Q_k(a_1, \ldots, a_k)$ for $Q_k(\mathbf{a})$. If $\alpha = [0; a_1, \ldots]$, then for $k \ge 0$, $p_k(\alpha) = P_k(a_1, \ldots, a_k)$ and $q_k(\alpha) = Q_k(a_1, \ldots, a_k)$.

For $k \in \mathbb{Z}_+$ and $\mathbf{a} \in \mathbb{N}^{k+1}$ let $J(\mathbf{a}) := \{\alpha \in \Omega : 1 \le j \le k+1 \Rightarrow a_j(\alpha) = a_j\}$. Then

$$J(\mathbf{a}) = \begin{cases} \left(\dfrac{P_{k+1}(\mathbf{a}) + P_k(\mathbf{a})}{Q_{k+1}(\mathbf{a}) + Q_k(\mathbf{a})}, \dfrac{P_{k+1}(\mathbf{a})}{Q_{k+1}(\mathbf{a})}\right) & \text{if } k \text{ is even,} \\ \left(\dfrac{P_{k+1}(\mathbf{a})}{Q_{k+1}(\mathbf{a})}, \dfrac{P_{k+1}(\mathbf{a}) + P_k(\mathbf{a})}{Q_{k+1}(\mathbf{a}) + Q_k(\mathbf{a})}\right) & \text{if } k \text{ is odd,} \end{cases}$$

but in any case, if λ denotes the Lebesgue measure on Ω,

$$\lambda(J(\mathbf{a})) = \frac{1}{Q_{k+1}(\mathbf{a})(Q_{k+1}(\mathbf{a}) + Q_k(\mathbf{a}))}.$$

For $x > 0$ we define $\log^+ x = \max\{\log x, 0\}$. For a real number x let $\{x\} := x - [x]$ be the fractional part of x.

3. Auxiliary results. We start by proving, for $b_i > 0$, the formulas

$$b_i = \begin{cases} a_{i+1}\{Nq_i\alpha\} + O(1) & \text{if } 2 \mid i, \\ a_{i+1}(1-\{Nq_i\alpha\}) + O(1) & \text{if } 2 \nmid i. \end{cases}$$

These formulas are valid for $\alpha \in \Omega$ except for $\alpha \in A_{N,i} = \{\alpha \in \Omega : A_i < 0\}$ if i is even, and except for $\alpha \in A_{N,i} = \{\alpha \in \Omega : A_i > 0\}$ if i is odd. Then we calculate the integral of $a_{i+1}\{Nq_i\alpha\}$ and $a_{i+1}(1-\{Nq_i\alpha\})$ over $[0,1]$. The rest of the section is devoted to obtaining an upper bound of the integral of $a_{i+1}(\alpha)$ over $A_{N,i}$.

For $i, j \in \mathbb{N}$, we define

$$s_{i,j} := q_{\min(i,j)}(q_{\max(i,j)}\alpha - p_{\max(i,j)}), \qquad A_i := \sum_{j=0}^{\infty} b_j s_{i,j}.$$

Noting that $a_{i+1}s_{i,j} = s_{i+1,j} - s_{i-1,j} + (-1)^i\delta_{i,j}$ for $i, j \geq 0$, we get

$$a_{i+1}A_i = A_{i+1} - A_{i-1} + (-1)^i b_i \quad \text{for all } i \geq 0. \tag{1}$$

LEMMA 1. *For $1 \leq i \leq m$ we have*:

(i) $\sum_{j=0}^{i-1} b_j q_j < q_i$.
(ii) $|\sum_{j=i}^{m} b_j(q_j\alpha - p_j)| \leq 1/q_i$, *and if* $b_i \neq 0$, *then* $\operatorname{sgn}(\sum_{j=i}^{m} b_j(q_j\alpha - p_j)) = (-1)^i$.
(iii) $|A_i| < 1$.
(iv) *If* $2 \mid i$ *and* $b_i > 0$, *then* $A_i > 0$.
(v) *If* $2 \nmid i$ *and* $b_i > 0$, *then* $A_i < 0$.

Proof. (i) We omit the simple proof.

(ii) and (iii). For a proof see [4, Section 3, Proposition 1], and note that the A_i there has to be replaced by A_i/q_i.

(iv) We have

$$A_i = \sum_{j=0}^{i-1} b_j q_j (q_i\alpha - p_i) + q_i \sum_{j=i}^{m} b_j(q_j\alpha - p_j).$$

The first sum is non-negative since $q_i\alpha - p_i \geq 0$ if $2 \mid i$, and using (ii) we find that the second sum is positive, giving $A_i > 0$.

(v) Similarly to (iv) noting that $q_i\alpha - p_i \leq 0$ if $2 \nmid i$. ∎

LEMMA 2. *For $i \in \mathbb{N}$ we have*:

(i) *If* $A_i \geq 0$ *then* $A_i = \{Nq_i\alpha\}$.
(ii) *If* $A_i < 0$ *then* $1 + A_i = \{Nq_i\alpha\}$.

Proof. We have

$$A_i - Nq_i\alpha = \sum_{j=0}^{i} b_j q_j(q_i\alpha - p_i) + \sum_{j=i+1}^{m} b_j q_i(q_j\alpha - p_j) - \sum_{j=0}^{m} b_j q_j q_i\alpha$$
$$= -\sum_{j=0}^{i} b_j q_j p_i - \sum_{j=i+1}^{m} b_j q_i p_j \in \mathbb{Z}.$$

Hence, as $|A_i| < 1$, we get $A_i = \{Nq_i\alpha\}$ if $A_i \geq 0$, and $A_i + 1 = \{Nq_i\alpha\}$ if $A_i < 0$. ∎

PROPOSITION 1. (i) *If* $2\,|\,i$ *and* $b_i > 0$ *then* $b_i = a_{i+1}\{Nq_i\alpha\} + O(1)$.

(ii) *If* $2\nmid i$ *and* $b_i > 0$ *then* $b_i = a_{i+1}(1 - \{Nq_i\alpha\}) + O(1)$.

In both cases the O*-constants do not depend on* α.

Proof. Define $\alpha_i(\alpha) = [a_i(\alpha); a_{i+1}(\alpha), a_{i+2}(\alpha), \ldots]$. It follows that $\alpha_i = a_i + 1/\alpha_{i+1}$.

(i) For $2\,|\,i$, we have

$$\sum_{j=0}^{i-1} s_{i,j} b_j = \sum_{j=0}^{i-1} (q_i\alpha - p_i) q_j b_j \leq (q_i\alpha - p_i) q_i \leq \frac{q_i}{q_{i+1}} \leq \frac{1}{a_{i+1}},$$
$$\Big|\sum_{j=i+1}^{\infty} s_{i,j} b_j\Big| \leq \frac{q_i}{q_{i+1}},$$

hence $\sum_{j=i+1}^{\infty} s_{i,j} b_j = O(1/a_{i+1})$ and furthermore

$$s_{i,i} = q_i(q_i\alpha - p_i) = \frac{(-1)^i q_i}{q_i\alpha_{i+1} + q_{i-1}} = \frac{q_i}{q_i a_{i+1} + q_i/\alpha_{i+2} + q_{i-1}} = \frac{1}{a_{i+1}} + O(1).$$

