ACTA ARITHMETICA
C.2 (2001)

The distribution of real-valued Q-additive functions modulo 1

by

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0. Introduction. Let $q \geq 2$ be an integer. A *q-additive function* $f : \mathbb{N} \to \mathbb{C}$ is a function of the form $f(n) = \sum_{j\geq 0} f_j(a_j(n))$ where $n = \sum_{j\geq 0} a_j(n)q^j$ is the base-q representation of n and the "component functions" f_j are functions defined on $\{0,1,\ldots,q-1\}$ and satisfying $f_j(0) = 0$. These functions were introduced by A. O. Gel'fond [Ge] in 1968, and have been studied by Coquet [Co1], Delange [De3], and others. They generalize the sum-of-digits functions $s_q(n)$ with respect to base q.

In 1977, Coquet [Co1] generalized q-additive functions to more general systems of numeration. Specifically, he considered so-called *mixed radix* representations (also called Cantor representations) defined as follows. Let $Q = \{Q_j\}_{j\geq 0}$ be a sequence of strictly increasing positive integers with $Q_0 = 1$ such that $Q_j \mid Q_{j+1}$ for all j. Note that the sequence Q is uniquely determined by the factors $q_j = Q_{j+1}/Q_j$. It is easily seen that each nonnegative integer n has a unique "base-Q" representation of the form $n = \sum_{j\geq 0} a_j(n)Q_j$, where the "digits" $a_j(n)$ satisfy $0 \leq a_j(n) < q_j$. Examples of such representations are the ordinary base-q representations ($q_j = q$) as well as the factorial representation ($q_j = j + 2$), the factorial-squared representation ($q_j = (j + 2)^2$), and the doubly-geometric representations ($q_j = q^j$). For a full discussion of these and other numeration systems see, for example, Grabner *et al.* [GLT] or the survey article by Fraenkel [Fr] and the references therein.

Given a mixed radix system Q, Coquet defined a *Q-additive function* $f : \mathbb{N} \to \mathbb{C}$ to be a function of the form $f(n) = \sum_{j\geq 0} f_j(a_j(n))$ where $n = \sum_{j\geq 0} a_j(n)Q_j$ is the base-Q representation of n and the component functions f_j are functions defined on $\{0,1,\ldots,q_j-1\}$ and satisfying $f_j(0) = 0$.

2000 *Mathematics Subject Classification*: Primary 11K16.

The results in this paper also appear in the author's doctoral thesis deposited in April of 1999 at the University of Illinois at Urbana-Champaign.

A simple example of a Q-additive function is the sum-of-digits function $s_Q(n) = \sum_{j\ge 0} a_j(n)$, which corresponds to the choice $f_j(a) = a$. This function has been studied by Kirschenhofer and Tichy [KT], among others. For recent work on general Q-additive functions see Manstavičius [Ma]. For generalizations of q-additive functions to other numeration systems see, for example, Barat and Grabner [BG].

Our main result, Theorem 1, characterizes those real-valued Q-additive functions that have a limit distribution (resp. uniform limit distribution) modulo 1. In order to prove this result, we consider so-called Q-multiplicative functions, which are defined in analogy to Q-additive functions as follows. A *Q-multiplicative function* is a function $g : \mathbb{N} \to \mathbb{C}$ of the form $g(n) = \prod_{j\ge 0} g_j(a_j(n))$, where $n = \sum_{j\ge 0} a_j(n) Q_j$ is the base-Q representation of n and the component functions g_j are functions defined on $\{0, 1, \ldots, q_j - 1\}$ and satisfying $g_j(0) = 1$. These functions have been studied by Coquet [Co1] and others, usually in conjunction with work on Q-additive functions. We establish mean value theorems for Q-multiplicative functions analogous to those of Delange and Wirsing (see, e.g., [El, Chapter 6]) for ordinary multiplicative functions.

Throughout this paper, we set $e(x) = e^{2\pi i x}$ and write $\|x\|$ to denote the distance from x to the nearest integer and $\{x\}$ to denote the fractional part of x.

1. Statement of results. Let $Q = \{Q_j\}_{j\ge 0}$ be a mixed radix system with factors $q_j = Q_{j+1}/Q_j$. Let f be a real-valued Q-additive function with component functions f_j. We say that f has a *limit distribution modulo* 1 if there is a distribution function F (i.e., F is right-continuous and monotonic with $F(x) = 0$ for $x < 0$ and $F(x) = 1$ for $x \ge 1$) such that the limit

$$\lim_{N\to\infty} \frac{1}{N} \#\{0 \le n < N : \{f(n)\} \le x\}$$

exists and equals $F(x)$ for every x at which F is continuous. We say that f has a *uniform limit distribution modulo* 1 if this holds with $F(x) = x$ for $0 \le x < 1$. Aside from its intrinsic interest, the study of the distribution modulo 1 of Q-additive functions is motivated in part by the results of Coquet [Co1] and Mendès France [MF] connecting the uniform distribution of certain Q-additive functions to so-called P-V numbers. Our main result is a complete characterization of real-valued Q-additive functions that have a limit distribution (resp. uniform limit distribution) modulo 1.

THEOREM 1. *A real-valued Q-additive function f has a limit distribution modulo* 1 *if and only if, for each integer $k \ne 0$, at least one of the following conditions holds*:

(i) *There exists $j \ge 0$ such that* $\sum_{0\le n<q_j} e(kf_j(n)) = 0$.

