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On the upper bound for $\pi_2(x)$

by

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1. Introduction. For sufficiently large x let

$$\pi_2(x) = \sum_{\substack{p \le x \\ p+2=p'}} 1,$$

where p and p' denote primes. It was showed by Brun [3] in 1919 that

$$\pi_2(x) \le b \frac{Cx}{\log^2 x},$$

for some constant $b > 0$, where

$$C = 2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right).$$

In 1949 Selberg [15] proved that we may take $b = 8 + o(1)$. In 1964 Pan Chengdong [13] proved that $b = 6 + o(1)$. In 1962 Wang Yuan [16] pointed out that under the GRH, $b = 4 + o(1)$, and in 1966 Bombieri and Davenport [1] obtained this result unconditionally.

On the other hand, Hardy and Littlewood [10] conjectured that the asymptotic formula

$$\pi_2(x) \sim \frac{Cx}{\log^2 x}$$

should hold, and the arguments in Hua Loo Keng [11] and in Pan Chengdong [14] lead essentially to the same conjecture. Thus the best possible value for b should be $1 + o(1)$.

It is rather difficult to reduce the coefficient $b = 4 + o(1)$. In 1978 Chen Jingrun [4] developed the weighted sieve techniques used in his famous work [5] and proved that

$$\pi_2(x) < \frac{3.9171Cx}{\log^2 x}.$$

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In recent years, with the development of mean value theorems to large moduli, the coefficient 3.9171 is reduced to

$$34/9;\quad 64/17;\quad 3.5;\quad 3.454;\quad 3.418$$

by Fouvry and Iwaniec [8], Fouvry [6], Bombieri, Friedlander and Iwaniec [2], Fouvry and Grupp [7], and Jie Wu [17], respectively. In [17] Chen Jingrun's techniques are applied.

In this paper we shall refine the techniques used in [17] by inserting a new weighted inequality of Chen's type to obtain

THEOREM.

$$\pi_2(x) < \frac{3.406Cx}{\log^2 x}.$$

2. $H(s)$ **and** $G(s)$**.** In this paper we denote by $\lambda(q)$ a well-factorable function of level Q and of order k. For the definition of well-factorable functions we refer the reader to [7].

LEMMA 1 ([7]). *For an arithmetical function λ' of level Q' and of order k', $Q' \le Q$, $\lambda \star \lambda'$ is a well-factorable function of level QQ' and of order $k+k'$.*

Let $\mathcal{A}$ denote a finite set of integers, $\mathcal{P}$ an infinite set of primes, and $\overline{\mathcal{P}}$ the set of primes that do not belong to $\mathcal{P}$. For $z \ge 2$, put

$$P(z) = \prod_{p<z,\, p\in\mathcal{P}} p, \qquad \mathcal{P}(q) = \{p \mid p \in \mathcal{P},\ (p,q)=1\},$$

$$S(\mathcal{A};\mathcal{P},z) = \sum_{a\in\mathcal{A},\,(a,P(z))=1} 1, \qquad \mathcal{A}_d = \{a \mid a\in\mathcal{A},\ a\equiv 0 \pmod d\}.$$

LEMMA 2 ([12]). *Suppose that*

$$|\mathcal{A}_d| = \frac{\omega(d)}{d}X + r(d), \qquad \mu(d)\ne 0, \qquad (d,\overline{\mathcal{P}})=1;$$

$$\frac{V(z_1)}{V(z_2)} \le \frac{\log z_2}{\log z_1}\left(1+\frac{K_1}{\log z_1}\right), \qquad K_1>1,\ z_2>z_1\ge 2;$$

$$\sum_{\substack{z_1\le p<z_2\\ p\in\overline{\mathcal{P}}}} \sum_{\alpha\ge 2} \frac{\omega(p^\alpha)}{p^\alpha} \le \frac{K_2}{\log 3z_1}, \qquad K_2>1,$$

where $\omega(d)$ is a multiplicative function, $0\le\omega(p)<p$, $X>1$ is independent of d, and

$$V(z) = \prod_{p\mid P(z)}\left(1-\frac{\omega(p)}{p}\right).$$

Then for $0 < \varepsilon < 10^{-5}, 2 \le z \le Q^{1/2}$, *we have*

$$S(\mathcal{A},\mathcal{P},z) \ge XV(z)(f(s)-E) - \sum_{l<L}\sum_{q} \lambda_l^{-}(q) r(q),$$

$$S(\mathcal{A},\mathcal{P},z) \le XV(z)(F(s)+E) + \sum_{l<L}\sum_{q} \lambda_l^{+}(q) r(q),$$

where $\lambda_l^{\pm}$ *are well-factorable functions of level* Q *and*

$$L = \exp(8\varepsilon^{-3}), \qquad E \ll \varepsilon + \varepsilon^{-8}\exp(K_1+K_2)\log^{-1/3} Q,$$
$$s = \frac{\log Q}{\log z}, \qquad |\lambda_l^{\pm}(q)| \le 1, \qquad \lambda_l^{\pm}(q) = 0 \quad \text{for } (q, P(z)) = 1.$$

$f(s)$ *and* $F(s)$ *are determined by the following differential-difference equation*:

$$\begin{cases} F(s) = 2e^{\gamma}/s, \quad f(s) = 0, \quad 0 < s \le 2, \\ (sF(s))' = f(s-1), \quad (sf(s))' = F(s-1), \quad s \ge 2; \end{cases}$$

here and below, γ *is Euler's constant.*

LEMMA 3 ([9]). *We have*

$$F(s) = \begin{cases} 2e^{\gamma}/s, & 0 < s \le 3, \\ \dfrac{2e^{\gamma}}{s}\Big(1 + \displaystyle\int_2^{s-1} \frac{\log(t-1)}{t}\,dt\Big), & 3 \le s \le 5; \end{cases}$$

$$f(s) = \frac{2e^{\gamma}\log(s-1)}{s}, \qquad 2 \le s \le 4.$$

LEMMA 4 ([2]). *Let* $Q = x^{4/7-\varepsilon}, \varepsilon > 0$. *For any given* $A > 0$ *and* $|a| \le \log^A x$,

$$\sum_{(q,a)=1} \lambda(q)\Big(\pi(x;q,a) - \frac{\mathrm{Li}\, x}{\varphi(q)}\Big) = O_{A,\varepsilon,a}\Big(\frac{x}{\log^A x}\Big).$$

Throughout this paper we shall take

$$\mathcal{A} = \{p+2 \mid p \le x\}, \qquad \mathcal{P} = \{p \mid p > 2\}.$$

Then

$$V(z) = C\,\frac{e^{-\gamma}}{\log z}\Big(1 + O\Big(\frac{1}{\log z}\Big)\Big).$$

For the definitions of $H(s)$ and $G(s)$ we refer the reader to [17], but for the sake of completeness we repeat them here.

For $\varepsilon > 0$ let

$$Q = x^{4/7-\varepsilon}, \qquad W = x^{\varepsilon}.$$

By $\Pi_{[Y,Z)}$ we denote the characteristic function of the set of the primes in the interval $[Y,Z)$. Let $\mathcal{U}_0$ denote the set which consists of the characteristic function χ of the set $\{1\}$. For $k = 1, 2, \ldots$, we denote by $\mathcal{U}_k$ the set of arithmetical functions χ of the form

$$\chi = \Pi_{[V_1, \Delta V_1)} \star \ldots \star \Pi_{[V_i, \Delta V_i)}$$

with

$$\begin{aligned}&1 \le i \le k, \quad 1 + \log^{-4} x \le \Delta \le 1 + 2\log^{-4} x,\\ &V_1 \ge \ldots \ge V_i \ge W,\\ &V_1^2 \le Q, \quad V_1 V_2^2 \le Q, \ldots, \quad V_1 \ldots V_{i-1} V_i^2 \le Q.\end{aligned}$$

By Lemma 1 of [12] we know that for any $\chi \in \mathcal{U}_k$, χ is a well-factorable function of level Q and of order k.