So, $A_i = b_i/a_{i+1} + O(1/a_{i+1}) + O(1/a_{i+1})$ and hence by Lemmas 1 and 2,

$$b_i = a_{i+1}\{Nq_i\alpha\} + O(1). \tag{2}$$

(ii) Similarly, $-A_i = b_i/a_{i+1} + O(1/a_{i+1})$, and again from Lemmas 1 and 2,

$$b_i = a_{i+1}(1 - \{Nq_i\alpha\}) + O(1). \tag{3}$$ ∎

By Proposition 1 we have explicit formulas for b_i except in the case $2\,|\,i$ and $A_i < 0$ and the case $2\nmid i$ and $A_i > 0$. Therefore we define the exceptional sets

$$A_{N,i} = \begin{cases} \{\alpha \in \Omega : A_i < 0\} & \text{for } 2\,|\,i, \\ \{\alpha \in \Omega : A_i > 0\} & \text{for } 2\nmid i. \end{cases}$$

Note that if $2\,|\,i$ and $A_i < 0$, then $1 + A_i = \{Nq_i\alpha\}$, hence $0 < 1 - \{Nq_i\alpha\} = -A_i < 1/a_{i+1}$. Therefore, $A_{N,i} \subseteq \{\alpha \in \Omega : 1 - \{Nq_i\alpha\} < 1/a_{i+1}\}$ for $2\,|\,i$. Analogously, $A_{N,i} \subseteq \{\alpha \in \Omega : \{Nq_i\alpha\} < 1/a_{i+1}\}$ if $2\nmid i$. For $x \in \mathbb{R}$, consider $B(x) = (\{x\} - \{x\}^2)/2$. Then we have

LEMMA 3. *If a and $i \neq 0$ are real numbers, we have*:

(i) $\int_0^a \{i\alpha\}\, d\alpha = a/2 - i^{-1}B(ia)$.

(ii) $\int_0^a (1 - \{i\alpha\})\, d\alpha = a/2 + i^{-1}B(-ia)$.

We omit the simple proof.

LEMMA 4. (i) *For even i, we have*

$$\int_{J(\mathbf{a})} \{Nq_i(\alpha)\alpha\}\, d\alpha = \frac{1}{2Q_{i+1}(\mathbf{a})(Q_{i+1}(\mathbf{a}) + Q_i(\mathbf{a}))} - \frac{1}{NQ_i(\mathbf{a})}\left(B\left(\frac{N}{Q_{i+1}(\mathbf{a})}\right) - B\left(\frac{N}{Q_{i+1}(\mathbf{a}) + Q_i(\mathbf{a})}\right)\right).$$

(ii) *For odd i, we have*

$$\int_{J(\mathbf{a})} (1 - \{Nq_i(\alpha)\alpha\})\, d\alpha = \frac{1}{2Q_{i+1}(\mathbf{a})(Q_{i+1}(\mathbf{a}) + Q_i(\mathbf{a}))} - \frac{1}{NQ_i(\mathbf{a})}\left(B\left(\frac{N}{Q_{i+1}(\mathbf{a})}\right) - B\left(\frac{N}{Q_{i+1}(\mathbf{a}) + Q_i(\mathbf{a})}\right)\right).$$

Proof. We note that

$$NQ_i(\mathbf{a})\,\frac{P_{i+1}(\mathbf{a})}{Q_{i+1}(\mathbf{a})} = N\,\frac{(-1)^i + P_i(\mathbf{a})Q_{i+1}(\mathbf{a})}{Q_{i+1}(\mathbf{a})} \equiv (-1)^i\,\frac{N}{Q_{i+1}(\mathbf{a})} \pmod 1 \tag{4}$$

and

$$NQ_i(\mathbf{a})\,\frac{P_{i+1}(\mathbf{a}) + P_i(\mathbf{a})}{Q_{i+1}(\mathbf{a}) + Q_i(\mathbf{a})} = N\,\frac{(-1)^i + P_i(\mathbf{a})Q_{i+1}(\mathbf{a}) + P_i(\mathbf{a})Q_i(\mathbf{a})}{Q_{i+1}(\mathbf{a}) + Q_i(\mathbf{a})} \equiv (-1)^i\,\frac{N}{Q_{i+1}(\mathbf{a}) + Q_i(\mathbf{a})} \pmod 1. \tag{5}$$

Consider the case of even i. From Lemma 3 we get

$$\begin{aligned}
&\int_{J(\mathbf{a})} \{Nq_i(\alpha)\alpha\}\, d\alpha \\
&= \int_0^{\frac{P_{i+1}(\mathbf{a})}{Q_{i+1}(\mathbf{a})}} \{Nq_i(\alpha)\alpha\}\, d\alpha - \int_0^{\frac{P_{i+1}(\mathbf{a})+P_i(\mathbf{a})}{Q_{i+1}(\mathbf{a})+Q_i(\mathbf{a})}} \{Nq_i(\alpha)\alpha\}\, d\alpha \\
&= \frac{P_{i+1}(\mathbf{a})}{2Q_{i+1}(\mathbf{a})} - \frac{1}{NQ_i(\mathbf{a})}\, B\left(NQ_i(\mathbf{a})\,\frac{P_{i+1}(\mathbf{a})}{Q_{i+1}(\mathbf{a})}\right) \\
&\quad - \left[\frac{P_{i+1}(\mathbf{a}) + P_i(\mathbf{a})}{2(Q_{i+1}(\mathbf{a}) + Q_i(\mathbf{a}))} - \frac{1}{NQ_i(\mathbf{a})}\, B\left(NQ_i(\mathbf{a})\,\frac{P_{i+1}(\mathbf{a}) + P_i(\mathbf{a})}{Q_{i+1}(\mathbf{a}) + Q_i(\mathbf{a})}\right)\right].
\end{aligned}$$

So, from formulas (4) and (5),

$$B\Big(NQ_i(\mathbf{a})\,\frac{P_{i+1}(\mathbf{a})}{Q_{i+1}(\mathbf{a})}\Big)=B\Big(\frac{N}{Q_{i+1}(\mathbf{a})}\Big),$$
$$B\Big(NQ_i(\mathbf{a})\,\frac{P_{i+1}(\mathbf{a})+P_i(\mathbf{a})}{Q_{i+1}(\mathbf{a})+Q_i(\mathbf{a})}\Big)=B\Big(\frac{N}{Q_{i+1}(\mathbf{a})+Q_i(\mathbf{a})}\Big).$$

Thus

$$\int\limits_{J(\mathbf{a})}\{Nq_i(\alpha)\alpha\}\,d\alpha=\frac{1}{2Q_{i+1}(\mathbf{a})(Q_{i+1}(\mathbf{a})+Q_i(\mathbf{a}))}-\frac{1}{NQ_i(\mathbf{a})}\Big(B\Big(\frac{N}{Q_{i+1}(\mathbf{a})}\Big)-B\Big(\frac{N}{Q_{i+1}(\mathbf{a})+Q_i(\mathbf{a})}\Big)\Big).$$

This proves (i). The proof of (ii) is completely similar. ∎

Proposition 2. (i) *For even i we have*

$$\int_0^1 a_{i+1}(\alpha)\{Nq_i(\alpha)\alpha\}\,d\alpha=\frac{1}{2}\int_0^1\Big(1+\frac{q_{i-1}(\alpha)}{q_i(\alpha)}\Big)\log^+\Big(\frac{N}{q_i(\alpha)}\Big)\,d\alpha+O(1).\tag{6}$$

(ii) *For odd i we have*

$$\int_0^1 a_{i+1}(\alpha)(1-\{Nq_i(\alpha)\alpha\})\,d\alpha=\frac{1}{2}\int_0^1\Big(1+\frac{q_{i-1}(\alpha)}{q_i(\alpha)}\Big)\log^+\Big(\frac{N}{q_i(\alpha)}\Big)\,d\alpha+O(1).\tag{7}$$

