ACTA ARITHMETICA
110.3 (2003)

On additive properties of two special sequences

by

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1. Introduction. Let A be an infinite sequence of positive integers. For each positive integer n, let $R_1(A,n)$, $R_2(A,n)$ and $R_3(A,n)$ denote the number of solutions of

$$\begin{aligned} x+y&=n, \quad x,y\in A,\\ x+y&=n, \quad x<y,\ x,y\in A,\\ x+y&=n, \quad x\le y,\ x,y\in A,\end{aligned}$$

respectively. A. Sárközy asked whether there exist two sets A and B of positive integers with infinite symmetric difference, i.e.

$$|(A\cup B)\setminus(A\cap B)|=\infty,$$

and

$$R_i(A,n)=R_i(B,n), \quad n\ge n_0,$$

for $i=1,2,3$. For $i=1$, the answer is no. For $i=2$, G. Dombi [1] proved that the set $\mathbb{N}$ of positive integers can be partitioned into two subsets A and B such that $R_2(A,n)=R_2(B,n)$ for all $n\in\mathbb{N}$. For $i=3$, G. Dombi [1] conjectured that the answer is no. For other related results, the reader is referred to [2–4]. Let

$$\begin{aligned} U(A,n)&=\{(x,y)\mid x+y=n,\ x,y\in A,\ x\le y\},\\ U_0(A,n)&=\{(x,y)\mid (x,y)\in U(A,n),\ 2\mid x\},\\ U_1(A,n)&=\{(x,y)\mid (x,y)\in U(A,n),\ 2\nmid x\}.\end{aligned}$$

Then $R_3(A,n)=|U(A,n)|=|U_0(A,n)|+|U_1(A,n)|$.

In this note, we prove the following theorem.

2000 *Mathematics Subject Classification*: 11B34, 11B83, 11B85.

Supported by the National Natural Science Foundation of China, grant no.10171046 and the Teaching and Research Award Program for Outstanding Young Teachers in Nanjing Normal University.

THEOREM. *The set $\mathbb{N}$ of positive integers can be partitioned into two subsets A and B such that $R_3(A,n) = R_3(B,n)$ for all $n \geq 3$.*

Proof. Let $T(n)$ denote the number of zero digits in the dyadic representation of $n \geq 0$ (thus $T(0) = 1$, $T(1) = 0$, etc.). Then $T(2n+1) = T(2n) - 1$ $(n \geq 0)$ and $T(2n) = T(n) + 1$ $(n \geq 1)$. Let

$$A = \{n \in \mathbb{N} \mid 2 \mid T(n-1)\}, \qquad B = \{n \in \mathbb{N} \mid 2 \nmid T(n-1)\}.$$

We use induction on n to prove that $|U(A,n)| = |U(B,n)|$ for all $n \geq 3$. By calculation we have $|U(A,n)| = |U(B,n)|$ for $n = 3, 4, 5$. Now suppose that $|U(A,n)| = |U(B,n)|$ for $3 \leq n \leq k-1$ $(k \geq 6)$.

CASE 1: $2 \mid k$. Define a map

$$f : U_0(A,k) \setminus \{(2, k-2)\} \to U\left(A, \frac{k}{2}\right) \Big\backslash \left\{\left(1, \frac{k-2}{2}\right)\right\}$$

by

$$f(a,b) = \left(\frac{a}{2}, \frac{b}{2}\right).$$

Noting that $b \geq a \geq 4$, $2 \mid a$ and $2 \mid b$, we have

$$T\left(\frac{a}{2} - 1\right) = T(a-2) - 1 = T(a-1),$$
$$T\left(\frac{b}{2} - 1\right) = T(b-2) - 1 = T(b-1).$$

Hence, f is well defined. It is easy to verify that f is bijective. Thus

$$|U_0(A,k) \setminus \{(2,k-2)\}| = \left|U\left(A, \frac{k}{2}\right) \Big\backslash \left\{\left(1, \frac{k-2}{2}\right)\right\}\right|. \tag{1}$$