For any $q \le QW^{-1}$ put $\underline{q} = Q/q$. For any $\chi \in \mathcal{U}_k$ we have

$$\Xi(\chi; x) = \sum \frac{2\chi(q)Cx}{\varphi(q)\log x \log \underline{q}} > \frac{2Cx}{4^k \log^{5k+2} x}. \tag{2.1}$$

By Lemmas 1, 2 and some routine arguments we get

$$\sum \chi(q) S(\mathcal{A}_q, \mathcal{P}(q), \underline{q}^{1/s}) \le \Xi(\chi; x)(A(s) + o(1)), \tag{2.2}$$

$$\sum \chi(q) S(\mathcal{A}_q, \mathcal{P}(q), \underline{q}^{1/s}) \ge \Xi(\chi; x)(B(s) + o(1)), \tag{2.3}$$

where

$$A(s) = \frac{sF(s)}{2e^\gamma}, \quad B(s) = \frac{sf(s)}{2e^\gamma}.$$

In view of (2.2) and (2.3) it is reasonable to define $H_{x,\chi}(s)$ and $G_{x,\chi}(s)$ as the supremum of $h > -\infty$ such that

$$\begin{aligned}\sum \chi(q) S(\mathcal{A}_q, \mathcal{P}(q), \underline{q}^{1/s}) &\le \Xi(\chi; x)(A(s) - h),\\ \sum \chi(q) S(\mathcal{A}_q, \mathcal{P}(q), \underline{q}^{1/s}) &\ge \Xi(\chi; x)(B(s) + h),\end{aligned}$$

respectively. Let

$$H(s) = \sup_{x > x_0} \sup_{k \ge 0} \sup_{\chi \in \mathcal{U}_k} H_{x,\chi}(s), \quad G(s) = \sup_{x > x_0} \sup_{k \ge 0} \sup_{\chi \in \mathcal{U}_k} G_{x,\chi}(s).$$

Then we have

PROPOSITION 1 ([17]). (1) *$H(s)$ is decreasing for $1 \le s \le 3$.*

(2) *We have the inequalities*

$$\begin{aligned}\sum \chi(q) S(\mathcal{A}_q, \mathcal{P}(q), \underline{q}^{1/s}) &\le \Xi(\chi; x)(A(s) - H(s)),\\ \sum \chi(q) S(\mathcal{A}_q, \mathcal{P}(q), \underline{q}^{1/s}) &\ge \Xi(\chi; x)(B(s) + G(s)).\end{aligned}$$

(3) $0 \le H(s) \le 1, 0 \le G(s) \le 1$.

In what follows, we shall denote by χ a well-factorable function of level Q and of order k such that $\chi \in \mathcal{U}_k$ and

$$\Xi(\chi; x) = \sum \frac{2\chi(q)Cx}{\varphi(q)\log x \log \underline{q}} > \frac{2Cx}{4^k \log^{5k+2} x}.$$

PROPOSITION 2 ([17]). *For $2 \le s \le s' \le 10$ we have*

$$H(s) \ge H(s') + \int_{s-1}^{s'-1} \frac{G(t)}{t}\,dt - \varepsilon, \qquad G(s) \ge G(s') + \int_{s-1}^{s'-1} \frac{H(t)}{t}\,dt - \varepsilon.$$

PROPOSITION 3. *We have*

$$\begin{aligned} G(4) \ge\ & 0.02238H(2.2) + 0.00443H(2.3) + 0.00479H(2.4) \\ & + 0.00515H(2.5) + 0.00550H(2.6) + 0.00583H(2.7) \\ & + 0.00615H(2.8) + 0.00645H(2.9) + 0.00676H(3) - \varepsilon. \end{aligned}$$

Proof. By Propositions 1, 2 we have

$$H(s) \ge H(5) + \int_{s-1}^{4} \frac{G(t)}{t}\,dt - \varepsilon \ge \int_{s-1}^{4} \frac{G(t)}{t}\,dt - \varepsilon, \tag{2.4}$$

$$G(4) \ge G(6) + \int_{3}^{5} \frac{H(t)}{t}\,dt - \varepsilon \ge \int_{3}^{5} \frac{H(t)}{t}\,dt - \varepsilon,$$

$$G(s) \ge G(4) + \int_{s-1}^{3} \frac{H(t)}{t}\,dt - \varepsilon, \tag{2.5}$$

$$\begin{aligned} G(4) &\ge \int_{3}^{5} \frac{H(t)}{t}\,dt - \varepsilon \ge \int_{3}^{5} \frac{dt}{t} \int_{t-1}^{4} \frac{G(s)}{s}\,ds - \varepsilon \\ &\ge \int_{3}^{5} \frac{dt}{t} \int_{t-1}^{4} \frac{1}{s}\Big(G(4) + \int_{s-1}^{3} \frac{H(u)}{u}\,du\Big)\,ds - \varepsilon \\ &= G(4)\int_{3}^{5} \frac{\log \frac{4}{t-1}}{t}\,dt + \int_{1}^{3} \frac{H(u)}{u}\,du \int_{3}^{u+2} \frac{\log \frac{u+1}{t-1}}{t}\,dt - \varepsilon \\ &\ge 0.17168G(4) + 0.01854H(2.2) + 0.00367H(2.3) \\ &\quad + 0.00397H(2.4) + 0.00427H(2.5) + 0.00456H(2.6) \\ &\quad + 0.00483H(2.7) + 0.00510H(2.8) + 0.00535H(2.9) \\ &\quad + 0.00560H(3) - \varepsilon, \end{aligned}$$

and Proposition 3 follows easily.

LEMMA 5 ([17]). *Let* $x > 1$, $z = x^{1/u}$, $Q(z) = \prod_{p<z} p$. *Then*

$$\sum_{\substack{n\le x\\(n,Q(z))=1}} 1 = w(u)\,\frac{x}{\log z} + O\left(\frac{x}{\log^2 z}\right),$$

where $w(u)$ *is determined by the following differential-difference equation*:

$$\begin{cases} w(u) = 0, \quad 0 < u < 1,\\ w(u) = 1/u, \quad 1 \le u \le 2,\\ (uw(u))' = w(u-1), \quad u \ge 2.\end{cases}$$

Moreover,

$$w(u) < \frac{1}{1.763} \qquad \text{for } u \ge 1.763.$$

LEMMA 6 ([17]). *For any* $\chi \in \mathcal{U}_k$, $k \ge 0$ *and* $x^{1/2} \le q\chi(q)M$, $2q\chi(q)M \le x$, $M < M' \le 2M$, *we have*

$$\sum \chi(q) \sum_{M\le m\le M'} a(m) S(\mathcal{A}_{qm}, \mathcal{P}(qm), z(m))$$
$$\le (1+o(1))\,\frac{3.5C}{\log x} \sum \chi(q) \sum_{M\le m\le M'} a(m) \sum_{\substack{n\le x/(qm)\\(n,Q(z(m)))=1}} 1 + O\left(\frac{x}{\log^{5k+100} x}\right),$$

where $0 \le a(m) \ll 1$ *for* $M \le m \le M'$.

LEMMA 7 ([17]). *For* $2.5 \le u < 3 < v < 4$ *we have*

$$\begin{aligned} H(u) \ge H(v) &- \int_2^{v-1} \frac{\log(t-1)}{t}\,dt\\ &+ \int_{u-1}^{v-1} \frac{\log\left(v-1-\frac{v}{t+1}\right)}{2t}\,dt + \int_{u-1}^{v-1} \frac{G\left(v-\frac{v}{t+1}\right)}{2t}\,dt\\ &- \frac{7}{8\cdot 1.763}\left((u+v)\log\frac{v}{u} - 2(v-u)\right) - \varepsilon. \end{aligned}$$

LEMMA 8. *We have*

$$\begin{aligned} H(2.5) \ge\ & 0.013948 + 0.15751H(2.2) + 0.02478H(2.3)\\ &+ 0.02544H(2.4) + 0.02653H(2.6) + 0.02699H(2.7)\\ &+ 0.02739H(2.8) + 0.02774H(2.9) + 0.02806H(3);\\ H(2.6) \ge\ & 0.01091 + 0.15796H(2.2) + 0.02420H(2.3)\\ &+ 0.02487H(2.4) + 0.02547H(2.5) + 0.02647H(2.7)\\ &+ 0.02688H(2.8) + 0.02724H(2.9) + 0.02757H(3); \end{aligned}$$

$$\begin{aligned}H(2.7) \geq 0.00733 &+ 0.16626H(2.2) + 0.02388H(2.3)\\ &+ 0.02455H(2.4) + 0.02516H(2.5) + 0.02570H(2.6)\\ &+ 0.02659H(2.8) + 0.02692H(2.9) + 0.02729H(3);\\ H(2.8) \geq 0.00383 &+ 0.1675H(2.2) + 0.02389H(2.3)\\ &+ 0.02457H(2.4) + 0.02518H(2.5) + 0.02572H(2.6)\\ &+ 0.02620H(2.7) + 0.02699H(2.9) + 0.02732H(3);\\ H(2.9) \geq 0.0011 &+ 0.17847H(2.2) + 0.02432H(2.3)\\ &+ 0.02498H(2.4) + 0.02559H(2.5) + 0.02612H(2.6)\\ &+ 0.02658H(2.7) + 0.02699H(2.8) + 0.02768H(3).\end{aligned}$$

Proof. By (2.4), (2.5) we have

$$(2.6)\qquad \begin{aligned}H(v) &\geq \int_{v-1}^{4} \frac{G(t)}{t}\,dt - \varepsilon \geq \int_{v-1}^{4} \left(G(4) + \int_{s-1}^{3} \frac{H(s)}{s}\,ds\right)\frac{dt}{t} - \varepsilon\\ &= G(4)\log\frac{4}{v-1} + \int_{v-2}^{3} \frac{H(s)}{s}\log\frac{s+1}{v-1}\,ds - \varepsilon,\end{aligned}$$

$$(2.7)\qquad \begin{aligned}\int_{u-1}^{v-1} \frac{G\big(v - \frac{v}{t+1}\big)}{t}\,dt &= v\int_{v-v/u}^{v-1} \frac{G(s)}{s(v-s)}\,ds\\ &\geq \int_{v-v/u}^{v-1} \left(G(4) + \int_{s-1}^{3} \frac{H(t)}{t}\,dt\right)\frac{v}{s(v-s)}\,ds - \varepsilon\\ &= G(4)\log\frac{v-1}{u-1} + \int_{v-2}^{3} \frac{H(t)}{t}\log\frac{v-1}{u-1}\,dt\\ &\quad + \int_{v-v/u-1}^{v-2} \frac{H(t)}{t}\log\frac{t+1}{(u-1)(v-1-t)}\,dt - \varepsilon.\end{aligned}$$