Proof. For $\alpha\in\mathbb{N}^i$ and $a\in\mathbb{N}$ let $J(\mathbf{a},a):=\{\alpha\in J(\mathbf{a}):a_{i+1}(\alpha)=a\}$. Then $J(\mathbf{a})=\bigcup_{a=1}^{\infty}J(\mathbf{a},a)$ and therefore

$$\int\limits_{J(\mathbf{a})}a_{i+1}(\alpha)\{Nq_i(\alpha)\alpha\}\,d\alpha=\sum_{a=1}^{\infty}a\int\limits_{J(\mathbf{a},a)}\{Nq_i(\alpha)\alpha\}\,d\alpha,\tag{8}$$

since, when α runs through $J(\mathbf{a},a)$, $a_{i+1}(\alpha)=a$ does not depend on α. Analogously,

$$\int\limits_{J(\mathbf{a})}a_{i+1}(1-\{Nq_i(\alpha)\alpha\})\,d\alpha=\sum_{a=1}^{\infty}a\int\limits_{J(\mathbf{a},a)}(1-\{Nq_i(\alpha)\alpha\})\,d\alpha.$$

Furthermore, if we put $\mathbf{a}'=(\mathbf{a},a)$ then $Q_k(\mathbf{a}')=Q_k(\mathbf{a})$ for $0\le k\le i$ and $Q_{i+1}(\mathbf{a}')=aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a})$. In order to calculate the sum in (8), we use Lemma 4 and the Abel summation formula. Note also that if $N\ge Q_i(\mathbf{a})$

and $T = \left[\frac{N-Q_{i-1}(\mathbf{a})}{Q_i(\mathbf{a})}\right]$, then

$$\begin{aligned}
&\sum_{a=T+1}^{\infty} \frac{1}{(aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))^2} \\
&\quad = [(T+1)Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a})]^{-2} + \sum_{a=T+2}^{\infty} \int_{a-1}^{a} \frac{dx}{(aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))^2} \\
&\quad \le \left[\frac{N-Q_{i-1}(\mathbf{a})}{Q_i(\mathbf{a})} \cdot Q_i(\mathbf{a}) + Q_{i-1}(\mathbf{a})\right]^{-2} + \sum_{a=T+2}^{\infty} \int_{a-1}^{a} \frac{dx}{(aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))^2} \\
&\quad \le N^{-2} + \sum_{a=T+2}^{\infty} \int_{a-1}^{a} \frac{dx}{(xQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))^2} \\
&\quad \le N^{-1}Q_i^{-1}(\mathbf{a}) + \left[\frac{-1}{Q_i(\mathbf{a})(xQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))}\right]_{T+1}^{\infty} = O\left(\frac{1}{NQ_i(\mathbf{a})}\right).
\end{aligned}$$

If $Q_{i-1}(\mathbf{a}) < N < Q_i(\mathbf{a})$, we have $T = 0$, so

$$\begin{aligned}
&\sum_{a=1}^{\infty} \frac{1}{(aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))^2} \\
&\quad = [Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a})]^{-2} + \sum_{a=2}^{\infty} \int_{a-1}^{a} \frac{dx}{(aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))^2} \\
&\quad \le Q_i(\mathbf{a})^{-2} + \sum_{a=2}^{\infty} \int_{a-1}^{a} \frac{dx}{(xQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))^2} \\
&\quad \le N^{-1}Q_i(\mathbf{a})^{-1} + \left[\frac{-1}{Q_i(\mathbf{a})(xQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))}\right]_{1}^{\infty} = O\left(\frac{1}{NQ_i(\mathbf{a})}\right).
\end{aligned}$$

Next we estimate the sum (8) from $a = T+1$ to ∞ from above. As $a \ge T+1$ we get

$$a > \frac{N-Q_{i-1}(\mathbf{a})}{Q_i(\mathbf{a})} \quad \text{and hence} \quad 0 \le \frac{N}{aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a})} < 1,$$

which implies that $N/Q_{i+1}(\mathbf{a})$ and $N/(Q_{i+1}(\mathbf{a})+Q_i(\mathbf{a}))$ lie in the interval $(0,1)$. Note also that for $x \in (0,1)$, $B(x) = (x-x^2)/2$. It follows that

$$\begin{aligned}
&\sum_{a=T+1}^{\infty} a \int_{J(\mathbf{a},a)} \{Nq_i(\alpha)\alpha\}\, d\alpha \\
&\qquad = \sum_{a=T+1}^{\infty} a \left(\frac{1}{2(aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))((a+1)Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))}\right)
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{a=T+1}^{\infty} a\biggl(\frac{1}{NQ_i(\mathbf{a})}\biggl(B\biggl(\frac{N}{aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a})}\biggr) \\
&\qquad\qquad - B\biggl(\frac{N}{(a+1)Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a})}\biggr)\biggr)\biggr) \\
&= \sum_{a=T+1}^{\infty} \frac{aN}{2Q_i(\mathbf{a})}\biggl(\frac{1}{(aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))^2} - \frac{1}{((a+1)Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))^2}\biggr) \\
&= \frac{N}{2Q_i(\mathbf{a})}\biggl(\sum_{a=T+1}^{\infty} \frac{1}{(aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))^2} + \frac{T}{((T+1)Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))^2}\biggr) \\
&= O\biggl(\frac{N}{Q_i(\mathbf{a})}\biggl(\frac{1}{NQ_i(\mathbf{a})} + \frac{N/Q_i(\mathbf{a})}{N^2}\biggr)\biggr) = O\biggl(\frac{1}{Q_i(\mathbf{a})^2}\biggr) = O(\lambda(J(\mathbf{a}))).
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
&\sum_{a=1}^{T} \frac{a}{NQ_i(\mathbf{a})}\biggl(B\biggl(\frac{N}{aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a})}\biggr) - B\biggl(\frac{N}{(a+1)Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a})}\biggr)\biggr) \\
&= \frac{1}{NQ_i(\mathbf{a})}\biggl(\sum_{a=1}^{T} B\biggl(\frac{N}{aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a})}\biggr) - TB\biggl(\frac{N}{(T+1)Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a})}\biggr)\biggr) \\
&= O\biggl(\frac{T}{NQ_i(\mathbf{a})}\biggr) = O\biggl(\frac{1}{Q_i(\mathbf{a})^2}\biggr) = O(\lambda(J(\mathbf{a}))).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\int\limits_{J(\mathbf{a})} a_{i+1}(\alpha)\{Nq_i(\alpha)\alpha\}\,d\alpha \\
&= \frac{1}{2}\sum_{a=1}^{T} \frac{a}{Q_i(\mathbf{a})}\biggl(\frac{1}{aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a})} - \frac{1}{(a+1)Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a})}\biggr) \\
&\quad - \sum_{a=1}^{T} \frac{a}{NQ_i(\mathbf{a})}\biggl(B\biggl(\frac{N}{aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a})}\biggr) - B\biggl(\frac{N}{(a+1)Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a})}\biggr)\biggr) \\
&\quad + \sum_{a=T+1}^{\infty} a \int\limits_{J(\mathbf{a},a)} \{Nq_i(\alpha)\alpha\}\,d\alpha \\
&= \frac{1}{2Q_i(\mathbf{a})}\sum_{a=1}^{T} \frac{1}{aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a})} - \frac{T}{2Q_i(\mathbf{a})}\,\frac{1}{(T+1)Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a})} \\
&\quad + O(\lambda(J(\mathbf{a})))
\end{aligned}
$$

$$= \frac{1}{2Q_i(\mathbf{a})}\biggl(\sum_{a=1}^{T}\frac{1}{aQ_i(\mathbf{a})} - \sum_{a=1}^{T}\frac{Q_{i-1}(\mathbf{a})}{aQ_i(\mathbf{a})(aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))}\biggr)$$
$$- \frac{T}{2Q_i(\mathbf{a})}\,\frac{1}{(T+1)Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a})} + O(\lambda(J(\mathbf{a})))$$
$$= \frac{1}{2Q_i(\mathbf{a})}\sum_{a=1}^{T}\frac{1}{aQ_i(\mathbf{a})}$$
$$+ O\biggl(\frac{1}{Q_i(\mathbf{a})}\sum_{a=1}^{T}\frac{Q_{i-1}(\mathbf{a})}{aQ_i(\mathbf{a})(aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))} + \lambda(J(\mathbf{a}))\biggr)$$
$$= \frac{1}{2Q_i(\mathbf{a})^2}\sum_{a=1}^{T}\frac{1}{a} + O\biggl(\frac{1}{Q_i(\mathbf{a})^2} + \lambda(J(\mathbf{a}))\biggr).$$