(ii) *The series*

$$\sum_{j\ge 0}\Bigl(1-\frac{1}{q_j}\Bigl|\sum_{0\le n<q_j} e(kf_j(n))\Bigr|\Bigr)$$

diverges.

(iii) *The series*

$$\sum_{j\ge 0}\Bigl(1-\frac{1}{q_j}\sum_{0\le n<q_j} e(kf_j(n))\Bigr)$$

converges, *and*

$$\lim_{j\to\infty}\Bigl(\max_{0<n\le q_j}\frac{1}{n}\sum_{0\le m<n}\|kf_j(m)\|^2\Bigr)=0.$$

Furthermore, f *is uniformly distributed modulo* 1 *if and only if*, *for all integers* $k\ne 0$, *at least one of conditions* (i) *or* (ii) *holds.*

This result generalizes the characterization given by Kim [Ki, p. 27] for the special case of q-additive functions.

We apply Theorem 1 to derive several corollaries that deal with special cases. We first consider numeration systems in which the factors q_j are bounded. In particular, these systems include the ordinary base-q representations generated by $q_j = q$ for all j.

Corollary 1. *Suppose the factors* q_j *are bounded. Then* f *is uniformly distributed modulo* 1 *if and only if*, *for all* $k\ne 0$, *either the series*

$$\sum_{j\ge 0}\sum_{0\le n<q_j}\|kf_j(n)\|^2$$

diverges, *or for some* $j\ge 0$ *we have*

$$\sum_{0\le n<q_j} e(kf_j(n))=0.$$

Let $\alpha\in\mathbb{R}$. We call an integer-valued arithmetic function f *normal* if the function αf is uniformly distributed modulo 1 if and only if α is irrational. Coquet [Co2] showed that for any base $q\ge 2$, the associated sum-of-digits function $s_q(n)$ is normal. General criteria for the normality of arithmetic functions have been given by Drmota and Tichy [DT, Section 1.4.3]. In Corollaries 2 and 3 below, we apply Theorem 1 to show that two classes of integer-valued Q-additive functions are normal.

Corollary 2. *For any mixed radix numeration system* Q, *the function* $s_Q(n)$, *the sum of digits in the base-*Q *representation of* n, *is normal.*

We call a Q-additive function f *completely* Q*-additive* if there exists a function $g:\mathbb{N}\to\mathbb{C}$ such that, for all $j\ge 0$ and $0\le n<q_j$, $f_j(n)=g(n)$,

i.e., if the component functions f_j are independent of j on their respective domains. The following corollary generalizes a result of Drmota and Tichy [DT, Theorem 1.99].

Corollary 3. *Suppose that the series $\sum_{j\ge 0} 1/q_j$ diverges. Let f be a completely Q-additive, integer-valued function such that there is some integer $1 \le a < \min_j q_j$ with $f_0(a) > 0$. Then f is normal.*

In the next two corollaries we investigate the normality of two particular integer-valued Q-additive functions that have been previously considered in [Ho, examples (c) and (a)] and [KT, Theorem 3]. These results provide examples of functions that have a non-uniform limit distribution modulo 1 as well as functions that have no limit distribution modulo 1.

Corollary 4. *Let α be an irrational number. Let $M(n)$ be the number of digits in the base-Q representation of n which are maximal, and set $f(n) = \alpha M(n)$. Then f has a limit distribution modulo 1. Moreover, the limit distribution is uniform if and only if the series $\sum_{j\ge 0} 1/q_j$ diverges.*

Corollary 5. *Let $a > 0$ be a fixed integer and let α be an irrational number. Let $N_a(n)$ be the number of digits a in the base-Q representation of n, and set $f(n) = \alpha N_a(n)$. Then f is uniformly distributed modulo 1 if and only if $\sum_{q_j > a} 1/q_j$ diverges. The function f has a non-uniform limit distribution modulo 1 if and only if $q_j \le a$ for all but at most finitely many j.*

2. Lemmas. The first lemma is a well known result on the distribution modulo 1 of real-valued arithmetic functions (see, e.g., [De2, p. 216]). The second assertion of the lemma is known as *Weyl's Criterion* [We].

Lemma 1. *A real-valued arithmetic function f has a limit distribution modulo 1 if and only if, for each integer $k \ne 0$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} e(kf(n))$$

exists. Further, the distribution is uniform modulo 1 if and only if, for each integer $k \ne 0$, the above limit is 0.

Throughout the remainder of this section, we fix a mixed radix system Q with factors $\{q_j\}_{j\ge 0}$. We denote by g a Q-multiplicative function satisfying $|g| \le 1$ with component functions g_j, and define

$$\mu_j(g) = \frac{1}{q_j} \sum_{0\le n<q_j} g_j(n).$$

Thus, $\mu_j(g)$ is the mean value of g_j on its domain $\{0, 1, \ldots, q_j - 1\}$.

The following lemma relates the mean value of g on $\{0, 1, \ldots, rQ_j - 1\}$ to that of the functions g_i.

LEMMA 2. *For $j \geq 0$ and any positive integer r with $1 \leq r \leq q_j$ we have*

$$\frac{1}{rQ_j}\sum_{n=0}^{rQ_j-1} g(n) = \left(\frac{1}{r}\sum_{n=0}^{r-1} g_j(n)\right)\left(\frac{1}{Q_j}\sum_{n=0}^{Q_j-1} g(n)\right). \tag{2.1}$$

Moreover, for any $j \geq 0$,

$$\frac{1}{Q_j}\sum_{n=0}^{Q_j-1} g(n) = \prod_{i=0}^{j-1} \mu_i(g). \tag{2.2}$$

Proof. We first note that, since $Q_0 = 1$ and $g(0) = 1$, we have

$$\frac{1}{Q_0}\sum_{n=0}^{Q_0-1} g(n) = g(0) = 1.$$

Thus, relation (2.2) follows from (2.1) by applying (2.1) with $r = q_j$ and iterating the identity. Hence it suffices to prove (2.1).