By (2.6), (2.7) and Lemma 7 we get

$$(2.8)\qquad \begin{aligned}H(u) &\geq J(u,v) + \frac{G(4)}{2}\log\frac{16}{(v-1)(u-1)}\\ &\quad + \int_{v-v/u-1}^{v-2} \frac{H(t)}{2t}\log\frac{t+1}{(u-1)(v-1-t)}\,dt\\ &\quad + \int_{v-2}^{3} \frac{H(t)}{2t}\log\frac{(t+1)^2}{(v-1)(u-1)}\,dt,\end{aligned}$$

where

$$J(u,v)=\int_{u-1}^{v-1}\frac{\log\left(v-1-\frac{v}{t+1}\right)}{2t}\,dt-\int_{2}^{v-1}\frac{\log(t-1)}{t}\,dt$$
$$-\frac{7}{8\cdot 1.763}\left((u+v)\log\frac{v}{u}-2(v-u)\right)-\varepsilon.$$

By (2.8) with

$$(u,v)=(2.5,3.72),\qquad (u,v)=(2.6,3.61),$$
$$(u,v)=(2.7,3.49),\qquad (u,v)=(2.8,3.35),$$
$$(u,v)=(2.9,3.19)$$

and some computation we get the assertion of Lemma 8.

LEMMA 9. *We have*

$$H(3)\geq 0.19649H(2.2)+0.02531H(2.3)+0.02598H(2.4)$$
$$+0.02654H(2.5)+0.02705H(2.6)+0.02749H(2.7)$$
$$+0.02788H(2.8)+0.02822H(2.9)-\varepsilon.$$

Proof. By (2.6) and Proposition 3 we have

$$H(3)\geq G(4)\log 2+\int_{1}^{3}\frac{H(s)}{s}\log\frac{s+1}{2}\,ds-\varepsilon \tag{2.9}$$
$$\geq 0.19104H(2.2)+0.02461H(2.3)+0.02526H(2.4)$$
$$+0.02581H(2.5)+0.02630H(2.6)+0.02673H(2.7)$$
$$+0.02711H(2.8)+0.02744H(2.9)+0.02774H(3)-\varepsilon.$$

By (2.9) we get Lemma 9 easily.

LEMMA 10 ([17]). *For* $2<u\leq 2.5<3<v<5$ *we have*

$$H(u)\geq H(v)+K(u,v)+\int_{u-1}^{v-1}\frac{G\left(v-1-\frac{v}{t+1}\right)}{t}\,dt$$
$$+\int_{2}^{v-1}\frac{G\left(v-1-\frac{v}{t+1}\right)}{t}\,dt$$
$$+H(2.2)\int_{2}^{v-1}\frac{\log\left(v-1-\frac{v}{t+1}\right)-\log(t-1)}{3t}\,dt-\varepsilon,$$

where

$$K(u,v)=\int_{u-1}^{v-1}\frac{\log\left(v-1-\frac{v}{t+1}\right)}{3t}\,dt-2\int_{2}^{v-1}\frac{\log(t-1)}{3t}\,dt$$
$$-\frac{7}{12\cdot 1.763}\left(v-3-3\log\frac{v}{3}\right)\log\frac{3}{u}$$

$$- \frac{7}{12 \cdot 1.763}\left(u - 3 + 3\log\frac{3}{u}\right)\log\frac{v}{3}$$
$$- \frac{7}{24}\left(u\log^2\frac{3}{u} + (6+4u)\log\frac{3}{u} + 6u - 18\right).$$

REMARK. It should be pointed out that there is a mistake in Lemme 11 of [17] where the term

$$\int_2^{v-1} \frac{G\left(v-1-\frac{v}{t+1}\right)}{t}\,dt$$

is missing.

LEMMA 11. *We have*

$$\begin{aligned}
H(2.3) \ge\ & 0.01770 + 0.14833H(2.2) + 0.02567H(2.4)\\
&+ 0.02624H(2.5) + 0.02675H(2.6) + 0.02720H(2.7)\\
&+ 0.02759H(2.8) + 0.02793H(2.9) + 0.02825H(3);\\
H(2.4) \ge\ & 0.01559 + 0.14941H(2.2) + 0.02444H(2.3)\\
&+ 0.02570H(2.5) + 0.02622H(2.6) + 0.02668H(2.7)\\
&+ 0.02709H(2.8) + 0.02744H(2.9) + 0.02777H(3).
\end{aligned}$$

Proof. By (2.6), (2.7) and Lemma 10 we get

$$\begin{aligned}
H(u) \ge\ & K(u,v) + \frac{G(4)}{3}\log\frac{32}{(v-1)(u-1)}\\
&+ \int_{v-v/u-1}^{2v/3-1} \frac{H(t)}{3t}\log\frac{t+1}{(u-1)(v-1-t)}\,dt\\
&+ \int_{2v/3-1}^{v-2} \frac{H(t)}{3t}\log\frac{(t+1)^2}{2(u-1)(v-1-t)^2}\,dt\\
&+ \int_{v-2}^{3} \frac{H(t)}{3t}\log\frac{(t+1)^3}{2(v-1)(u-1)}\,dt\\
&+ H(2.2)\int_2^{v-1} \frac{\log\left(v-1-\frac{v}{t+1}\right) - \log(t-1)}{3t}\,dt - \varepsilon.
\end{aligned} \tag{2.10}$$

By (2.10) with $(u,v) = (2.3, 4.12), (u,v) = (2.4, 4)$ and some computation we get the assertion.

3. Proof of the Theorem. By Buchstab's identity, we have

$$S(\mathcal{A}_q, \mathcal{P}(q), \underline{q}^{1/2.2}) = S(\mathcal{A}_q, \mathcal{P}(q), \underline{q}^{1/4.4}) - \sum_{\underline{q}^{1/4.4} \le p < \underline{q}^{1/2.2}} S(\mathcal{A}_{pq}, \mathcal{P}(q), p), \tag{3.1}$$

$$(3.2)\quad S(\mathcal{A}_q,\mathcal{P}(q),\underline{q}^{1/2.2})$$
$$= S(\mathcal{A}_q,\mathcal{P}(q),\underline{q}^{1/4.4}) - \sum_{\underline{q}^{1/4.4}\le p<\underline{q}^{1/3}} S(\mathcal{A}_{pq},\mathcal{P}(q),\underline{q}^{1/4.4})$$
$$+ \sum_{\underline{q}^{1/4.4}\le p_1<p_2<\underline{q}^{1/3}} S(\mathcal{A}_{p_1p_2q},\mathcal{P}(q),p_1) - \sum_{\underline{q}^{1/3}\le p<\underline{q}^{1/2.2}} S(\mathcal{A}_{pq},\mathcal{P}(q),p),$$

$$(3.3)\quad S(\mathcal{A}_q,\mathcal{P}(q),\underline{q}^{1/2.2})$$
$$= S(\mathcal{A}_q,\mathcal{P}(q),\underline{q}^{1/3.6}) - \sum_{\underline{q}^{1/3.6}\le p<\underline{q}^{1/2.2}} S(\mathcal{A}_{pq},\mathcal{P}(q),p).$$

By (3.1)–(3.3) we have

$$(3.4)\quad 3S(\mathcal{A}_q,\mathcal{P}(q),\underline{q}^{1/2.2})$$
$$= 2S(\mathcal{A}_q,\mathcal{P}(q),\underline{q}^{1/4.4}) + S(\mathcal{A}_q,\mathcal{P}(q),\underline{q}^{1/3.6})$$
$$- \sum_{\underline{q}^{1/4.4}\le p<\underline{q}^{1/3}} S(\mathcal{A}_{pq},\mathcal{P}(q),\underline{q}^{1/4.4}) - \sum_{\underline{q}^{1/4.4}\le p<\underline{q}^{1/2.2}} S(\mathcal{A}_{pq},\mathcal{P}(q),p)$$
$$- \sum_{\underline{q}^{1/3.6}\le p<\underline{q}^{1/2.2}} S(\mathcal{A}_{pq},\mathcal{P}(q),p) + \sum_{\underline{q}^{1/4.4}\le p_1<p_2<\underline{q}^{1/3}} S(\mathcal{A}_{p_1p_2q},\mathcal{P}(q),p_1)$$
$$- \sum_{\underline{q}^{1/3}\le p<\underline{q}^{1/2.2}} S(\mathcal{A}_{pq},\mathcal{P}(q),p).$$