Observe that

$$\sum_{a\le T}\frac{1}{a} = \sum_{aQ_i(\mathbf{a})\le N}\frac{1}{a} - \sum_{N-Q_{i-1}(\mathbf{a})<aQ_i(\mathbf{a})\le N}\frac{1}{a} = \sum_{aQ_i(\mathbf{a})\le N}\frac{1}{a} + O(1),$$

as the condition $N - Q_{i-1}(\mathbf{a}) < a \le N$ is satisfied for at most one a. Then

$$\int_{J(\mathbf{a})} a_{i+1}(\alpha)\{Nq_i(\alpha)\alpha\}\,d\alpha \tag{9}$$
$$= \frac{1}{2Q_i(\mathbf{a})^2}\sum_{a=1}^{T}\frac{1}{a} + O\biggl(\frac{1}{Q_i(\mathbf{a})^2} + \lambda(J(\mathbf{a}))\biggr)$$
$$= \frac{1}{2Q_i(\mathbf{a})^2}\sum_{aQ_i(\mathbf{a})\le N}\frac{1}{a} + O(\lambda(J(\mathbf{a})))$$
$$= \frac{1}{2Q_i(\mathbf{a})(Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))}\biggl(1+\frac{Q_{i-1}(\mathbf{a})}{Q_i(\mathbf{a})}\biggr)\sum_{aQ_i(\mathbf{a})\le N}\frac{1}{a} + O(\lambda(J(\mathbf{a})))$$
$$= \frac{1}{2}\int_{J(\mathbf{a})}\biggl(1+\frac{q_{i-1}(\alpha)}{q_i(\alpha)}\biggr)\sum_{aQ_i(\mathbf{a})\le N}\frac{1}{a}\,d\alpha + O(\lambda(J(\mathbf{a})))$$
$$= \frac{1}{2}\int_{J(\mathbf{a})}\biggl(1+\frac{q_{i-1}(\alpha)}{q_i(\alpha)}\biggr)\biggl(\log^+\biggl(\frac{N}{q_i(\alpha)}\biggr) + O(1)\biggr)\,d\alpha + O(\lambda(J(\mathbf{a})))$$
$$= \frac{1}{2}\int_{J(\mathbf{a})}\biggl(1+\frac{q_{i-1}(\alpha)}{q_i(\alpha)}\biggr)\log^+\biggl(\frac{N}{q_i(\alpha)}\biggr)\,d\alpha + O(\lambda(J(\mathbf{a}))).$$

If $N \le Q_{i-1}(\mathbf{a})$ we have

$$\frac{N}{Q_{i+1}(\mathbf{a})} \le 1, \qquad \frac{N}{Q_{i+1}(\mathbf{a})+Q_i(\mathbf{a})} \le 1.$$

Hence

$$\begin{aligned}\sum_{a=1}^{\infty} a \int\limits_{J(\mathbf{a},a)} \{Nq_i(\alpha)\alpha\}\,d\alpha &= \frac{N}{2Q_i(\mathbf{a})}\sum_{a=1}^{\infty}\frac{1}{(aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))^2}\\ &\le \frac{N}{2Q_i(\mathbf{a})}\sum_{a=1}^{\infty}\frac{1}{a^2NQ_i(\mathbf{a})} = \frac{1}{2Q_i(\mathbf{a})^2}\sum_{a=1}^{\infty}\frac{1}{a^2}\\ &= O(\lambda(J(\mathbf{a}))).\end{aligned}$$

As $N \le Q_{i-1}(\mathbf{a})$ we have $\log^+(N/q_i(\alpha)) = 0$ and formula (9) is valid in this case also. By summing up in (9) over all $\mathbf{a} \in \mathbb{N}^i$ we get (6). Analogously we obtain (7) for odd i. ■

PROPOSITION 3. *There exists a constant $C > 0$ such that for all N and i,*

$$\int\limits_{A_{N,i}} a_{i+1}(\alpha)\,d\alpha \le C.$$

Proof. First we treat the case of i odd. We have

$$\begin{aligned}\int\limits_{A_{N,i}} a_{i+1}(\alpha)\,d\alpha &= \sum_{\mathbf{a}\in\mathbb{N}^i}\sum_{a=1}^{\infty}\int\limits_{A_{N,i}\cap J(\mathbf{a},a)} a_{i+1}(\alpha)\,d\alpha\\ &= \sum_{\mathbf{a}\in\mathbb{N}^i}\sum_{a=1}^{\infty}\int\limits_{A_{N,i}\cap J(\mathbf{a},a)} a\,d\alpha = \sum_{\mathbf{a}\in\mathbb{N}^i}\sum_{a=1}^{\infty} a\lambda(A_{N,i}\cap J(\mathbf{a},a)).\end{aligned}$$

Now, for odd i,

$$A_{N,i}\cap J(\mathbf{a},a) \subseteq \{\alpha\in[0,1] : \{Nq_i(\alpha)\alpha\} < 1/a\} \cap J(\mathbf{a},a).$$

Consider the set

$$M_k(\mathbf{a}) = \{\alpha \in J(\mathbf{a}) : k = [Nq_i(\alpha)\alpha]\}.$$

Then

$$A_{N,i}\cap J(\mathbf{a},a) = \bigcup_{k=0}^{NQ_i(\mathbf{a})-1} (A_{N,i}\cap J(\mathbf{a},a)\cap M_k(\mathbf{a}))$$

and hence

$$\int\limits_{A_{N,i}} a_{i+1}(\alpha)\,d\alpha = \sum_{\mathbf{a}\in\mathbb{N}^i}\sum_{a=1}^{\infty} a \sum_{0\le k\le NQ_i(\mathbf{a})} \lambda(A_{N,i}\cap J(\mathbf{a},a)\cap M_k(\mathbf{a})).$$

If $\alpha \in A_{N,i}\cap J(\mathbf{a},a)\cap M_k(\mathbf{a})$ then

$$\frac{k}{NQ_i(\mathbf{a})} < \alpha < \frac{k}{NQ_i(\mathbf{a})} + \frac{1}{aNQ_i(\mathbf{a})}$$

and

$$\frac{aP_i(\mathbf{a})+P_{i-1}(\mathbf{a})}{aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a})} \le \alpha \le \frac{(a+1)P_i(\mathbf{a})+P_{i-1}(\mathbf{a})}{(a+1)Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a})}.$$