Observe that any non-negative integer $n < rQ_j$ can be written uniquely (via the division algorithm) in the form $n = sQ_j + t$ with $0 \leq t < Q_j$ and $0 \leq s < r$. By the Q-multiplicativity of g, we have, with this representation,

$$g(n) = g_j(s)g(t).$$

As n runs through the set $\{0, 1, \ldots, rQ_j - 1\}$, s and t run independently through the sets $\{0, 1, \ldots, r-1\}$ and $\{0, 1, \ldots, Q_j-1\}$, respectively. It follows that

$$\frac{1}{rQ_j}\sum_{n=0}^{rQ_j-1} g(n) = \frac{1}{rQ_j}\sum_{s=0}^{r-1}\sum_{t=0}^{Q_j-1} g_j(s)g(t) = \frac{1}{r}\sum_{s=0}^{r-1} g_j(s)\frac{1}{Q_j}\sum_{t=0}^{Q_j-1} g(t),$$

which is (2.1).

To obtain necessary and sufficient conditions for the convergence of the product in (2.2), we will use the following lemma, a proof of which can be found in many elementary texts on complex variables (see, e.g., [LR, pp. 383–384]).

LEMMA 3. *Let $z_0, z_1, \ldots$ be complex numbers satisfying $|z_j| \leq 1$, and let $P_i = \prod_{j=0}^{i} z_j$. Then $\lim_{i\to\infty} P_i = 0$ if and only if at least one of the following two conditions holds*:

(i) *There is some $j \geq 0$ such that $z_j = 0$.*
(ii) $\sum_{j=0}^{\infty}(1 - |z_j|) = \infty$.

Then $\lim_{i\to\infty} P_i$ exists and is non-zero if and only if the following two conditions are both satisfied:

(iii) *$z_j \neq 0$ for all j.*
(iv) *$\sum_{j=0}^{\infty}(1 - z_j)$ converges.*

The next lemma relates the mean value of g on $\{0, 1, \ldots, N-1\}$ for general integers N to the mean values of the functions g_j.

LEMMA 4. *Let N be a positive integer and let $\sum_{j=0}^{i} a_j Q_j$, with $a_i > 0$, be the base-Q representation of N. Then*

$$\frac{1}{N}\sum_{n=0}^{N-1} g(n) = \sum_{j=0}^{i} \frac{a_j Q_j}{N}\Big(\prod_{m=j+1}^{i} g_m(a_m)\Big)\bigg(\frac{1}{a_j}\sum_{n=0}^{a_j-1} g_j(n)\bigg)\Big(\prod_{k=0}^{j-1}\mu_k(g)\Big), \tag{2.3}$$

where empty products and empty sums are to be interpreted as 1 *and* 0, *respectively. Furthermore, for any positive integer $h \le i$, we have*

$$\sum_{j=0}^{i-h} \frac{a_j Q_j}{N} < 2^{1-h}. \tag{2.4}$$

Proof. We begin by dividing the interval $0 \le n < N$ into the subintervals $0 \le n < a_i Q_i$ and $a_i Q_i \le n < N$, to obtain

$$\frac{1}{N}\sum_{n=0}^{N-1} g(n) = \frac{1}{N}\sum_{0\le n<a_iQ_i} g(n) + \frac{1}{N}\sum_{a_iQ_i\le n<N} g(n).$$

We have, by Lemma 2,

$$\begin{aligned}\frac{1}{N}\sum_{0\le n<a_iQ_i} g(n) &= \frac{a_iQ_i}{N}\bigg(\frac{1}{a_i}\sum_{n=0}^{a_i-1} g_i(n)\bigg)\bigg(\frac{1}{Q_i}\sum_{n=0}^{Q_i-1} g(n)\bigg)\\ &= \frac{a_iQ_i}{N}\bigg(\frac{1}{a_i}\sum_{n=0}^{a_i-1} g_i(n)\bigg)\Big(\prod_{k=0}^{i-1}\mu_k(g)\Big).\end{aligned}$$

Furthermore, by the Q-multiplicativity of g, we also have, for all n with $a_iQ_i \le n < N$, $g(n) = g(n - a_iQ_i)g_i(a_i)$. Thus,

$$\begin{aligned}\frac{1}{N}\sum_{a_iQ_i\le n<N} g(n) &= \frac{1}{N}\sum_{a_iQ_i\le n<N} g(n-a_iQ_i)g_i(a_i)\\ &= g_i(a_i)\frac{1}{N}\sum_{a_iQ_i\le n<N} g(n-a_iQ_i)\\ &= g_i(a_i)\frac{1}{N}\sum_{0\le n<N-a_iQ_i} g(n)\\ &= g_i(a_i)\frac{N-a_iQ_i}{N}\bigg(\frac{1}{N-a_iQ_i}\sum_{0\le n<N-a_iQ_i} g(n)\bigg).\end{aligned}$$

It follows that

$$\frac{1}{N}\sum_{n=0}^{N-1} g(n) = \frac{a_iQ_i}{N}\Big(\frac{1}{a_i}\sum_{n=0}^{a_i-1} g_i(n)\Big)\Big(\prod_{k=0}^{i-1}\mu_k(g)\Big) + g_i(a_i)\frac{N-a_iQ_i}{N}\bigg(\frac{1}{N-a_iQ_i}\sum_{0\le n<N-a_iQ_i} g(n)\bigg).$$