Similarly, we have

$$|U_0(B,k) \setminus \{(2,k-2)\}| = \left|U\left(B, \frac{k}{2}\right) \Big\backslash \left\{\left(1, \frac{k-2}{2}\right)\right\}\right|. \tag{2}$$

Define a map $g : U_1(A,k) \setminus \{(1, k-1)\} \to U\big(B, \frac{k+2}{2}\big) \setminus \big\{\big(1, \frac{k}{2}\big)\big\}$ by $f(a,b) = \big(\frac{a+1}{2}, \frac{b+1}{2}\big)$. Noting that $b \geq a \geq 2$, $2 \nmid a$ and $2 \nmid b$, we have

$$T\left(\frac{a+1}{2} - 1\right) = T\left(\frac{a-1}{2}\right) = T(a-1) - 1,$$
$$T\left(\frac{b+1}{2} - 1\right) = T\left(\frac{b-1}{2}\right) = T(b-1) - 1.$$

Hence, g is well defined. It is easy to verify that g is bijective. Thus

$$|U_1(A,k) \setminus \{(1,k-1)\}| = \left|U\left(B, \frac{k+2}{2}\right) \Big\backslash \left\{\left(1, \frac{k}{2}\right)\right\}\right|. \tag{3}$$

Similarly, we have

$$|U_1(B,k)\setminus\{(1,k-1)\}| = \left|U\left(A,\frac{k+2}{2}\right)\Big\backslash\left\{\left(1,\frac{k}{2}\right)\right\}\right|. \tag{4}$$

Noting that $1\notin A$ and $2\notin B$, by (1)–(4), we have

$$|U(A,k)\setminus\{(2,k-2)\}| = \left|U\left(A,\frac{k}{2}\right)\right| + \left|U\left(B,\frac{k+2}{2}\right)\Big\backslash\left\{\left(1,\frac{k}{2}\right)\right\}\right|, \tag{5}$$

$$\begin{aligned}&|U(B,k)\setminus\{(1,k-1)\}|\\ &\qquad = \left|U\left(B,\frac{k}{2}\right)\Big\backslash\left\{\left(1,\frac{k-2}{2}\right)\right\}\right| + \left|U\left(A,\frac{k+2}{2}\right)\right|.\end{aligned} \tag{6}$$

Noting that $T(k-2) = T\big(\frac{k}{2}-1\big)+1$ and $T(k-3) = T(k-4)-1 = T\big(\frac{k-4}{2}\big) = T\big(\frac{k-2}{2}-1\big)$, we have the following possibilities:

(i) If $2\nmid T\big(\frac{k}{2}-1\big)$ and $2\nmid T\big(\frac{k-2}{2}-1\big)$, then

$$\left(1,\frac{k}{2}\right)\in U\left(B,\frac{k+2}{2}\right),\qquad \left(1,\frac{k-2}{2}\right)\in U\left(B,\frac{k}{2}\right),$$
$$(2,k-2)\notin U(A,k),\qquad (1,k-1)\notin U(B,k).$$

In this case, by (5) and (6), we have

$$|U(A,k)| = \left|U\left(A,\frac{k}{2}\right)\right| + \left|U\left(B,\frac{k+2}{2}\right)\right| - 1,$$
$$|U(B,k)| = \left|U\left(B,\frac{k}{2}\right)\right| + \left|U\left(A,\frac{k+2}{2}\right)\right| - 1.$$

(ii) If $2\nmid T\big(\frac{k}{2}-1\big)$ and $2\,|\,T\big(\frac{k-2}{2}-1\big)$, then

$$\left(1,\frac{k}{2}\right)\in U\left(B,\frac{k+2}{2}\right),\qquad \left(1,\frac{k-2}{2}\right)\notin U\left(B,\frac{k}{2}\right),$$
$$(2,k-2)\in U(A,k),\qquad (1,k-1)\notin U(B,k).$$