We now define, omitting the dependence on $\mathbf{a}$ in P_i and Q_i,

$$E_1(\mathbf{a},a) = \left\{k \in \mathbb{Z}_+ : \frac{k}{NQ_i} \le \frac{aP_i+P_{i-1}}{aQ_i+Q_{i-1}} \le \frac{(a+1)P_i+P_{i-1}}{(a+1)Q_i+Q_{i-1}} \le \frac{k}{NQ_i} + \frac{1}{aNQ_i}\right\},$$
$$E_2(\mathbf{a},a) = \left\{k \in \mathbb{Z}_+ : \frac{aP_i+P_{i-1}}{aQ_i+Q_{i-1}} \le \frac{k}{NQ_i} \le \frac{k}{NQ_i} + \frac{1}{aNQ_i} \le \frac{(a+1)P_i+P_{i-1}}{(a+1)Q_i+Q_{i-1}}\right\},$$
$$E_3(\mathbf{a},a) = \left\{k \in \mathbb{Z}_+ : \frac{k}{NQ_i} \le \frac{aP_i+P_{i-1}}{aQ_i+Q_{i-1}} \le \frac{k}{NQ_i} + \frac{1}{aNQ_i} \le \frac{(a+1)P_i+P_{i-1}}{(a+1)Q_i+Q_{i-1}}\right\},$$
$$E_4(\mathbf{a},a) = \left\{k \in \mathbb{Z}_+ : \frac{aP_i+P_{i-1}}{aQ_i+Q_{i-1}} \le \frac{k}{NQ_i} \le \frac{(a+1)P_i+P_{i-1}}{(a+1)Q_i+Q_{i-1}} \le \frac{k}{NQ_i} + \frac{1}{aNQ_i}\right\}.$$

Then

$$\int_{A_{N,i}} a_{i+1}(\alpha)\,d\alpha \le \sum_{\mathbf{a}\in\mathbb{N}^i} \sum_{a=1}^{\infty} a \sum_{j=1}^{4} \sum_{k\in E_j(\mathbf{a},a)} \lambda(A_{N,i} \cap J(\mathbf{a},a) \cap M_k(\mathbf{a})). \tag{10}$$

We first derive an upper bound for

$$\sum_{a=1}^{\infty} a \sum_{k\in E_1(\mathbf{a},a)} \lambda(A_{N,i} \cap J(\mathbf{a},a) \cap M_k(\mathbf{a})).$$

From the conditions for k in $E_1(\mathbf{a},a)$ we obtain

$$k \le NQ_i(\mathbf{a})\,\frac{aP_i(\mathbf{a})+P_{i-1}(\mathbf{a})}{aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a})}, \qquad k \ge NQ_i(\mathbf{a})\,\frac{(a+1)P_i(\mathbf{a})+P_{i-1}(\mathbf{a})}{(a+1)Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a})} - \frac{1}{a}.$$

Thus

$$k \in \left[NQ_i(\mathbf{a})\,\frac{(a+1)P_i(\mathbf{a})+P_{i-1}(\mathbf{a})}{(a+1)Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a})} - \frac{1}{a},\, NQ_i(\mathbf{a})\,\frac{aP_i(\mathbf{a})+P_{i-1}(\mathbf{a})}{aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a})}\right],$$

which is an interval of length

$$\frac{NQ_i(\mathbf{a})}{(aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))((a+1)Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))} + \frac{1}{a}. \tag{11}$$

If $E_1(\mathbf{a},a)$ is not empty we have

$$\begin{aligned}\frac{1}{aNQ_i(\mathbf{a})} &= \frac{k}{NQ_i(\mathbf{a})} + \frac{1}{aNQ_i(\mathbf{a})} - \left(\frac{k}{NQ_i(\mathbf{a})}\right)\\ &\ge \frac{(a+1)P_i(\mathbf{a})+P_{i-1}(\mathbf{a})}{(a+1)Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a})} - \frac{aP_i(\mathbf{a})+P_{i-1}(\mathbf{a})}{aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a})}\\ &= \frac{1}{(aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))((a+1)Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))}\end{aligned}$$

and hence

$$
\begin{aligned}
N &\le \frac{(aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))((a+1)Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))}{aQ_i(\mathbf{a})} \\
&\le \frac{(a+1)Q_i(\mathbf{a})(a+2)Q_i(\mathbf{a})}{aQ_i(\mathbf{a})} \le \frac{(a+a)Q_i(\mathbf{a})(a+2a)Q_i(\mathbf{a})}{aQ_i(\mathbf{a})} \\
&\le 6aQ_i(\mathbf{a}).
\end{aligned}
\tag{12}
$$

We have $P_i(\mathbf{a})Q_{i-1}(\mathbf{a}) - P_{i-1}(\mathbf{a})Q_i(\mathbf{a}) = 1$. It follows for $k \in E_1(\mathbf{a}, a)$ that

$$
\begin{aligned}
k &\le NQ_i(\mathbf{a})\,\frac{aP_i(\mathbf{a})+P_{i-1}(\mathbf{a})}{aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a})} \\
&= N\,\frac{aP_i(\mathbf{a})Q_i(\mathbf{a})+P_{i-1}(\mathbf{a})Q_i(\mathbf{a})}{aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a})} \\
&= N\,\frac{aP_i(\mathbf{a})Q_i(\mathbf{a})-1+Q_{i-1}(\mathbf{a})P_i(\mathbf{a})}{aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a})} \\
&= NP_i(\mathbf{a})\,\frac{aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a})}{aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a})} - \frac{N}{aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a})} \\
&= NP_i(\mathbf{a}) - \frac{N}{aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a})}.
\end{aligned}
$$

Similarly, using

$$
k \ge NQ_i(\mathbf{a})\,\frac{(a+1)P_i(\mathbf{a})+P_{i-1}(\mathbf{a})}{(a+1)Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a})} - \frac{1}{a},
$$

we have

$$
k \ge NP_i(\mathbf{a}) - \frac{N}{(a+1)Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a})} - \frac{1}{a}
$$

and hence

$$
\begin{aligned}
NP_i(\mathbf{a}) - \frac{N}{(a+1)Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a})} - \frac{1}{a} &\le k \\
&\le NP_i(\mathbf{a}) - \frac{N}{aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a})}.
\end{aligned}
\tag{13}
$$

Therefore $|E_1(\mathbf{a}, a)| \le 1$. Now, if

$$
\frac{2N}{Q_i(\mathbf{a})} < a, \qquad \frac{N}{aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a})} < \frac{1}{2}
$$

we directly get, for $a > 1$,

$$
\frac{N}{(a+1)Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a})} + \frac{1}{a} < \frac{1}{a} + \frac{1}{2} < 1,
$$

which means that the interval (13) does not contain an integer in this case.

Furthermore

$$\begin{aligned}\lambda(J(\mathbf{a},a)) &= \frac{1}{(aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))((a+1)Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))}\\ &\le \frac{1}{a^2(Q_i(\mathbf{a}))^2} = \frac{2}{2a^2(Q_i(\mathbf{a}))^2}\\ &\le \frac{2}{a^2Q_i(\mathbf{a})(Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))} = 2\frac{\lambda(J(\mathbf{a}))}{a^2}.\end{aligned}$$

Thus, since for positive integers k and m,

$$\sum_{a=k}^{mk}\frac{1}{a} \le \sum_{a=k}^{mk}\frac{1}{k} = (mk-k+1)\,\frac{1}{k} \le m,$$

we have

$$\begin{aligned}&\sum_{a=1}^{\infty} a \sum_{k\in E_1(\mathbf{a},a)} \lambda(A_{N,i}\cap J(\mathbf{a},a)\cap M_k(\mathbf{a})) \le \sum_{a=1}^{\infty} a \sum_{k\in E_1(\mathbf{a},a)} \lambda(J(\mathbf{a},a))\\ &\le 2\lambda(J(\mathbf{a},1)) + \sum_{N/(6Q_i(\mathbf{a}))\le a\le 2N/Q_i(\mathbf{a})} a\lambda(J(\mathbf{a},a))\\ &\le \frac{2}{(Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))(2Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))} + \sum_{N/(6Q_i(\mathbf{a}))\le a\le 2N/Q_i(\mathbf{a})} 2\,\frac{\lambda(J(\mathbf{a}))}{a}\\ &\le \frac{2}{(Q_i(\mathbf{a}))^2} + 2\cdot 12\cdot \lambda(J(\mathbf{a})) = O(\lambda(J(\mathbf{a}))).\end{aligned}$$