Iterating the last expression $i-1$ times gives (2.3). Inequality (2.4) follows from the chain of inequalities

$$\frac{1}{N}\sum_{j=0}^{i-h} a_jQ_j \le \frac{1}{Q_i}\sum_{j=0}^{i-h} a_jQ_j \le \frac{Q_{i-h+1}}{Q_i} = \prod_{j=i-h+1}^{i-1}\frac{1}{q_j} \le 2^{1-h}.$$

3. Mean value theorems for Q-multiplicative functions. Throughout this section, we let Q be a mixed radix system with factors $\{q_j\}_{j\ge 0}$. For a given Q-multiplicative function g with component functions g_j, we define the mean value of g by

$$M(g) = \lim_{N\to\infty}\frac{1}{N}\sum_{0\le n<N} g(n),$$

provided this limit exists. We set

$$\sigma_j(g) = \max_{0<n\le q_j}\frac{1}{n}\sum_{0\le m<n}(1-\mathrm{Re}(g_j(m))),$$

and recall the notation

$$\mu_j(g) = \frac{1}{q_j}\sum_{0\le n<q_j} g_j(n)$$

introduced in the previous section.

The following theorem, due to Coquet [Co1, Lemma 1], gives a characterization of Q-multiplicative functions of modulus at most 1 which have mean value 0. We present a proof here for completeness.

THEOREM 2. *Let g be a Q-multiplicative function satisfying $|g|\le 1$. The mean value $M(g)$ exists and is equal to* 0 *if and only if at least one of the following two conditions holds*:

(i) *For some $j\ge 0$, $\mu_j(g)=0$.*
(ii) *The series $\sum_{j\ge 0}(1-|\mu_j(g)|)$ diverges.*

Proof. Assume first that $M(g)=0$. Then, by (2.2) of Lemma 2,

$$\prod_{i=0}^{\infty}\mu_i(g) = \lim_{j\to\infty}\prod_{i=0}^{j}\mu_i(g) = \lim_{j\to\infty}\frac{1}{Q_{j+1}}\sum_{n=0}^{Q_{j+1}-1} g(n) = 0.$$

By Lemma 3, this implies that at least one of conditions (i) or (ii) holds.

Conversely, assume that at least one of conditions (i) or (ii) holds. Applying Lemma 3 again, we conclude that $\prod_{j=0}^{\infty} \mu_j(g) = 0$. We now show that $M(g)$ exists and is equal to 0. Let N be a positive integer with base-Q representation $\sum_{j=0}^{i} a_j Q_j$, where $a_i > 0$. Applying Lemma 4 with $h = \lfloor i/2 \rfloor$, we obtain

$$\begin{aligned}\Bigg|\frac{1}{N}\sum_{0\le n<N} g(n)\Bigg| &< \sum_{j=0}^{i} \frac{a_j Q_j}{N}\Bigg|\prod_{k=0}^{j-1}\mu_k(g)\Bigg| < 2^{1-\lfloor i/2\rfloor} + \sum_{j=i-\lfloor i/2\rfloor+1}^{i} \frac{a_j Q_j}{N}\Bigg|\prod_{k=0}^{j-1}\mu_k(g)\Bigg| \\ &\le 2^{2-i/2} + \sum_{j=\lfloor i/2\rfloor}^{i} \frac{a_j Q_j}{N}\Bigg|\prod_{k=0}^{j-1}\mu_k(g)\Bigg|.\end{aligned}$$

Since i tends to infinity as N tends to infinity and $\sum_{j=\lfloor i/2\rfloor}^{i} a_j Q_j \le N$, the right-hand side tends to 0 as N tends to infinity. Thus, $M(g) = 0$. This completes the proof of Theorem 2.

We now characterize those Q-multiplicative functions of modulus at most 1 having a non-zero mean value, a case that was not considered by Coquet. This characterization is the content of the following theorem which represents an analog of the well known mean value theorem of Delange [De1], and generalizes a result of Delange [De3] for the case of ordinary base-q expansions.

Theorem 3. *Let g be a Q-multiplicative function satisfying $|g| \le 1$. The mean value $M(g)$ exists and is non-zero if and only if the following three conditions all hold*:

(i) *For each $j \ge 0$, $\mu_j(g) \ne 0$.*
(ii) $\sum_{j\ge 0}(1 - \mu_j(g))$ *converges.*
(iii) $\lim_{j\to\infty} \sigma_j(g) = 0$.

Proof. For simplicity of notation, we will write $\mu_j = \mu_j(g)$ and $\sigma_j = \sigma_j(g)$ throughout the proof.

Assume first that $M(g) = L$ for some number $L \ne 0$. Then, in particular, we have

$$\lim_{j\to\infty} \frac{1}{Q_j} \sum_{0\le n<Q_j} g(n) = L.$$

By (2.2) of Lemma 2, this implies that $\prod_{j=0}^{\infty} \mu_j = L$. By Lemma 3, the convergence of the product $\prod_{j=0}^{\infty} \mu_j$ to a non-zero value implies that conditions (i) and (ii) of the theorem hold.