In this case, by (5) and (6), we have

$$|U(A,k)| = \left|U\left(A,\frac{k}{2}\right)\right| + \left|U\left(B,\frac{k+2}{2}\right)\right|,$$
$$|U(B,k)| = \left|U\left(B,\frac{k}{2}\right)\right| + \left|U\left(A,\frac{k+2}{2}\right)\right|.$$

(iii) If $2\,|\,T\big(\frac{k}{2}-1\big)$ and $2\nmid T\big(\frac{k-2}{2}-1\big)$, then

$$\left(1,\frac{k}{2}\right)\notin U\left(B,\frac{k+2}{2}\right),\qquad \left(1,\frac{k-2}{2}\right)\in U\left(B,\frac{k}{2}\right),$$
$$(2,k-2)\notin U(A,k),\qquad (1,k-1)\in U(B,k).$$

In this case, by (5) and (6), we have

$$|U(A,k)| = \left|U\left(A,\frac{k}{2}\right)\right| + \left|U\left(B,\frac{k+2}{2}\right)\right|,$$
$$|U(B,k)| = \left|U\left(B,\frac{k}{2}\right)\right| + \left|U\left(A,\frac{k+2}{2}\right)\right|.$$

(iv) If $2\,|\,T\left(\frac{k}{2}-1\right)$ and $2\,|\,T\left(\frac{k-2}{2}-1\right)$, then

$$\left(1,\frac{k}{2}\right)\notin U\left(B,\frac{k+2}{2}\right),\qquad \left(1,\frac{k-2}{2}\right)\notin U\left(B,\frac{k}{2}\right),$$
$$(2,k-2)\in U(A,k),\qquad (1,k-1)\in U(B,k).$$

In this case, by (5) and (6), we have

$$|U(A,k)| = \left|U\left(A,\frac{k}{2}\right)\right| + \left|U\left(B,\frac{k+2}{2}\right)\right| + 1,$$
$$|U(B,k)| = \left|U\left(B,\frac{k}{2}\right)\right| + \left|U\left(A,\frac{k+2}{2}\right)\right| + 1.$$

Since $3\le k/2<k$, $3\le (k+2)/2<k$, by the induction hypothesis, we have

$$\left|U\left(A,\frac{k}{2}\right)\right| = \left|U\left(B,\frac{k}{2}\right)\right|,\qquad \left|U\left(A,\frac{k+2}{2}\right)\right| = \left|U\left(B,\frac{k+2}{2}\right)\right|.$$

By (i)–(iv), we have

$$|U(A,k)| = |U(B,k)|.$$

Case 2: $2\nmid k$. Define a map $h: U_0(A,k)\to U_1(B,k)$ by

$$h(a,b) = (a-1,b+1).$$

Since $2\,|\,a$, we have $2\nmid b$, $b+1\ge a-1\ge 1$ and $2\nmid a-1$. By $T(a-2)=T(a-1)+1$ and $T(b)=T(b-1)-1$, we know that h is well defined. It is clear that h is injective. Now we show that h is surjective. Assume that $(u,v)\in U_1(B,k)$. Let $a'=u+1$ and $b'=v-1$. Then $2\,|\,u+1$, $2\,|\,v-2$, $T(a'-1)=T(u)=T(u-1)-1$ and $T(b'-1)=T(v-2)=T(v-1)+1$. To prove that $(a',b')\in U_0(A,k)$, it is sufficient to prove that $a'\le b'$. If $a'>b'$, then, since $u\le v$, $2\nmid u$ and $2\,|\,v$, we have $a'-1=b'$. But

$$T(a'-1)=T(u-1)-1\equiv T(v-1)-1=T(b')-1 \pmod 2,$$

a contradiction. So $a'\le b'$ and then $(a',b')\in U_0(A,k)$. Hence h is bijective. Thus

$$|U_0(A,k)| = |U_1(B,k)|.$$

Similarly, $|U_0(B,k)| = |U_1(A,k)|$. Therefore $|U(A,k)| = |U(B,k)|$. This completes the proof.

References

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Received on 26.8.2002
and in revised form on 3.2.2003 (4355)