We have established

$$\sum_{a=1}^{\infty} a \sum_{k\in E_1(\mathbf{a},a)} \lambda(A_{N,i}\cap J(\mathbf{a},a)\cap M_k(\mathbf{a})) = O(\lambda(J(\mathbf{a}))).$$

In order to estimate

$$\sum_{a=1}^{\infty} a \sum_{k\in E_2(\mathbf{a},a)} \lambda(A_{N,i}\cap J(\mathbf{a},a)\cap M_k(\mathbf{a})),$$

we start by observing that for k in $E_2(\mathbf{a},a)$ we have

$$\begin{aligned}&k \ge NQ_i(\mathbf{a})\,\frac{aP_i(\mathbf{a})+P_{i-1}(\mathbf{a})}{aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a})},\\ &k \le NQ_i(\mathbf{a})\,\frac{(a+1)P_i(\mathbf{a})+P_{i-1}(\mathbf{a})}{(a+1)Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a})} - \frac{1}{a},\end{aligned} \tag{14}$$

so it follows that

$$k \in \left[NQ_i(\mathbf{a})\,\frac{aP_i(\mathbf{a})+P_{i-1}(\mathbf{a})}{aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a})},\, NQ_i(\mathbf{a})\,\frac{(a+1)P_i(\mathbf{a})+P_{i-1}(\mathbf{a})}{(a+1)Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a})} - \frac{1}{a}\right],$$

which is an interval of length

$$\frac{NQ_i(\mathbf{a})}{(aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))((a+1)Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))}-\frac{1}{a}. \tag{15}$$

As

$$A_{N,i}\cap J(\mathbf{a},a)\cap M_k(\mathbf{a})\subseteq\left[\frac{k}{NQ_i(\mathbf{a})},\frac{k}{NQ_i(\mathbf{a})}+\frac{1}{aNQ_i(\mathbf{a})}\right],$$

we get

$$\lambda(A_{N,i}\cap J(\mathbf{a},a)\cap M_k(\mathbf{a}))\le\frac{1}{aNQ_i(\mathbf{a})}.$$

If $E_2(\mathbf{a},a)$ is not empty we have

$$\frac{1}{aNQ_i(\mathbf{a})}\le\frac{1}{(aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))((a+1)Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))}$$

and hence $2N\ge aQ_i(\mathbf{a})$. Moreover, from (14) we get

$$\begin{aligned} NP_i(\mathbf{a})-\frac{N}{aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a})}&\le k \\ &\le NP_i(\mathbf{a})-\frac{N}{(a+1)Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a})}-\frac{1}{a}. \end{aligned} \tag{16}$$

From this we infer that $|E_2(\mathbf{a},a)|$ is at most

$$\begin{aligned} &\frac{N}{aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a})}-\frac{N}{(a+1)Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a})}-\frac{1}{a}+1 \\ &\quad=O\left(\frac{NQ_i(\mathbf{a})}{(aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))((a+1)Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))}+1\right) \\ &\quad=O\left(\frac{N}{a^2Q_i(\mathbf{a})}+1\right). \end{aligned} \tag{17}$$

Consider the set

$$A(\mathbf{a})=\left\{a\in\mathbb{N}:\frac{NQ_i(\mathbf{a})}{(aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))((a+1)Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))}\ge\frac{1}{2}\right\}.$$

In the case $a\in\mathbb{N}\backslash A(\mathbf{a})$, there is no k satisfying (16), which means that in this case $E_2(\mathbf{a},a)=\emptyset$. If $a\in A(\mathbf{a})$, we have

$$2NQ_i(\mathbf{a})\ge(aQ_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))((a+1)Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a}))\ge a^2Q_i^2(\mathbf{a})$$

and then $N/(a^2Q_i(\mathbf{a}))\ge 1/2$. This implies that $1=O(N/(a^2Q_i(\mathbf{a})))$ and from (17) we get $|E_2(\mathbf{a},a)|=O(N/(a^2Q_i(\mathbf{a})))$. Thus

$$\begin{aligned} &\sum_{a=1}^{\infty}a\sum_{k\in E_2(\mathbf{a},a)}\lambda(A_{N,i}\cap J(\mathbf{a},a)\cap M_k(\mathbf{a})) \\ &\qquad=\sum_{a\le 2N/Q_i(\mathbf{a})}a\sum_{k\in E_2(\mathbf{a},a)}\lambda(A_{N,i}\cap J(\mathbf{a},a)\cap M_k(\mathbf{a})) \end{aligned}$$

$$= \sum_{a \in A(\mathbf{a}) \wedge a \le 2N/Q_i(\mathbf{a})} a \sum_{k \in E_2(\mathbf{a},a)} \lambda(A_{N,i} \cap J(\mathbf{a},a) \cap M_k(\mathbf{a}))$$
$$= O\biggl(\sum_{a \in A(\mathbf{a})} a \,\frac{N}{a^2 Q_i(\mathbf{a})} \cdot \frac{1}{aNQ_i(\mathbf{a})}\biggr) = O\biggl(\sum_{a=1}^{\infty} \frac{1}{a^2 (Q_i(\mathbf{a}))^2}\biggr)$$
$$= O\biggl(\frac{1}{(Q_i(\mathbf{a}))^2}\biggr) = O(\lambda(J(\mathbf{a}))).$$

Hence,

$$\sum_{a=1}^{\infty} a \sum_{k \in E_2(\mathbf{a},a)} \lambda(A_{N,i} \cap J(\mathbf{a},a) \cap M_k(\mathbf{a})) = O(\lambda(J(\mathbf{a}))).$$

In order to estimate

$$\sum_{a=1}^{\infty} a \sum_{k \in E_3(\mathbf{a},a)} \lambda(A_{N,i} \cap J(\mathbf{a},a) \cap M_k(\mathbf{a})),$$

we notice that for $k \in E_3(\mathbf{a},a)$ we have

$$k \le NQ_i(\mathbf{a}) \,\frac{aP_i(\mathbf{a}) + P_{i-1}(\mathbf{a})}{aQ_i(\mathbf{a}) + Q_{i-1}(\mathbf{a})}, \qquad k \ge NQ_i(\mathbf{a}) \,\frac{aP_i(\mathbf{a}) + P_{i-1}(\mathbf{a})}{aQ_i(\mathbf{a}) + Q_{i-1}(\mathbf{a})} - \frac{1}{a}.$$

From these conditions we deduce that $E_3(\mathbf{a},a)$ is an interval of length at most $1/a$ and (as before) that if $k \in E_3(\mathbf{a},a)$, then

$$NP_i(\mathbf{a}) - \frac{N}{aQ_i(\mathbf{a}) + Q_{i-1}(\mathbf{a})} - \frac{1}{a} \le k \le NP_i(\mathbf{a}) - \frac{N}{aQ_i(\mathbf{a}) + Q_{i-1}(\mathbf{a})}. \tag{18}$$

We have seen above that for $2N/Q_i(\mathbf{a}) < a$, $E_3(\mathbf{a},a)$ is empty. So,

$$\sum_{a=1}^{\infty} a \sum_{k \in E_3(\mathbf{a},a)} \lambda(A_{N,i} \cap J(\mathbf{a},a) \cap M_k(\mathbf{a}))$$
$$\le \sum_{a=1}^{\infty} a \sum_{k \in E_3(\mathbf{a},a)} \lambda(J(\mathbf{a},a))$$
$$\le \sum_{1 \le a \le 2N/Q_i(\mathbf{a})} a\lambda(J(\mathbf{a},a)) \le \sum_{1 \le a \le 2N/Q_i(\mathbf{a})} a \,\frac{1}{aNQ_i(\mathbf{a})}$$
$$\le \frac{2N}{Q_i(\mathbf{a})} \cdot \frac{1}{NQ_i(\mathbf{a})} = \frac{2}{Q_i^2(\mathbf{a})} = O(\lambda(J(\mathbf{a}))).$$