By Buchstab's identity, we have

$$(3.5)\quad S(\mathcal{A}_q,\mathcal{P}(q),\underline{q}^{1/2.2})$$
$$= S(\mathcal{A}_q,\mathcal{P}(q),\underline{q}^{1/4.5}) - \frac{1}{2}\sum_{\underline{q}^{1/4.5}\le p<\underline{q}^{1/4.4}} S(\mathcal{A}_{pq},\mathcal{P}(q),p)$$
$$- \frac{1}{2}\sum_{\underline{q}^{1/4.5}\le p<\underline{q}^{1/4.4}} S(\mathcal{A}_{pq},\mathcal{P}(q),\underline{q}^{1/4.5})$$
$$+ \frac{1}{2}\sum_{\underline{q}^{1/4.5}\le p_1<p_2<\underline{q}^{1/4.4}} S(\mathcal{A}_{p_1p_2q},\mathcal{P}(q),p_1)$$
$$- \sum_{\underline{q}^{1/4.4}\le p<\underline{q}^{1/3}} S(\mathcal{A}_{pq},\mathcal{P}(q),\underline{q}^{1/4.4})$$
$$+ \sum_{\underline{q}^{1/4.4}\le p_1<p_2<\underline{q}^{1/3.6}} S(\mathcal{A}_{p_1p_2q},\mathcal{P}(q),p_1)$$

$$+\sum_{\underline{q}^{1/4.4}\le p_1<\underline{q}^{1/3.6}\le p_2<\underline{q}^{1/3}} S(\mathcal{A}_{p_1p_2q},\mathcal{P}(q),p_1)$$

$$+\sum_{\underline{q}^{1/3.6}\le p_1<p_2<\underline{q}^{1/3}} S(\mathcal{A}_{p_1p_2q},\mathcal{P}(q),p_1)$$

$$-\sum_{\underline{q}^{1/3}\le p<\underline{q}^{1/2.4}} S(\mathcal{A}_{pq},\mathcal{P}(q),\underline{q}^{1/4.5})$$

$$+\sum_{\underline{q}^{1/4.5}\le p_1<\underline{q}^{1/4.4}<\underline{q}^{1/3}\le p_2<\underline{q}^{1/2.4}} S(\mathcal{A}_{p_1p_2q},\mathcal{P}(q),p_1)$$

$$+\sum_{\underline{q}^{1/4.4}\le p_1<\underline{q}^{1/3.6}<\underline{q}^{1/3}\le p_2<\underline{q}^{1/2.4}} S(\mathcal{A}_{p_1p_2q},\mathcal{P}(q),\underline{q}^{1/4.5})$$

$$-\sum_{\underline{q}^{1/4.5}\le p_1<\underline{q}^{1/4.4}\le p_2<\underline{q}^{1/3.6}<\underline{q}^{1/3}\le p_3<\underline{q}^{1/2.4}} S(\mathcal{A}_{p_1p_2p_3q},\mathcal{P}(q),p_1)$$

$$-\sum_{\underline{q}^{1/4.4}\le p_1<p_2<\underline{q}^{1/3.6}<\underline{q}^{1/3}\le p_3<\underline{q}^{1/2.4}} S(\mathcal{A}_{p_1p_2p_3q},\mathcal{P}(q),p_1)$$

$$+\sum_{\underline{q}^{1/3.6}\le p_1<\underline{q}^{1/3}\le p_2<\underline{q}^{1/2.4}} S(\mathcal{A}_{p_1p_2q},\mathcal{P}(q),p_1)$$

$$+\sum_{\underline{q}^{1/3}\le p_1<p_2<\underline{q}^{1/2.4}} S(\mathcal{A}_{p_1p_2q},\mathcal{P}(q),p_1)$$

$$-\sum_{\underline{q}^{1/2.4}\le p<\underline{q}^{1/2.2}} S(\mathcal{A}_{pq},\mathcal{P}(q),p),$$

and

$$\begin{aligned}(3.6)\quad & S(\mathcal{A}_q,\mathcal{P}(q),\underline{q}^{1/2.2}) \\ &= S(\mathcal{A}_q,\mathcal{P}(q),\underline{q}^{1/4.5}) - \sum_{\underline{q}^{1/4.5}\le p<\underline{q}^{1/2.2}} S(\mathcal{A}_{pq},\mathcal{P}(q),\underline{q}^{1/4.5}) \\ &\quad + \frac{1}{2}\sum_{\underline{q}^{1/4.5}\le p_1<p_2<\underline{q}^{1/4.4}} S(\mathcal{A}_{p_1p_2q},\mathcal{P}(q),p_1) \\ &\quad + \frac{1}{2}\sum_{\underline{q}^{1/4.5}\le p_1<p_2<\underline{q}^{1/4.4}} S(\mathcal{A}_{p_1p_2q},\mathcal{P}(q),\underline{q}^{1/4.5}) \\ &\quad - \frac{1}{2}\sum_{\underline{q}^{1/4.5}\le p_1<p_2<p_3<\underline{q}^{1/4.4}} S(\mathcal{A}_{p_1p_2p_3q},\mathcal{P}(q),p_1)\end{aligned}$$

$$+\frac{1}{2}\sum_{\underline{q}^{1/4.5}\le p_1<\underline{q}^{1/4.4}\le p_2<\underline{q}^{1/2.2}} S(\mathcal{A}_{p_1p_2q},\mathcal{P}(q),p_1)$$
$$+\frac{1}{2}\sum_{\underline{q}^{1/4.5}\le p_1<\underline{q}^{1/4.4}\le p_2<\underline{q}^{1/2.2}} S(\mathcal{A}_{p_1p_2q},\mathcal{P}(q),\underline{q}^{1/4.5})$$
$$-\frac{1}{2}\sum_{\underline{q}^{1/4.5}\le p_1<p_2<\underline{q}^{1/4.4}\le p_3<\underline{q}^{1/2.2}} S(\mathcal{A}_{p_1p_2p_3q},\mathcal{P}(q),p_1)$$
$$+\sum_{\underline{q}^{1/4.4}\le p_1<p_2<\underline{q}^{1/3}} S(\mathcal{A}_{p_1p_2q},\mathcal{P}(q),\underline{q}^{1/4.5})$$
$$-\sum_{\underline{q}^{1/4.5}\le p_1<\underline{q}^{1/4.4}\le p_2<p_3<\underline{q}^{1/3}} S(\mathcal{A}_{p_1p_2p_3q},\mathcal{P}(q),p_1)$$
$$-\sum_{\underline{q}^{1/4.4}\le p_1<p_2<p_3<\underline{q}^{1/3}} S(\mathcal{A}_{p_1p_2p_3q},\mathcal{P}(q),p_1)$$
$$+\sum_{\underline{q}^{1/4.4}\le p_1<\underline{q}^{1/3}\le p_2<\underline{q}^{1/2.2}} S(\mathcal{A}_{p_1p_2q},\mathcal{P}(q),p_1)$$
$$+\sum_{\underline{q}^{1/3}\le p_1<p_2<\underline{q}^{1/2.2}} S(\mathcal{A}_{p_1p_2q},\mathcal{P}(q),p_1).$$

By (3.4)–(3.6) we get

$$5\sum\chi(q)S(\mathcal{A}_q,\mathcal{P}(q),\underline{q}^{1/2.2})=\sum\chi(q)(S_q^1+S_q^2+S_q^3), \tag{3.7}$$

where

$$S_q^1=2S(\mathcal{A}_q,\mathcal{P}(q),\underline{q}^{1/4.5})+2S(\mathcal{A}_q,\mathcal{P}(q),\underline{q}^{1/4.4}) \tag{3.8}$$
$$+S(\mathcal{A}_q,\mathcal{P}(q),\underline{q}^{1/3.6})-2\sum_{\underline{q}^{1/4.4}\le p<\underline{q}^{1/3}} S(\mathcal{A}_{pq},\mathcal{P}(q),\underline{q}^{1/4.4})$$
$$-\frac{1}{2}\sum_{\underline{q}^{1/4.5}\le p<\underline{q}^{1/4.4}} S(\mathcal{A}_{pq},\mathcal{P}(q),\underline{q}^{1/4.5})$$
$$-\sum_{\underline{q}^{1/3}\le p<\underline{q}^{1/2.4}} S(\mathcal{A}_{pq},\mathcal{P}(q),\underline{q}^{1/4.5})$$
$$-\sum_{\underline{q}^{1/4.5}\le p<\underline{q}^{1/2.2}} S(\mathcal{A}_{pq},\mathcal{P}(q),\underline{q}^{1/4.5})$$
$$+\frac{1}{2}\sum_{\underline{q}^{1/4.5}\le p_1<p_2<\underline{q}^{1/4.4}} S(\mathcal{A}_{p_1p_2q},\mathcal{P}(q),\underline{q}^{1/4.5})$$

$$+\frac{1}{2}\sum_{\underline{q}^{1/4.5}\le p_1<p_2<\underline{q}^{1/4.4}} S(\mathcal{A}_{p_1p_2q},\mathcal{P}(q),p_1)$$

$$+\sum_{\underline{q}^{1/4.4}\le p_1<p_2<\underline{q}^{1/3}} S(\mathcal{A}_{p_1p_2q},\mathcal{P}(q),\underline{q}^{1/4.5})$$

$$+\frac{1}{2}\sum_{\underline{q}^{1/4.5}\le p_1<\underline{q}^{1/4.4}\le p_2<\underline{q}^{1/2.2}} S(\mathcal{A}_{p_1p_2q},\mathcal{P}(q),\underline{q}^{1/4.5})$$