It remains to show that condition (iii) also holds, i.e., we wish to show that the quantity

$$\sigma_j = \max_{0<n\le q_j} \frac{1}{n} \sum_{0\le m<n} (1 - \mathrm{Re}(g_j(m)))$$

tends to zero as j tends to infinity. Let $\{n_j\}_{j=0}^{\infty}$ be a sequence of integers such that the maximum in the definition of σ_j is attained at $n = n_j$, so that

$$\sigma_j = 1 - \mathrm{Re}\left(\frac{1}{n_j} \sum_{0\le m<n_j} g_j(m)\right).$$

Applying (2.1) of Lemma 2 with $r = n_j$, we obtain

$$\lim_{j\to\infty} \frac{1}{n_j} \sum_{0\le m<n_j} g_j(m) = \lim_{j\to\infty} \frac{(1/n_jQ_j)\sum_{0\le n<n_jQ_j} g(n)}{(1/Q_j)\sum_{0\le n<Q_j} g(n)} = \frac{L}{L} = 1.$$

Therefore, σ_j tends to 0 as j tends to infinity, which proves condition (iii).

Conversely, assume that conditions (i), (ii), and (iii) all hold. The first two conditions imply that the infinite product $\prod_{j=0}^{\infty} \mu_j$ converges to a non-zero value, by Lemma 3. Let L denote this value. We will show that $M(g)$ exists and is equal to L.

We first note that, by the bound $|g_j| \le 1$ and the general inequality

$$|1-z|^2 = 1 + |z|^2 - 2\,\mathrm{Re}\, z \le 2(1 - \mathrm{Re}\, z) \qquad (|z| \le 1),$$

condition (iii) is equivalent to

$$\text{(iii)}' \qquad \lim_{j\to\infty} \max_{0<n\le q_j} \left|1 - \frac{1}{n} \sum_{0\le m<n} g_j(m)\right| = 0.$$

Furthermore, (iii)$'$ implies that

$$\text{(3.1)} \qquad \lim_{j\to\infty} g_j(m) = 1$$

for any fixed m.

Let $\varepsilon > 0$ be given and choose h and i_0 such that $2^{1-h} < \varepsilon$, and for $i \ge i_0$ we have the following three conditions:

(a) $|\prod_{j=0}^{i-1} \mu_j - L| < \varepsilon$.

(b) $|(1/n)\sum_{0\le m<n} g_i(m) - 1| < \varepsilon \quad (0 < n \le q_i)$.

(c) $|\prod_{k<j\le i} g_j(m_j) - 1| < \varepsilon \quad (i - h \le k \le i,\ 0 \le m_j < 1/\varepsilon)$.

Condition (a) is possible since $\prod_{j=0}^{\infty} \mu_j = L$, while conditions (b) and (c) can be met in view of condition (iii)$'$ and (3.1).

Let N be a positive integer with base-Q representation $N=\sum_{j=0}^{i} a_jQ_j$ where $a_i>0$, and suppose that N is sufficiently large and $i>i_0+h$. Applying Lemma 4, we have

$$\begin{aligned}
&\left|\frac{1}{N}\sum_{0\le n<N} g(n)-L\right| \\
&\quad=\left|\sum_{j=0}^{i}\frac{a_jQ_j}{N}\left(\Bigl(\prod_{m=j+1}^{i} g_m(a_m)\Bigr)\left(\frac{1}{a_j}\sum_{0\le n<a_j} g_j(n)\right)\Bigl(\prod_{k=0}^{j-1}\mu_k\Bigr)-L\right)\right| \\
&\quad\le\left|\sum_{j=i-h+1}^{i}\frac{a_jQ_j}{N}\left(\Bigl(\prod_{m=j+1}^{i} g_m(a_m)\Bigr)\left(\frac{1}{a_j}\sum_{0\le n<a_j} g_j(n)\right)\Bigl(\prod_{k=0}^{j-1}\mu_k\Bigr)-L\right)\right| \\
&\qquad+2\sum_{j=0}^{i-h}\frac{a_jQ_j}{N},
\end{aligned}$$

where in the last step we have used the fact that g_m, g, μ_k, and L are at most 1 in modulus. By inequality (2.4) of Lemma 4, the second term on the right hand side is at most $2(2^{1-h})<2\varepsilon$. Moreover, by the triangle inequality, the first term is bounded by

$$\begin{aligned}
&\sum_{j=i-h+1}^{i}\frac{a_jQ_j}{N}\left|\Bigl(\prod_{m=j+1}^{i} g_m(a_m)\Bigr)-1\right|\cdot\left|\left(\frac{1}{a_j}\sum_{0\le n<a_j} g_j(n)\right)\Bigl(\prod_{k=0}^{j-1}\mu_k\Bigr)\right| \\
&\qquad+\sum_{j=i-h+1}^{i}\frac{a_jQ_j}{N}\left(\left|\frac{1}{a_j}\sum_{0\le n<a_j} g_j(n)-1\right|\cdot\left|\prod_{k=0}^{j-1}\mu_k\right|+\left|\prod_{k=0}^{j-1}\mu_k-L\right|\right) \\
&\quad=\Sigma_1+\Sigma_2,
\end{aligned}$$

say. Since $i-h>i_0$, we have, by assumptions (a) and (b) above,

$$\Sigma_2<2\varepsilon\sum_{j=i-h+1}^{i}\frac{a_jQ_j}{N}\le 2\varepsilon.$$

To estimate Σ_1, we distinguish two cases. If $a_j<1/\varepsilon$ for all j with $i-h<j\le i$, then by assumption (c), $|(\prod_{m=j+1}^{i} g_m(a_m))-1|<\varepsilon$ for all j and therefore $\Sigma_1<\varepsilon$. Otherwise, let j_0 be the largest value of j in the range $i-h<j\le i$ for which $a_{j_0}\ge 1/\varepsilon$. The contribution of terms with $j_0\le j\le i$ to Σ_1 is, as before, at most ε. Thus,