Hence

$$\sum_{a=1}^{\infty} a \sum_{k \in E_3(\mathbf{a},a)} \lambda(A_{N,i} \cap J(\mathbf{a},a) \cap M_k(\mathbf{a})) = O(\lambda(J(\mathbf{a}))).$$

In order to estimate

$$\sum_{a=1}^{\infty} a \sum_{k\in E_4(\mathbf{a},a)} \lambda(A_{N,i}\cap J(\mathbf{a},a)\cap M_k(\mathbf{a})),$$

we observe that for $k\in E_4(\mathbf{a},a)$ we have

$$k \le NQ_i(\mathbf{a})\,\frac{(a+1)P_i(\mathbf{a})+P_{i-1}(\mathbf{a})}{(a+1)Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a})},$$

$$k \ge NQ_i(\mathbf{a})\,\frac{(a+1)P_i(\mathbf{a})+P_{i-1}(\mathbf{a})}{(a+1)Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a})} - \frac{1}{a}.$$

From these conditions we deduce that $E_4(\mathbf{a},a)$ is an interval of length at most $1/a$ and, as before, if $k\in E_4(\mathbf{a},a)$ then

$$NP_i(\mathbf{a}) - \frac{N}{(a+1)Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a})} - \frac{1}{a} \le k \le NP_i(\mathbf{a}) - \frac{N}{(a+1)Q_i(\mathbf{a})+Q_{i-1}(\mathbf{a})}. \tag{19}$$

Once again, if $2N/Q_i(\mathbf{a}) < a$, $E_4(\mathbf{a},a)$ is empty. Then

$$\begin{aligned}\sum_{a=1}^{\infty} a \sum_{k\in E_4(\mathbf{a},a)} \lambda(A_{N,i}\cap J(\mathbf{a},a)\cap M_k(\mathbf{a})) &\le \sum_{a=1}^{\infty} a \sum_{k\in E_4(\mathbf{a},a)} \lambda(J(\mathbf{a},a))\\ &= O(\lambda(J(\mathbf{a}))).\end{aligned}$$

It follows that

$$\sum_{a=1}^{\infty} a \sum_{k\in E_4(\mathbf{a},a)} \lambda(A_{N,i}\cap J(\mathbf{a},a)\cap M_k(\mathbf{a})) = O(\lambda(J(\mathbf{a}))).$$

So, if $2\nmid i$, we have proved that

$$\begin{aligned}\int_{A_{N,i}} a_{i+1}(\alpha)\,d\alpha &= \sum_{\mathbf{a}\in\mathbb{N}^i}\sum_{a=1}^{\infty} a \sum_{j=1}^{4}\sum_{k\in E_j(\mathbf{a},a)} \lambda(A_{N,i}\cap J(\mathbf{a},a)\cap M_k(\mathbf{a}))\\ &= \sum_{\mathbf{a}\in\mathbb{N}^i} 4\cdot O(\lambda(J(\mathbf{a}))) = O(1).\end{aligned}$$

The case $2\,|\,i$ can be proved either similarly or by a change of variable $\alpha\to 1-\alpha$. In fact, for $0<\alpha\le 1/2$ we have $a_{i+1}(1-\alpha)=a_i(\alpha)$ and $q_{i+1}(1-\alpha)=q_i(\alpha)$ for $i>0$, and if $1/2<\alpha\le 1$ we have $a_i(1-\alpha)=a_{i+1}(\alpha)$ and $q_i(1-\alpha)=q_{i+1}(\alpha)$ for $i\ge 0$. Hence,

- if $0 < \alpha < 1/2$ and $i > 0$ then

$$\begin{aligned} 1-\alpha \in A_{N,i} &\Leftrightarrow 1-\{Nq_i(1-\alpha)(1-\alpha)\} < \frac{1}{a_{i+1}(1-\alpha)} \\ &\Leftrightarrow \{Nq_i(1-\alpha)\alpha\} < \frac{1}{a_{i+1}(1-\alpha)} \\ &\Leftrightarrow \{Nq_{i-1}(\alpha)\alpha\} < \frac{1}{a_i(\alpha)} \Leftrightarrow \alpha \in A_{N,i-1}; \end{aligned}$$

- if $1/2 < \alpha < 1$ and $i > 0$ then

$$\begin{aligned} 1-\alpha \in A_{N,i} &\Leftrightarrow 1-\{Nq_i(1-\alpha)(1-\alpha)\} < \frac{1}{a_{i+1}(1-\alpha)} \\ &\Leftrightarrow \{Nq_i(1-\alpha)\alpha\} < \frac{1}{a_{i+1}(1-\alpha)} \\ &\Leftrightarrow \{Nq_{i+1}(\alpha)\alpha\} < \frac{1}{a_{i+2}(\alpha)} \Leftrightarrow \alpha \in A_{N,i+1}; \end{aligned}$$

- if $1/2 < \alpha < 1$ and $i = 0$ then

$$\begin{aligned} 1-\alpha \in A_{N,0} &\Leftrightarrow 1-\{Nq_0(1-\alpha)(1-\alpha)\} < \frac{1}{a_1(1-\alpha)} \\ &\Leftrightarrow \{Nq_0(1-\alpha)\alpha\} < \frac{1}{a_1(1-\alpha)} \\ &\Leftrightarrow \{Nq_1(\alpha)\alpha\} < \frac{1}{a_2(\alpha)+1} \\ &\Rightarrow \{Nq_1(\alpha)\alpha\} < \frac{1}{a_2(\alpha)} \Rightarrow \alpha \in A_{N,1}. \end{aligned}$$

Let C_X be the characteristic function of the set X. Then for $i > 0$,

$$\begin{aligned} \int\limits_{A_{N,i}} a_{i+1}(\alpha)\,d\alpha &= \int\limits_0^1 a_{i+1}(\alpha)C_{A_{N,i}}(\alpha)\,d\alpha \\ &= \int\limits_0^{1/2} a_{i+1}(1-\alpha)C_{A_{N,i}}(1-\alpha)\,d\alpha + \int\limits_{1/2}^1 a_{i+1}(1-\alpha)C_{A_{N,i}}(1-\alpha)\,d\alpha \\ &= \int\limits_0^{1/2} a_i(\alpha)C_{A_{N,i-1}}(\alpha)\,d\alpha + \int\limits_{1/2}^1 a_{i+2}(\alpha)C_{A_{N,i+1}}(\alpha)\,d\alpha = O(1). \end{aligned}$$

Similarly,

$$\begin{aligned} \int\limits_{A_{N,0}} a_1(\alpha)\,d\alpha &= \int\limits_0^1 a_1(\alpha)C_{A_{N,0}}(\alpha)\,d\alpha \\ &= \int\limits_0^{1/2} a_1(1-\alpha)C_{A_{N,0}}(1-\alpha)\,d\alpha + \int\limits_{1/2}^1 a_1(1-\alpha)C_{A_{N,0}}(1-\alpha)\,d\alpha \end{aligned}$$

$$\leq \frac{1}{2} + \int_{1/2}^{1} (a_2(\alpha)+1) C_{A_{N,1}}(\alpha)\, d\alpha \leq 1 + \int_{1/2}^{1} a_2(\alpha) C_{A_{N,1}}(\alpha)\, d\alpha = O(1),$$

and the result follows. ∎

4. Proof of the Theorem. In order to prove the Theorem we start by proving the following result:

Proposition 4. *Given $\alpha \in \Omega$ and $N \in \mathbb{N}$ with $N = \sum_{i=0}^{m} b_i q_i$ we have for $0 \leq i \leq m$,*

$$\int_0^1 b_i(\alpha)\, d\alpha = \frac{1}{2}\int_0^1 \left(1 + \frac{q_{i-1}(\alpha)}{q_i(\alpha)}\right) \log^+\left(\frac{N}{q_i(\alpha)}\right) d\alpha + O(1).$$