$$+\sum_{\underline{q}^{1/4.4}\le p_1<p_2<\underline{q}^{1/3.6}} S(\mathcal{A}_{p_1p_2q},\mathcal{P}(q),p_1)$$

$$+\sum_{\underline{q}^{1/4.4}\le p_1<\underline{q}^{1/3.6}\le p_2<\underline{q}^{1/3}} S(\mathcal{A}_{p_1p_2q},\mathcal{P}(q),p_1)$$

$$+\sum_{\underline{q}^{1/4.5}\le p_1<\underline{q}^{1/4.4}<\underline{q}^{1/3}\le p_2<\underline{q}^{1/2.4}} S(\mathcal{A}_{p_1p_2q},\mathcal{P}(q),p_1)$$

$$+\sum_{\underline{q}^{1/4.4}\le p_1<\underline{q}^{1/3.6}<\underline{q}^{1/3}\le p_2<\underline{q}^{1/2.4}} S(\mathcal{A}_{p_1p_2q},\mathcal{P}(q),\underline{q}^{1/4.5}),$$

$$S_q^2=\sum_{\underline{q}^{1/4.4}\le p_1<p_2<\underline{q}^{1/2.2}} S(\mathcal{A}_{p_1p_2q},\mathcal{P}(q),p_1) \tag{3.9}$$

$$+\sum_{\underline{q}^{1/3.6}\le p_1<p_2<\underline{q}^{1/2.4}} S(\mathcal{A}_{p_1p_2q},\mathcal{P}(q),p_1)$$

$$-\sum_{\underline{q}^{1/4.4}\le p<\underline{q}^{1/2.2}} S(\mathcal{A}_{pq},\mathcal{P}(q),p)$$

$$-\sum_{\underline{q}^{1/3.6}\le p<\underline{q}^{1/2.2}} S(\mathcal{A}_{pq},\mathcal{P}(q),p)$$

$$-\sum_{\underline{q}^{1/4.4}\le p_1<p_2<p_3<\underline{q}^{1/3}} S(\mathcal{A}_{p_1p_2p_3q},\mathcal{P}(q),p_1)$$

$$-\sum_{\underline{q}^{1/4.4}\le p_1<p_2<\underline{q}^{1/3.6}<\underline{q}^{1/3}\le p_3<\underline{q}^{1/2.4}} S(\mathcal{A}_{p_1p_2p_3q},\mathcal{P}(q),p_1)$$

$$-\sum_{\underline{q}^{1/3}\le p<\underline{q}^{1/2.2}} S(\mathcal{A}_{pq},\mathcal{P}(q),p)$$

$$-\sum_{\underline{q}^{1/2.4}\le p<\underline{q}^{1/2.2}} S(\mathcal{A}_{pq},\mathcal{P}(q),p)$$

$$=U_1(q)+U_2(q)-V_1(q)-V_2(q)-V_3(q)$$

$$-V_4(q)-V_5(q)-V_6(q),$$

$$
\begin{aligned}
(3.10)\qquad S_q^3 = {}& \frac{1}{2} \sum_{\underline{q}^{1/4.5} \le p_1 < p_2 < \underline{q}^{1/4.4}} S(\mathcal{A}_{p_1p_2q}, \mathcal{P}(q), p_1) \\
&+ \frac{1}{2} \sum_{\underline{q}^{1/4.5} \le p_1 < \underline{q}^{1/4.4} \le p_2 < \underline{q}^{1/2.2}} S(\mathcal{A}_{p_1p_2q}, \mathcal{P}(q), p_1) \\
&- \frac{1}{2} \sum_{\underline{q}^{1/4.5} \le p < \underline{q}^{1/4.4}} S(\mathcal{A}_{pq}, \mathcal{P}(q), p) \\
&- \frac{1}{2} \sum_{\underline{q}^{1/4.5} \le p_1 < p_2 < \underline{q}^{1/4.4} \le p_3 < \underline{q}^{1/2.2}} S(\mathcal{A}_{p_1p_2p_3q}, \mathcal{P}(q), p_1) \\
&- \frac{1}{2} \sum_{\underline{q}^{1/4.5} \le p_1 < p_2 < p_3 < \underline{q}^{1/4.4}} S(\mathcal{A}_{p_1p_2p_3q}, \mathcal{P}(q), p_1) \\
&- \sum_{\underline{q}^{1/4.5} \le p_1 < \underline{q}^{1/4.4} \le p_2 < \underline{q}^{1/3.6} < \underline{q}^{1/3} \le p_3 < \underline{q}^{1/2.4}} S(\mathcal{A}_{p_1p_2p_3q}, \mathcal{P}(q), p_1) \\
&- \sum_{\underline{q}^{1/4.5} \le p_1 < \underline{q}^{1/4.4} \le p_2 < p_3 < \underline{q}^{1/3}} S(\mathcal{A}_{p_1p_2p_3q}, \mathcal{P}(q), p_1).
\end{aligned}
$$

We shall denote $O\left(\frac{x}{q\log^{5k+100} x}\right)$ by $E(q)$ below. By Buchstab's identity we have

$$
\begin{aligned}
V_1(q) &= \sum_{\underline{q}^{1/4.4} \le p < \underline{q}^{1/2.2}} S(\mathcal{A}_{pq}, \mathcal{P}(q), \underline{q}^{1/2.2}) \\
&\quad + \sum_{\underline{q}^{1/4.4} \le p_1 \le p_2 < \underline{q}^{1/2.2}} S(\mathcal{A}_{p_1p_2q}, \mathcal{P}(p_1q), p_2) \\
&\ge \sum_{\underline{q}^{1/4.4} \le p_1 < p_2 < \underline{q}^{1/2.2}} S(\mathcal{A}_{p_1p_2q}, \mathcal{P}(p_1q), p_2),
\end{aligned}
$$

so that

$$
\begin{aligned}
(3.11)\quad U_1(q) - V_1(q) &\le \sum_{\underline{q}^{1/4.4} \le p_1 \le p_2 < p_3 < \underline{q}^{1/2.2}} S(\mathcal{A}_{p_1p_2p_3q}, \mathcal{P}(p_1q), p_2) \\
&\le \sum_{\underline{q}^{1/4.4} \le p_1 < p_2 < p_3 < \underline{q}^{1/2.2}} S(\mathcal{A}_{p_1p_2p_3q}, \mathcal{P}(p_1q), p_2) + E(q) \\
&= U_3(q) + E(q).
\end{aligned}
$$

In a similar way, we have

$$
(3.12)\qquad U_2(q) - V_2(q) \le \sum_{\underline{q}^{1/3.6} \le p_1 < p_2 < p_3 < \underline{q}^{1/2.4}} S(\mathcal{A}_{p_1p_2p_3q}, \mathcal{P}(p_1q), p_2)
$$

$$- \sum_{\underline{q}^{1/3.6} \le p_1 < \underline{q}^{1/2.4} \le p_2 < \underline{q}^{1/2.2}} S(\mathcal{A}_{p_1 p_2 q}, \mathcal{P}(p_1 q), p_2)$$

$$- \sum_{\underline{q}^{1/2.4} \le p < \underline{q}^{1/2.2}} S(\mathcal{A}_{pq}, \mathcal{P}(q), p) + E(q)$$

$$= W_1(q) - V_7(q) - V_6(q) + E(q),$$

$$U_3(q) - V_3(q) \tag{3.13}$$

$$\le \sum_{\underline{q}^{1/4.4} \le p_1 < p_2 < \underline{q}^{1/3} \le p_3 < \underline{q}^{1/2.2}} S(\mathcal{A}_{p_1 p_2 p_3 q}, \mathcal{P}(p_1 q), p_2)$$

$$+ \sum_{\underline{q}^{1/4.4} \le p_1 < \underline{q}^{1/3} \le p_2 < p_3 < \underline{q}^{1/2.2}} S(\mathcal{A}_{p_1 p_2 p_3 q}, \mathcal{P}(p_1 q), p_2)$$

$$+ \sum_{\underline{q}^{1/3} \le p_1 < p_2 < p_3 < \underline{q}^{1/2.2}} S(\mathcal{A}_{p_1 p_2 p_3 q}, \mathcal{P}(p_1 q), p_2) + E(q)$$

$$= U_4(q) + U_5(q) + U_6(q) + E(q),$$

$$U_4(q) - V_4(q) \tag{3.14}$$

$$\le \sum_{\underline{q}^{1/4.4} \le p_1 < p_2 < \underline{q}^{1/3.6} < \underline{q}^{1/2.4} \le p_3 < \underline{q}^{1/2.2}} S(\mathcal{A}_{p_1 p_2 p_3 q}, \mathcal{P}(p_1 q), p_2)$$

$$+ \sum_{\underline{q}^{1/4.4} \le p_1 < \underline{q}^{1/3.6} \le p_2 < \underline{q}^{1/3} \le p_3 < \underline{q}^{1/2.2}} S(\mathcal{A}_{p_1 p_2 p_3 q}, \mathcal{P}(p_1 q), p_2)$$

$$+ \sum_{\underline{q}^{1/3.6} \le p_1 < p_2 < \underline{q}^{1/3} \le p_3 < \underline{q}^{1/2.2}} S(\mathcal{A}_{p_1 p_2 p_3 q}, \mathcal{P}(p_1 q), p_2) + E(q)$$