$$\Sigma_1<\varepsilon+\sum_{j=i-h+1}^{j_0-1}\frac{a_jQ_j}{N}\left|\Bigl(\prod_{m=j+1}^{i} g_m(a_m)\Bigr)-1\right|\cdot\left|\left(\frac{1}{a_j}\sum_{0\le n<a_j} g(n)\right)\Bigl(\prod_{k=0}^{j-1}\mu_k\Bigr)\right|$$

$$\le \varepsilon + 2\sum_{j=i-h+1}^{j_0-1} \frac{a_j Q_j}{N} \le \varepsilon + 2\sum_{j=0}^{j_0-1} \frac{(q_j-1)Q_j}{N} = \varepsilon + 2\sum_{j=0}^{j_0-1} \frac{Q_{j+1}-Q_j}{N}$$
$$< \varepsilon + \frac{2Q_{j_0}}{N} \le \varepsilon + \frac{2Q_{j_0}}{a_{j_0}Q_{j_0}} \le 3\varepsilon.$$

In either case, we have

$$\left|\frac{1}{N}\sum_{0\le n<N} g(n) - L\right| < 7\varepsilon.$$

Since ε was arbitrary, we have shown that $M(g) = L$. This completes the proof of the theorem.

The following result is an immediate consequence of Theorems 2 and 3.

THEOREM 4. *Let g be a Q-multiplicative function satisfying $|g| \le 1$. The mean value $M(g)$ exists if and only if at least one of the following three conditions holds*:

(i) *For some $j \ge 0$, $\mu_j(g) = 0$.*
(ii) *The series $\sum_{j\ge 0}(1-|\mu_j(g)|)$ diverges.*
(iii) $\sum_{j\ge 0}(1-\mu_j(g))$ *converges*, *and* $\lim_{j\to\infty}\sigma_j(g) = 0$.

The mean value is zero if either condition (i) *or* (ii) *holds.*

4. Proof of Theorem 1. Let $Q = \{Q_j\}_{j\ge 0}$ be a mixed radix system, with factors $q_j = Q_{j+1}/Q_j$, and let f be a real-valued Q-additive function with component functions f_j.

For each integer $k \ne 0$, we set $g^{(k)}(n) = e(kf(n))$. Then each function $g^{(k)}$ is a Q-multiplicative function with component functions $g_j^{(k)}(n) = e(kf_j(n))$. We write

$$\mu_j^{(k)} = \mu_j(g^{(k)}) = \frac{1}{q_j}\sum_{0\le n<q_j} g_j^{(k)}(n),$$

and

$$\sigma_j^{(k)} = \sigma_j(g^{(k)}) = \max_{0<n\le q_j} \frac{1}{n}\sum_{0\le m<n}(1-\mathrm{Re}(g_j^{(k)}(n))),$$

and denote the mean value of $g^{(k)}$ by M_k, whenever this mean value exists. By Lemma 1, f has a limit distribution modulo 1 if and only if, for each integer $k \ne 0$, the mean value M_k exists, and the distribution is uniform if and only if, for each integer $k \ne 0$, $M_k = 0$. By Theorem 4, for each $k \ne 0$, M_k exists if and only if at least one of the following three conditions holds:

(i)$_k$ For some $j \ge 0$, $\mu_j^{(k)} = 0$.
(ii)$_k$ The series $\sum_{j\ge 0}(1-|\mu_j^{(k)}|)$ diverges.

(iii)$_k$ $\sum_{j\geq 0}(1-\mu_j^{(k)})$ converges, and $\lim_{j\to\infty}\sigma_j^{(k)}=0$.

Further, $M_k=0$ if and only if either condition (i)$_k$ or (ii)$_k$ holds. Therefore, it remains only to show that, for each integer $k\neq 0$, conditions (i)$_k$, (ii)$_k$, and (iii)$_k$ are equivalent to conditions (i), (ii), and (iii) of Theorem 1, respectively.

To prove this, we fix an integer $k\neq 0$. Conditions (i)$_k$ and (ii)$_k$ are identical to conditions (i) and (ii) of Theorem 1, respectively, by the definition of $\mu_j^{(k)}$. The equivalence between condition (iii)$_k$ and condition (iii) of Theorem 1 follows from the definition of $\mu_j^{(k)}$ and the relation

$$\sigma_j^{(k)}=\max_{0<n\leq q_j}\frac{1}{n}\sum_{0\leq m<n}(1-\mathrm{Re}(g_j^{(k)}(n)))\asymp\max_{0<n\leq q_j}\frac{1}{n}\sum_{0\leq m<n}\|kf_j(n)\|^2,$$

which holds since

$$1-\mathrm{Re}\,e(x)=1-\cos(2\pi x)\asymp\|x\|^2$$

for any real number x. This completes the proof of Theorem 1.