Proof. For even i we have, by Section 2 and Propositions 1–3,

$$\begin{aligned}\int_0^1 b_i(\alpha)\, d\alpha &= \int_{[0,1]\setminus A_{N,i}} b_i(N,\alpha)\, d\alpha + \underbrace{\int_{A_{N,i}} b_i(N,\alpha)\, d\alpha}_{=\,0} \\ &= \int_{[0,1]\setminus A_{N,i}} (a_{i+1}(\alpha)\{N q_i(\alpha)\alpha\} + O(1))\, d\alpha \\ &= \int_0^1 a_{i+1}(\alpha)\{N q_i(\alpha)\alpha\}\, d\alpha + O(1) \\ &= \frac{1}{2}\int_0^1 \left(1 + \frac{q_{i-1}(\alpha)}{q_i(\alpha)}\right) \log^+\left(\frac{N}{q_i(\alpha)}\right) d\alpha + O(1).\end{aligned}$$

The proof for the odd case is entirely similar. ∎

Theorem 1. *For $N \in \mathbb{N}$,*

$$\int_0^1 s_N(\alpha)\, d\alpha = \frac{1}{2}\sum_{i=0}^{\infty}\int_0^1 \left(1 + \frac{q_{i-1}(\alpha)}{q_i(\alpha)}\right) \log^+\left(\frac{N}{q_i(\alpha)}\right) d\alpha + O(\log N).$$

Proof. Let $(F_k)_{k\geq 0}$ be the sequence of Fibonacci numbers: $F_0 = F_1 = 1$, $F_{k+1} = F_k + F_{k-1}$. Then there is a $c > 0$ such that $\log F_k \geq k/c$ for $k \geq 2$. For every $\alpha \in \Omega$ we have $q_k(\alpha) \geq F_k$, and $N \leq q_k(\alpha)$ for $c \log N \leq k$, and therefore for $2 \mid k$,

$$a_{k+1}\{N q_k \alpha\} \leq a_{k+1} N |q_k \alpha - p_k| < a_{k+1} N / q_{k+1} \leq N/q_k.$$

Similarly for $2 \nmid k$,

$$a_{k+1}(1 - \{N q_k \alpha\}) \leq a_{k+1} N |q_k \alpha - p_k| < N/q_k.$$

Hence

$$\sum_{2|k\geq c\log N}^{\infty} a_{k+1}\{Nq_k\alpha\} = \sum_{2\nmid k\geq c\log N}^{\infty} a_{k+1}(1-\{Nq_k\alpha\}) = O(1).$$

So, we can calculate $\int_0^1 s_N(\alpha)\,d\alpha$ as follows:

$$\begin{aligned}\int_0^1 s_N(\alpha)\,d\alpha &= \int_0^1 \sum_{i=0}^{\infty} b_i(N,\alpha)\,d\alpha \\ &= \sum_{\substack{2|i\\ i\leq c\log N}} \int_{[0,1]\setminus A_{N,i}} a_{i+1}(\alpha)\{Nq_i(\alpha)\alpha\}\,d\alpha \\ &\quad + \sum_{\substack{2\nmid i\\ i\leq c\log N}} \int_{[0,1]\setminus A_{N,i}} a_{i+1}(\alpha)(1-\{Nq_i(\alpha)\alpha\})\,d\alpha + O(1) \\ &= \sum_{\substack{2|i\\ i\leq c\log N}} \Big(\int_0^1 a_{i+1}(\alpha)\{Nq_i(\alpha)\alpha\}\,d\alpha + O(1)\Big) \\ &\quad + \sum_{\substack{2\nmid i\\ i\leq c\log N}} \Big(\int_0^1 a_{i+1}(\alpha)(1-\{Nq_i(\alpha)\alpha\})\,d\alpha + O(1)\Big) + O(1) \\ &= \sum_{\substack{2|i\\ i\leq c\log N}} \int_0^1 a_{i+1}(\alpha)\{Nq_i(\alpha)\alpha\}\,d\alpha \\ &\quad + \sum_{\substack{2\nmid i\\ i\leq c\log N}} \int_0^1 a_{i+1}(\alpha)(1-\{Nq_i(\alpha)\alpha\})\,d\alpha + O(\log N) + O(1) \\ &= \frac{1}{2}\sum_{i\leq c\log N} \int_0^1 \Big(1+\frac{q_{i-1}(\alpha)}{q_i(\alpha)}\Big)\log^+\Big(\frac{N}{q_i(\alpha)}\Big)\,d\alpha + O(\log N).\end{aligned}$$

Since $\log^+(N/q_i(\alpha)) = 0$ for $i > c\log N$, the result follows. ∎

Then, by Theorem 1 we have

$$\int_0^1 s_N(\alpha)\,d\alpha = \frac{1}{2}\sum_{i\leq c\log N} \int_0^1 \Big(1+\frac{q_{i-1}(\alpha)}{q_i(\alpha)}\Big)\log^+\Big(\frac{N}{q_i(\alpha)}\Big)\,d\alpha + O(\log N).$$

This sum has been asymptotically developed in [4] with the effect that it is equal to $(6/\pi^2)\log^2 N + O((\log N)^{3/2}\log\log N)$. ∎

5. Concluding remarks. The methods used here can be generalized to prove that

$$\sum_{i=0}^{\infty}\int_0^1 \frac{b_i^n(N,\alpha)}{a_{i+1}^{n-1}(\alpha)}\,d\alpha = \frac{6}{(n+1)\pi^2}\,\log^2 N + O((\log N)^{3/2}\log\log N), \quad n\in\mathbb{N}.$$

It seems to be hopeless to generalize this method to more general integrals, like $\int_0^1 s_N(\alpha)^L\,d\alpha$. On the other hand it might very well happen that there is a central limit law behind our main theorem.

Acknowledgements. The result of this paper is part of my Ph.D. studies under the supervision of Johannes Schoißengeier at the Universidade de Trás-os-Montes e Alto Douro. I am very indebted to him for his valuable comments. Likewise I thank the referee for his comments and suggestions.

References

[1] M. Drmota, *The distribution of patterns in digital expansions*, in: Algebraic Number Theory and Diophantine Analysis (Graz, 1998), de Gruyter, 2000, 103–121.

[2] M. Drmota and J. Gajdosik, *The distribution of the sum-of-digits function*, J. Théor. Nombres Bordeaux 10 (1998), 17–32.

[3] M. Drmota and J. Schoißengeier, *Digital expansions with respect to different bases*, Monatsh. Math. 138 (2003), 31–59.

[4] J. Schoißengeier, *On the discrepancy of* $(n\alpha)$, Acta Arith. 44 (1984), 241–279.

[5] —, *The integral mean of the discrepancy of the sequence* $(n\alpha)$, Monatsh. Math. 131 (2000), 227–234.

Departamento de Matemática
Universidade de Trás-os-Montes e Alto Douro
Quinta dos Prados, Vila Real
5001-911 Vila Real, Portugal
E-mail: rocadas@utad.pt

Received on 22.4.2002
and in revised form on 6.10.2002 (4270)