$$= W_2(q) + W_3(q) + W_4(q) + E(q),$$

$$U_5(q) - V_7(q) \tag{3.15}$$

$$\le \sum_{\underline{q}^{1/4.4} \le p_1 < \underline{q}^{1/3.6} < \underline{q}^{1/3} \le p_2 < p_3 < \underline{q}^{1/2.2}} S(\mathcal{A}_{p_1 p_2 p_3 q}, \mathcal{P}(p_1 q), p_2)$$

$$+ \sum_{\underline{q}^{1/3.6} \le p_1 < \underline{q}^{1/3} \le p_2 < p_3 < \underline{q}^{1/2.4}} S(\mathcal{A}_{p_1 p_2 p_3 q}, \mathcal{P}(p_1 q), p_2)$$

$$+ \sum_{\underline{q}^{1/3.6} \le p_1 < \underline{q}^{1/3} \le p_2 < \underline{q}^{1/2.4} \le p_3 < \underline{q}^{1/2.2}} S(\mathcal{A}_{p_1 p_2 p_3 q}, \mathcal{P}(p_1 q), p_2)$$

$$+ \sum_{\underline{q}^{1/3.6} \le p_1 < \underline{q}^{1/3} < \underline{q}^{1/2.4} \le p_2 < p_3 < p_4 < \underline{q}^{1/2.2}} S(\mathcal{A}_{p_1 p_2 p_3 q}, \mathcal{P}(p_1 p_2 q), p_3)$$

$$
\begin{aligned}
&- \sum_{\underline{q}^{1/3}\le p_1<\underline{q}^{1/2.4}\le p_2<\underline{q}^{1/2.2}} S(\mathcal{A}_{p_1p_2q}, \mathcal{P}(p_1q), p_2) + E(q) \\
&= W_5(q) + W_6(q) + W_7(q) + W_8(q) - V_8(q) + E(q),
\end{aligned}
$$

(3.16)
$$
\begin{aligned}
&U_6(q) - V_5(q) \\
&\le \sum_{\underline{q}^{1/3}\le p_1<p_2<p_3<p_4<\underline{q}^{1/2.2}} S(\mathcal{A}_{p_1p_2p_3p_4q}, \mathcal{P}(p_1p_2q), p_3) + E(q) \\
&= U_7(q) + E(q),
\end{aligned}
$$

(3.17)
$$
\begin{aligned}
&U_7(q) - 2V_6(q) - V_8(q) \\
&\le \sum_{\underline{q}^{1/3}\le p_1<p_2<p_3<p_4<\underline{q}^{1/2.4}} S(\mathcal{A}_{p_1p_2p_3p_4q}, \mathcal{P}(p_1p_2q), p_3) \\
&\quad + \sum_{\underline{q}^{1/3}\le p_1<p_2<p_3<\underline{q}^{1/2.4}\le p_4<\underline{q}^{1/2.2}} S(\mathcal{A}_{p_1p_2p_3p_4q}, \mathcal{P}(p_1p_2q), p_3) \\
&\quad + \sum_{\underline{q}^{1/3}\le p_1<p_2<\underline{q}^{1/2.4}\le p_3<p_4<\underline{q}^{1/2.2}} S(\mathcal{A}_{p_1p_2p_3p_4q}, \mathcal{P}(p_1p_2q), p_3) \\
&\quad + \sum_{\underline{q}^{1/3}\le p_1<\underline{q}^{1/2.4}\le p_2<p_3<p_4<p_5<\underline{q}^{1/2.2}} S(\mathcal{A}_{p_1p_2p_3p_4q}, \mathcal{P}(p_1p_2p_3q), p_4) \\
&\quad + \sum_{\Delta} S(\mathcal{A}_{p_1p_2p_3p_4p_5p_6q}, \mathcal{P}(p_1p_2p_3p_4q), p_5) + E(q) \\
&= \sum_{j=9}^{13} W_j(q) + E(q),
\end{aligned}
$$

where $\sum_{\Delta}$ means

$$
\sum_{\underline{q}^{1/2.4}\le p_1<p_2<p_3<p_4<p_5<p_6<\underline{q}^{1/2.2}} .
$$

By (3.9), (3.11)–(3.17), we get

$$
S_q^2 \le \sum_{j=1}^{13} W_j(q) + E(q). \tag{3.18}
$$

Similarly, we have

(3.19)
$$
\begin{aligned}
S_q^3 \le{}& \frac{1}{2} \sum_{\underline{q}^{1/4.5}\le p_1<\underline{q}^{1/4.4}<\underline{q}^{1/3}\le p_2<p_3<\underline{q}^{1/2.2}} S(\mathcal{A}_{p_1p_2p_3q}, \mathcal{P}(p_1q), p_2) \\
&+ \frac{1}{2} \sum_{\underline{q}^{1/4.5}\le p_1<\underline{q}^{1/4.4}\le p_2<\underline{q}^{1/3.6}<\underline{q}^{1/2.4}\le p_3<\underline{q}^{1/2.2}} S(\mathcal{A}_{p_1p_2p_3q}, \mathcal{P}(p_1q), p_2)
\end{aligned}
$$

$$+\frac{1}{2}\sum_{\underline{q}^{1/4.5}\le p_1<\underline{q}^{1/4.4}<\underline{q}^{1/3.6}\le p_2<\underline{q}^{1/3}\le p_3<\underline{q}^{1/2.2}} S(\mathcal{A}_{p_1p_2p_3q},\mathcal{P}(p_1q),p_2)+E(q)$$

$$=\frac{1}{2}(W_{14}(q)+W_{15}(q)+W_{16}(q))+E(q).$$

By Lemma 6, Lemma 5 and some routine arguments we get

$$\sum\chi(q)W_1(q) \tag{3.20}$$
$$\prec\frac{3.5C}{\log x}\sum\chi(q)\sum_{\underline{q}^{1/3.6}\le p_1<p_2<p_3<\underline{q}^{1/2.4}}\ \sum_{\substack{n\le\frac{x}{p_1p_2p_3q}\\(n,Q(p_2))=1}}1$$
$$\prec\frac{3.5C}{1.763\log x}\sum\chi(q)\sum_{\underline{q}^{1/3.6}\le p_1<p_2<p_3<\underline{q}^{1/2.4}}\frac{x}{p_1p_2p_3q\log p_2}$$
$$\prec\frac{3.5Cx}{1.763\log x}\sum\frac{\chi(q)}{\varphi(q)}\int_{\underline{q}^{1/3.6}}^{\underline{q}^{1/2.4}}\frac{dr}{r\log r}\int_{r}^{\underline{q}^{1/2.4}}\frac{ds}{s\log^2 s}\int_{s}^{\underline{q}^{1/2.4}}\frac{dt}{t\log t}$$
$$\prec\frac{7}{4\cdot 1.763}\left(6\log\frac{3.6}{2.4}-2.4\right)\Xi(\chi;x),$$

where $A\prec B$ means $A\le(1+o(1))B$. Similarly, we have

$$\sum\chi(q)W_2(q)\prec\frac{7}{4\cdot 1.763}\left(0.8-3.6\log\frac{4.4}{3.6}\right)\log\frac{2.4}{2.2}\,\Xi(\chi;x), \tag{3.21}$$

$$\sum\chi(q)W_3(q)\prec\frac{7}{4\cdot 1.763}\cdot 0.6\log\frac{4.4}{3.6}\log\frac{3}{2.2}\,\Xi(\chi;x), \tag{3.22}$$

$$\sum\chi(q)W_4(q)\prec\frac{7}{4\cdot 1.763}\left(0.6-3\log\frac{3.6}{3}\right)\log\frac{3}{2.2}\,\Xi(\chi;x), \tag{3.23}$$

$$\sum\chi(q)W_5(q)\prec\frac{7}{4\cdot 1.763}\left(3\log\frac{3}{2.2}-0.8\right)\log\frac{4.4}{3.6}\,\Xi(\chi;x), \tag{3.24}$$

$$\sum\chi(q)W_6(q)\prec\frac{7}{4\cdot 1.763}\left(3\log\frac{3}{2.4}-0.6\right)\log\frac{3.6}{3}\,\Xi(\chi;x), \tag{3.25}$$

$$\sum\chi(q)W_7(q)\prec\frac{7}{4\cdot 1.763}\cdot 0.6\log\frac{3.6}{3}\log\frac{2.4}{2.2}\,\Xi(\chi;x), \tag{3.26}$$

$$\sum\chi(q)W_8(q)\prec\frac{7}{8}\left(4.6\log\frac{2.4}{2.2}-0.4\right)\log\frac{3.6}{3}\,\Xi(\chi;x), \tag{3.27}$$

$$\sum\chi(q)W_9(q)\prec\frac{7}{8}\left(2.4\log^2\frac{3}{2.4}+15.6\log\frac{3}{2.4}-3.6\right)\Xi(\chi;x), \tag{3.28}$$

$$\sum\chi(q)W_{10}(q)\prec\frac{7}{8}\left(1.2-4.8\log\frac{3}{2.4}-2.4\log^2\frac{3}{2.4}\right)\log\frac{2.4}{2.2}\,\Xi(\chi;x), \tag{3.29}$$