5. Proof of the corollaries

Proof of Corollary 1. Fix an integer $k\neq 0$. For each $j\geq 0$, let n_j be such that $\max_{0\leq n<q_j}\|kf_j(n)\|^2$ is attained at $n=n_j$. First we note that by the elementary inequality

$$|1+e(x)|\leq 2-2\|x\|^2 \qquad (x\in\mathbb{R}),$$

we have, for all j,

$$\begin{aligned}\frac{1}{q_j}\Big|\sum_{0\leq n<q_j}e(kf_j(n))\Big| &\leq \frac{1}{q_j}\Big|\sum_{\substack{1\leq n<q_j\\ n\neq n_j}}e(kf_j(n))\Big|+\frac{1}{q_j}|1+e(kf_j(n_j))|\\ &\leq\frac{1}{q_j}((q_j-2)+|1+e(kf_j(n_j))|)\\ &\leq\frac{1}{q_j}((q_j-2)+(2-2\|kf_j(n_j)\|^2))\\ &=\frac{1}{q_j}(q_j-2\|kf_j(n_j)\|^2)\end{aligned}$$

and thus

$$1-\frac{1}{q_j}\Big|\sum_{0\leq n<q_j}e(kf_j(n))\Big|\geq\frac{2\|kf_j(n_j)\|^2}{q_j}\geq\frac{2}{q_j^2}\sum_{0\leq n<q_j}\|kf_j(n_j)\|^2.$$

Since, by assumption, the factors q_j are bounded, the divergence of the series

$$\sum_{j\geq 0}\sum_{0\leq n<q_j}\|kf_j(n)\|^2 \tag{5.1}$$

implies that condition (ii) of Theorem 1 holds. Hence, if for all $k \neq 0$ either the series in (5.1) diverges or condition (i) of Theorem 1 holds, then Theorem 1 implies that f has a uniform limit distribution modulo 1.

Conversely, assume that f is uniformly distributed modulo 1. Then, for each $k \neq 0$, either condition (i) or condition (ii) of Theorem 1 holds. We will show that if condition (ii) holds for some $k \neq 0$ then the series in (5.1) diverges. Fix $k \neq 0$. Since, for all real x,

$$1 - \operatorname{Re} e(x) = 1 - \cos(2\pi x) \leq 2\pi^2 \|x\|^2,$$

we have, for all j,

$$\begin{aligned} 1 - \frac{1}{q_j} \Big| \sum_{0 \leq n < q_j} e(kf_j(n)) \Big| &\leq 1 - \frac{1}{q_j} \operatorname{Re} \sum_{0 \leq n < q_j} e(kf_j(n)) \\ &= \frac{1}{q_j} \sum_{0 \leq n < q_j} (1 - \operatorname{Re} e(kf_j(n))) \\ &\leq \frac{1}{q_j} \sum_{0 \leq n < q_j} 2\pi^2 \|kf_j(n)\|^2 \\ &\leq \pi^2 \sum_{0 \leq n < q_j} \|kf_j(n)\|^2. \end{aligned}$$

Thus, condition (ii) of Theorem 1 implies the divergence of the series in (5.1), as claimed.

Proof of Corollary 2. Assume first that α is irrational. If the factors q_j are bounded, then, since $f_j(1) = \alpha$ for all j, it follows from Corollary 1 that f is uniformly distributed modulo 1. It remains to deal with the case when the factors q_j are unbounded.

Fix $k \neq 0$. Then we have, for all $j \geq 0$,

$$\begin{aligned} \frac{1}{q_j} \Big| \sum_{0 \leq n < q_j} e(kf_j(n)) \Big| &= \frac{1}{q_j} \Big| \sum_{0 \leq n < q_j} e(k\alpha n) \Big| \\ &= \frac{1}{q_j} \left| \frac{1 - e(k\alpha q_j)}{1 - e(k\alpha)} \right| \leq \frac{2}{q_j(1 - e(\alpha k))}. \end{aligned}$$

Since the factors q_j are unbounded and α is irrational, this quantity is $\leq 1/2$ for infinitely many j, and so condition (ii) of Theorem 1 is satisfied. Therefore, f has a uniform limit distribution modulo 1.

On the other hand, if α is rational, then f takes on only finitely many values modulo 1, and thus f cannot be uniformly distributed modulo 1.

Proof of Corollary 3. Let $F = \alpha f$. Then F is completely Q-additive with component functions $F_j = \alpha f_j$. As in Corollary 2, if α is rational then F cannot be uniformly distributed modulo 1. Assume therefore that α is irrational. We will show that condition (ii) of Theorem 1 is satisfied (with F in place of f) for all $k \neq 0$. By Theorem 1 it then follows that F is uniformly distributed modulo 1. Fix an integer $k \neq 0$ and let a be as in the statement of the corollary, so that $f_0(a) > 0$ and $a < q_j$ for all j. As in the proof of Corollary 1, we have, for all j,

$$1 - \frac{1}{q_j}\Big|\sum_{0 \le n < q_j} e(kF_j(n))\Big| \ge \frac{2\|kF_j(a)\|^2}{q_j} = \frac{2\|k\alpha f_j(a)\|^2}{q_j} = \frac{2\|k\alpha f_0(a)\|^2}{q_j}.$$

Since $\|k\alpha f_0(a)\| \neq 0$ by our assumptions that α is irrational, $k \neq 0$, and $f_0(a) \neq 0$, and since, by the hypothesis of Corollary 2, the series $\sum_{j \ge 0} 1/q_j$ diverges, condition (ii) of Theorem 1 is satisfied as claimed.

Proof of Corollary 4. We note first that the component functions $f_j(n)$ of $f(n) = \alpha M(n)$ are given by

$$f_j(n) = \begin{cases} \alpha, & n = q_j - 1, \\ 0, & 0 \le n < q_j - 1. \end{cases}$$

Thus we have, for any integer $k \neq 0$,

$$\sum_{0 \le n < q_j} e(kf_j(n)) = q_j - 1 + e(k\alpha). \tag{5.2}$$

It follows that condition (i) of Theorem 1 is satisfied if and only if, for some j, $q_j = 2$ and $\|k\alpha\| = 0$. Since α is irrational, this is impossible unless $k = 0$. Therefore, condition (i) of Theorem 1 does not hold for any $k \neq 0$.