$$\sum \chi(q) W_{11}(q) \prec \frac{7}{8}\left(2.4 \log \frac{2.4}{2.2} - 0.2\right) \log^2 \frac{3}{2.4}\, \Xi(\chi; x), \tag{3.30}$$

$$\sum \chi(q) W_{12}(q) \prec \frac{7}{8}\left(2.2 \log^2 \frac{2.4}{2.2} + 13.6 \log \frac{2.4}{2.2} - 1.2\right) \log \frac{3}{2.4}\, \Xi(\chi; x), \tag{3.31}$$

$$\sum \chi(q) W_{13}(q) \prec \frac{7}{4 \cdot 6!} \log^5 \frac{2.4}{2.2}\, \Xi(\chi; x), \tag{3.32}$$

$$\sum \chi(q) W_{14}(q) \prec \frac{7}{4 \cdot 1.763}\left(3 \log \frac{3}{2.2} - 0.8\right) \log \frac{4.5}{4.4}\, \Xi(\chi; x), \tag{3.33}$$

$$\sum \chi(q) W_{15}(q) \prec \frac{7}{4 \cdot 1.763} \cdot 0.8 \log \frac{4.5}{4.4} \log \frac{2.4}{2.2}\, \Xi(\chi; x), \tag{3.34}$$

$$\sum \chi(q) W_{16}(q) \prec \frac{7}{4 \cdot 1.763} \cdot 0.6 \log \frac{4.5}{4.4} \log \frac{3}{2.2}\, \Xi(\chi; x), \tag{3.35}$$

where we have used the prime number theorem and the formulae

$$\int_{y^{1/\alpha}}^{y^{1/\beta}} \frac{dr}{r \log^2 r} \int_{r}^{y^{1/\beta}} \frac{ds}{s \log s} = \frac{\alpha \log \frac{\alpha}{\beta} + \beta - \alpha}{\log y},$$

$$\int_{y^{1/\alpha}}^{y^{1/\beta}} \frac{dr}{r \log r} \int_{r}^{y^{1/\beta}} \frac{ds}{s \log^2 s} = \frac{\alpha - \beta - \beta \log \frac{\alpha}{\beta}}{\log y},$$

$$\int_{y^{1/\alpha}}^{y^{1/\beta}} \frac{dr}{r \log r} \int_{r}^{y^{1/\beta}} \frac{ds}{s \log s} \int_{s}^{y^{1/\beta}} \frac{dt}{t \log^2 t} = \frac{\alpha - \beta - \beta \log \frac{\alpha}{\beta} - \frac{\beta}{2} \log^2 \frac{\alpha}{\beta}}{\log y},$$

$$\int_{y^{1/\alpha}}^{y^{1/\beta}} \frac{dr}{r \log r} \int_{r}^{y^{1/\beta}} \frac{ds}{s \log^2 s} \int_{s}^{y^{1/\beta}} \frac{dt}{t \log t} = \frac{(\alpha + \beta) \log \frac{\alpha}{\beta} + 2(\beta - \alpha)}{\log y},$$

$$\int_{y^{1/\alpha}}^{y^{1/\beta}} \frac{dr}{r \log r} \int_{r}^{y^{1/\beta}} \frac{ds}{s \log s} \int_{s}^{y^{1/\beta}} \frac{dt}{t \log^2 t} \int_{w}^{y^{1/\beta}} \frac{dw}{w \log w} = \frac{\frac{\beta}{2} \log^2 \frac{\alpha}{\beta} + (\alpha + 2\beta) \log \frac{\alpha}{\beta} + 3(\beta - \alpha)}{\log y}.$$

By (3.18)–(3.35) we get

$$\sum \chi(q) S_q^2 \le \sum \chi(q) \sum_{j=1}^{13} (W_j(q) + E(q)) < 0.14228\,\Xi(\chi; x), \tag{3.36}$$

$$\sum \chi(q) S_q^3 \le \frac{1}{2} \sum \chi(q) \sum_{j=14}^{16} (W_j(q) + E(q)) < 0.00431\,\Xi(\chi; x). \tag{3.37}$$

By the definition of $H(s)$, $G(s)$, Lemma 3 and some routine arguments we get

$$\sum \chi(q) S(\mathcal{A}_q, \mathcal{P}(q), \underline{q}^{1/v}) \le \left(1 + \int_2^{v-1} \frac{\log(t-1)}{t}\, dt - H(v)\right) \Xi(\chi; x), \tag{3.38}$$

$$\sum \chi(q) \sum_{\underline{q}^{1/w} \le p < \underline{q}^{1/u}} S(\mathcal{A}_{pq}, \mathcal{P}(pq), \underline{q}^{1/v}) \ge \left(\int_{u-1}^{w-1} \frac{\log\left(v - 1 - \frac{v}{t+1}\right) + G\left(v - \frac{v}{t+1}\right)}{t}\, dt - o(1)\right) \Xi(\chi; x), \tag{3.39}$$

where $3 < u < w < v < 5$, and

$$\sum \chi(q) \sum_{\underline{q}^{1/4.5} \le p_1 < p_2 < \underline{q}^{1/4.4}} S(\mathcal{A}_{p_1 p_2 q}, \mathcal{P}(q), \underline{q}^{1/4.5}) \le \left(\int_{1/4.5}^{1/4.4} \int_{t_1}^{1/4.4} \frac{1 - H(2.5)}{t_1 t_2 (1 - t_1 - t_2)}\, dt_1\, dt_2 + o(1)\right) \Xi(\chi; x), \tag{3.40}$$

$$\sum \chi(q) \sum_{\underline{q}^{1/4.5} \le p_1 < p_2 < \underline{q}^{1/4.4}} S(\mathcal{A}_{p_1 p_2 q}, \mathcal{P}(q), p_1) \le \left(\int_{1/4.5}^{1/4.4} \int_{t_1}^{1/4.4} \frac{1 - H(2.5)}{t_1 t_2 (1 - t_1 - t_2)}\, dt_1\, dt_2 + o(1)\right) \Xi(\chi; x), \tag{3.41}$$

$$\sum \chi(q) \sum_{\underline{q}^{1/4.5} \le p_1 < \underline{q}^{1/4.4} < \underline{q}^{1/3} \le p_2 < \underline{q}^{1/2.4}} S(\mathcal{A}_{p_1 p_2 q}, \mathcal{P}(q), p_1) \le \left(\int_{1/4.5}^{1/4.4} \int_{1/3}^{1/2.4} \frac{1 - H(2.2)}{t_1 t_2 (1 - t_1 - t_2)}\, dt_1\, dt_2 + o(1)\right) \Xi(\chi; x), \tag{3.42}$$

$$\sum \chi(q) \sum_{\underline{q}^{1/4.4} \le p_1 < \underline{q}^{1/3.6} < \underline{q}^{1/3} \le p_2 < \underline{q}^{1/2.4}} S(\mathcal{A}_{p_1 p_2 q}, \mathcal{P}(q), \underline{q}^{1/4.5}) \le \left(\int_{1/4.4}^{1/3.6} \int_{1/3}^{1/2.4} \frac{1 - H(2.2)}{t_1 t_2 (1 - t_1 - t_2)}\, dt_1\, dt_2 + o(1)\right) \Xi(\chi; x), \tag{3.43}$$

$$\sum \chi(q) \sum_{\underline{q}^{1/4.4} \le p_1 < p_2 < \underline{q}^{1/3}} S(\mathcal{A}_{p_1 p_2 q}, \mathcal{P}(q), \underline{q}^{1/4.5}) \tag{3.44}$$

$$\leq \left(\int_{1/4.4}^{1/3} \int_{t_1}^{1/3} \frac{1 - H(4.5(1 - t_1 - t_2))}{t_1 t_2 (1 - t_1 - t_2)} \, dt_1 \, dt_2 + o(1) \right) \Xi(\chi; x),$$

$$(3.45) \quad \sum \chi(q) \sum_{\underline{q}^{1/4.5} \leq p_1 < \underline{q}^{1/4.4} \leq p_2 < \underline{q}^{1/2.2}} S(\mathcal{A}_{p_1 p_2 q}, \mathcal{P}(q), \underline{q}^{1/4.5})$$

$$\leq \left(\int_{1/4.5}^{1/4.4} \int_{1/4.4}^{1/2.2} \frac{1 - H(4.5(1 - t_1 - t_2))}{t_1 t_2 (1 - t_1 - t_2)} \, dt_1 \, dt_2 + o(1) \right) \Xi(\chi; x),$$

$$(3.46) \quad \sum \chi(q) \sum_{\underline{q}^{1/4.4} \leq p_1 < p_2 < \underline{q}^{1/3.6}} S(\mathcal{A}_{p_1 p_2 q}, \mathcal{P}(q), \underline{q}^{1/4.5})$$

$$\leq \left(\int_{1/4.4}^{1/3.6} \int_{t_1}^{1/3.6} \frac{1 - H\left(\frac{1 - t_1 - t_2}{t_1}\right)}{t_1 t_2 (1 - t_1 - t_2)} \, dt_1 \, dt_2 + o(1) \right) \Xi(\chi; x),$$

$$(3.47) \quad \sum \chi(q) \sum_{\underline{q}^{1/4.4} \leq p_1 < \underline{q}^{1/3.6} \leq p_2 < \underline{q}^{1/3}} S(\mathcal{A}_{p_1 p_2 q}, \mathcal{P}(q), \underline{q}^{1/4.5})$$

$$\leq \left(\int_{1/4.4}^{1/3.6} \int_{1/3.6}^{1/3} \frac{1 - H(4.5(1 - t_1 - t_2))}{t_1 t_2 (1 - t_1 - t_2)} \, dt_1 \, dt_2 + o(1) \right) \Xi(\chi; x).$$

By (3.38)–(3.47) and (3.8) we get

$$(3.48) \quad S_q^1 \leq (N_1 - N_2) \Xi(\chi; x),$$

where

$$(3.49) \quad N_1 = 5 + \int_2^{3.5} \frac{\log(t-1)}{t} \, dt + \int_2^{3.4} \frac{\log(t-1)}{t} \, dt + 2 \int_2^{2.6} \frac{\log(t-1)}{t} \, dt$$
$$- \frac{1}{2} \int_{1.2}^{3.5} \frac{\log\left(3.5 - \frac{4.5}{t+1}\right)}{t} \, dt - \frac{1}{2} \int_{1.2}^{3.4} \frac{\log\left(3.4 - \frac{4.4}{t+1}\right)}{t} \, dt$$
$$- \int_{1.4}^{2.6} \frac{\log\left(2.6 - \frac{3.6}{t+1}\right)}{t} \, dt$$
$$\leq 5 - 0.244058,$$

$$(3.50) \quad N_2 = 2H(4.5) + 2H(4.4) + H(3.6) + \int_{1.2}^{3.5} \frac{G\left(4.5 - \frac{4.5}{t+1}\right)}{t} \, dt$$
$$+ \frac{1}{2} \int_{3.4}^{3.5} \frac{G\left(4.5 - \frac{4.5}{t+1}\right)}{t} \, dt + 2 \int_2^{3.4} \frac{G\left(4.4 - \frac{4.4}{t+1}\right)}{t} \, dt$$

$$+\int_{1.4}^{2}\frac{G\big(4.5-\frac{4.5}{t+1}\big)}{t}\,dt+\int_{1/4.5}^{1/4.4}\int_{t_1}^{1/4.4}\frac{H(2.5)}{t_1t_2(1-t_1-t_2)}\,dt_1\,dt_2$$
$$+\frac{1}{2}\int_{1/4.5}^{1/4.4}\int_{1/4.4}^{1/2.2}\frac{H(4.5(1-t_1-t_2))}{t_1t_2(1-t_1-t_2)}\,dt_1\,dt_2$$
$$+\int_{1/4.4}^{1/3}\int_{t_1}^{1/3}\frac{H(4.5(1-t_1-t_2))}{t_1t_2(1-t_1-t_2)}\,dt_1\,dt_2$$
$$+\int_{1/4.4}^{1/3.6}\int_{1/3.6}^{1/3}\frac{H(4.5(1-t_1-t_2))}{t_1t_2(1-t_1-t_2)}\,dt_1\,dt_2$$
$$+\int_{1/4.4}^{1/3.6}\int_{t_1}^{1/3.6}\frac{H\big(\frac{1-t_1-t_2}{t_1}\big)}{t_1t_2(1-t_1-t_2)}\,dt_1\,dt_2$$
$$+\int_{1/4.4}^{1/3.6}\int_{1/3}^{1/2.4}\frac{H(2.2)}{t_1t_2(1-t_1-t_2)}\,dt_1\,dt_2$$
$$+\int_{1/4.5}^{1/4.4}\int_{1/3}^{1/2.4}\frac{H(2.2)}{t_1t_2(1-t_1-t_2)}\,dt_1\,dt_2.$$

By (2.6) we have

$$\begin{aligned}(3.51)\quad & 2H(4.5)+2H(4.4)+H(3.6)\\ &\ge 1.02288G(4)+0.03245H(2.2)+0.0099H(2.3)+0.01078H(2.4)\\ &\quad+0.01272H(2.5)+0.01668H(2.6)+0.02130H(2.7)\\ &\quad+0.02544H(2.8)+0.02916H(2.9)\\ &\quad+0.03252H(3)-\varepsilon.\end{aligned}$$

By (2.7) we have

$$\begin{aligned}(3.52)\quad & \Big(\int_{1.2}^{3.5}+\frac{1}{2}\int_{3.4}^{3.5}+\int_{1.4}^{2}\Big)\frac{G\big(4.5-\frac{4.5}{t+1}\big)}{t}\,dt+2\int_{2}^{3.4}\frac{G\big(4.4-\frac{4.4}{t+1}\big)}{t}\,dt\\ &\ge 2.50285G(4)+0.23871H(2.2)+0.08093H(2.3)\\ &\quad+0.09264H(2.4)+0.09912H(2.5)+0.09816H(2.6)\\ &\quad+0.09444H(2.7)+0.09102H(2.8)+0.08782H(2.9)\\ &\quad+0.08485H(3)-\varepsilon.\end{aligned}$$

By Proposition 1(1) we have

$$
\begin{aligned}
(3.53)\quad & \int_{1/4.5}^{1/4.4}\int_{t_1}^{1/4.4} \frac{H(2.5)}{t_1t_2(1-t_1-t_2)}\,dt_1\,dt_2 \\
&\quad + \frac{1}{2}\int_{1/4.5}^{1/4.4}\int_{1/4.4}^{1/2.2} \frac{H(4.5(1-t_1-t_2))}{t_1t_2(1-t_1-t_2)}\,dt_1\,dt_2 \\
&\quad + \int_{1/4.4}^{1/3}\int_{t_1}^{1/3} \frac{H(4.5(1-t_1-t_2))}{t_1t_2(1-t_1-t_2)}\,dt_1\,dt_2 \\
&\quad + \int_{1/4.4}^{1/3.6}\int_{1/2.4}^{1/3} \frac{H(2.2)}{t_1t_2(1-t_1-t_2)}\,dt_1\,dt_2 \\
&\quad + \int_{1/4.5}^{1/4.4}\int_{1/3}^{1/2.4} \frac{H(2.2)}{t_1t_2(1-t_1-t_2)}\,dt_1\,dt_2 \\
&\quad + \int_{1/4.4}^{1/3.6}\int_{1/3.6}^{1/3} \frac{H(4.5(1-t_1-t_2))}{t_1t_2(1-t_1-t_2)}\,dt_1\,dt_2 \\
&\quad + \int_{1/4.4}^{1/3.6}\int_{t_1}^{1/3.6} \frac{H\left(\frac{1-t_1-t_2}{t_1}\right)}{t_1t_2(1-t_1-t_2)}\,dt_1\,dt_2 \\
&\geq 0.39974H(2.2) + 0.04041H(2.5).
\end{aligned}
$$

By (3.50)–(3.53) we have

$$
\begin{aligned}
(3.54)\quad N_2 \geq\ & 0.74978H(2.2) + 0.10644H(2.3) + 0.1203H(2.4) \\
&+ 0.1704H(2.5) + 0.13423H(2.6) + 0.13629H(2.7) \\
&+ 0.13814H(2.8) + 0.13972H(2.9) + 0.1412H(3) - \varepsilon,
\end{aligned}
$$

where Proposition 3 is used.

By (3.7), (3.36), (3.37), (3.48) and (3.49) we have

$$
(3.55)\quad 5\sum \chi(q)S(\mathcal{A}_q, \mathcal{P}(q), \underline{q}^{1/2.2}) \leq (5 - 0.097465 - N_2)\varXi(\chi; x).
$$

By (3.54), (3.55) and the definition of $H(s)$ we get

$$
\begin{aligned}
H(2.2) \geq\ & 0.019493 + 0.14995H(2.2) + 0.02129H(2.3) \\
&+ 0.02406H(2.4) + 0.03408H(2.5) + 0.02684H(2.6) \\
&+ 0.02725H(2.7) + 0.02762H(2.8) + 0.02794H(2.9) \\
&+ 0.02824H(3),
\end{aligned}
$$

$$H(2.2) \geq 0.022931 + 0.02504H(2.3) + 0.02830H(2.4) + 0.04009H(2.5) + 0.03157H(2.6) + 0.03205H(2.7) + 0.03249H(2.8) + 0.03286H(2.9) + 0.03322H(3). \tag{3.56}$$

By Lemmas 8, 9 and 11 and (3.56) we get

$$H(2.2) \geq 0.02685 + 0.000073 + 0.00137H(2.2),$$

$$H(2.2) \geq 0.02695. \tag{3.57}$$

Proof of the Theorem. Let χ denote the characteristic function of the set $\{1\}$. By the definition of $H(s)$, Proposition 1(2) and (3.57) we have

$$\begin{aligned}\pi_2(x) &\leq S(\mathcal{A}, \mathcal{P}, \underline{1}^{1/2.2}) + O(\underline{1}^{1/2.2}) \\ &\leq 3.5(1+o(1))\,\frac{Cx}{\log^2 x}\,(1 - H(2.2)) + O(\underline{1}^{1/2.2}) \\ &< \frac{3.406Cx}{\log^2 x}.\end{aligned}$$

The proof of the Theorem is complete.

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