We next show that condition (ii) of Theorem 1 is equivalent to the divergence of $\sum_{j \ge 0} 1/q_j$. In view of (5.2), condition (ii) of Theorem 1 is equivalent to

$$\sum_{j \ge 0} \left(1 - \frac{1}{q_j}\,|q_j - 1 + e(k\alpha)|\right) = \infty. \tag{5.3}$$

To show the equivalence between (5.3) and the divergence of $\sum_{j \ge 0} 1/q_j$, we will establish the inequalities

$$\frac{4\|k\alpha\|^2}{q} \le 1 - \frac{1}{q}\,|q - 1 + e(k\alpha)| \le \frac{2}{q} \tag{5.4}$$

for any integer $q \ge 2$ and any real number α.

The upper bound in (5.4) is trivial. To prove the lower bound, we note that

$$\begin{aligned}\left(\frac{1}{q}\,|q - 1 + e(k\alpha)|\right)^2 &\le \frac{1}{q^2}((q-1)^2 + 1 + 2(q-1)\cos(2\pi k\alpha)) \\ &= \frac{1}{q^2}(q^2 - 2(q-1)(1 - \cos(2\pi k\alpha)))\end{aligned}$$

$$
\begin{aligned}
&= 1 - \frac{2(q-1)}{q^2}(1-\cos(2\pi k\alpha)) \\
&\le \left(1 - \frac{q-1}{q^2}(1-\cos(2\pi k\alpha))\right)^2 \\
&\le \left(1 - \frac{(q-1)(8\|k\alpha\|^2)}{q^2}\right)^2 \\
&\le \left(1 - \frac{4\|k\alpha\|^2}{q}\right)^2.
\end{aligned}
$$

It follows that

$$\frac{1}{q}|q-1+e(k\alpha)| \le 1 - \frac{4\|k\alpha\|^2}{q},$$

which implies the lower bound in (5.4). Since α is irrational, we have $\|k\alpha\| \neq 0$ for all non-zero integers k. Thus condition (ii) of Theorem 1 holds for all $k \neq 0$ if and only if $\sum_{j\ge 0} 1/q_j$ diverges. From the theorem it therefore follows that f is uniformly distributed modulo 1 if and only if $\sum_{j\ge 0} 1/q_j$ diverges.

It remains to show that f has a non-uniform limit distribution modulo 1 if and only if the series $\sum_{j\ge 0} 1/q_j$ converges. To this end we note that, by (5.2), the first part of condition (iii) of Theorem 1 is equivalent to the convergence of

$$\sum_{j\ge 0} \frac{1-e(k\alpha)}{q_j},$$

which in turn is equivalent to the convergence of $\sum_{j\ge 0} 1/q_j$, since α is irrational. Therefore it remains only to show that if $\sum_{j\ge 0} 1/q_j$ converges, then the second part of condition (iii) of Theorem 2.1 holds for all $k \neq 0$. This follows immediately from the observation that, for all $k \neq 0$,

$$\max_{0<n\le q_j} \frac{1}{n} \sum_{0\le m<n} \|kf_j(m)\|^2 = \frac{\|k\alpha\|^2}{q_j} \to 0,$$

as j tends to infinity, since the convergence of $\sum_{j\ge 0} 1/q_j$ implies that $1/q_j$ tends to 0.

Proof of Corollary 5. We note first that, for all j with $q_j > a$,

$$f_j(n) = \begin{cases} \alpha, & n = a, \\ 0, & \text{otherwise}. \end{cases} \tag{5.5}$$

Thus, we have

$$\sum_{0\le n<q_j} e(kf_j(n)) = \begin{cases} q_j - 1 + e(k\alpha), & q_j > a, \\ q_j, & q_j \le a. \end{cases} \tag{5.6}$$

As in the proof of Corollary 4, this implies that condition (i) of Theorem 1 does not hold for any $k \neq 0$. Moreover, using (5.4), we see that condition (ii) of Theorem 1 is satisfied for all $k \neq 0$ if and only if $\sum_{q_j > a} 1/q_j$ diverges. Therefore, f is uniformly distributed modulo 1 if and only if $\sum_{q_j > a} 1/q_j$ diverges. This proves the first assertion of the corollary.

To prove the second assertion of the corollary, we note that by (5.5), we have, for all $k \neq 0$,

$$\max_{0<n\le q_j} \frac{1}{n} \sum_{0\le m<n} \|kf_j(m)\|^2 = \begin{cases} \|k\alpha\|^2/a, & q_j > a, \\ 0, & q_j \le a. \end{cases}$$

Therefore, the limit in condition (iii) of Theorem 1 is 0 for all $k \neq 0$ if and only if $q_j \le a$ for all but at most finitely many j. It remains only to show that under the same condition, the series in condition (iii) of Theorem 1 converges for all $k \neq 0$. This follows immediately, since, by (5.6),

$$\sum_{j\ge 0} \left(1 - \frac{1}{q_j} \sum_{0\le n<q_j} e(kf_j(n))\right) = \sum_{q_j > a} \frac{1 - e(k\alpha)}{q_j}.$$

Acknowledgments. The author wishes to gratefully acknowledge the advice and encouragement provided by Professor A. J. Hildebrand in the preparation of this paper.

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Received on 19.7.1999
and in revised form on 11.4.2001 (3651)