The Tate conjecture for certain abelian varieties over finite fields

by

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In an earlier work, we showed that if the Hodge conjecture holds for all complex abelian varieties of CM-type, then the Tate conjecture holds for all abelian varieties over finite fields (Milne 1999b). In this article, we extract from the proof a statement (Theorem 1.1) that sometimes allows one to deduce the Tate conjecture for the powers of a single abelian variety A over a finite field from knowing that some Hodge classes on their lifts to characteristic zero are algebraic.

Tate's theorem (Tate 1966) implies that the Tate conjecture holds for any abelian variety over a finite field whose $\mathbb{Q}_\ell$-algebra of Tate classes is generated by those of degree 1. Examples are known of abelian varieties for which this condition (and hence the Tate conjecture) hold (Lenstra, Spiess, Zarhin; see the examples in A.7 below). Using Theorem 1.1 and a result of Schoen (1988, 1998), we construct examples of abelian varieties for which the condition fails, but for which we are nevertheless able to prove the Tate conjecture (see 1.7, 1.8).

The main results are stated in Section 1 and proved in Section 2. Appendix A summarizes the theories of abelian varieties of CM-type over $\mathbb{C}$ and of abelian varieties over finite fields, and how the reduction map relates the two. Appendix B sharpens a result of Clozel on the relation between numerical and homological equivalence for abelian varieties over finite fields.

Notations not introduced in Section 1 are listed at the start of Appendix A.

1. Statements. Let X be a smooth complete variety over an algebraic closure $\mathbb{F}$ of the field $\mathbb{F}_p$ of p elements. The choice of a model X_1 of X over a subfield $\mathbb{F}_{p^n}$ of $\mathbb{F}$ determines an action of $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_{p^n})$ on the étale cohomology group $H^{2r}(X, \mathbb{Q}_\ell(r))$, and we define

$$\mathcal{T}_\ell^r(X) = \bigcup_{X_1/\mathbb{F}_{p^n}} H^{2r}(X, \mathbb{Q}_\ell(r))^{\mathrm{Gal}(\mathbb{F}/\mathbb{F}_{p^n})}$$

(union over all models). The elements of $\mathcal{T}_\ell^r(X)$ are called the *ℓ-adic Tate classes of degree r* on X. We shall say that *the Tate conjecture holds for X* if the $\mathbb{Q}_\ell$-vector space $\mathcal{T}_\ell^r(X)$ is spanned by the classes of algebraic cycles for all r and all $\ell \neq p$.

A Tate class is said to be *exotic* if it is not in the $\mathbb{Q}_\ell$-algebra generated by the Tate classes of degree 1. For an abelian variety over $\mathbb{F}$, Tate (1966) showed that all Tate classes of degree 1 are divisor classes, and so the nonexotic Tate classes are algebraic.

Let A_0 be an abelian variety over $\mathbb{F}$. If the Tate conjecture holds for A_0, then the equivalent statements of Tate 1994, Theorem 2.9, hold for every model A_1 of A over a finite field. In particular, the Tate conjecture holds for $A_1/\mathbb{F}_q$, and for every r, the order of the pole of the zeta function $Z(A_1, t)$ of A_1 at $t = q^{-r}$ is equal to the rank of the group of numerical equivalence classes of algebraic cycles of codimension r on A_1.

Let A be an abelian variety with many endomorphisms (see A.2) over an algebraically closed field k. Then (see A.3) there is a group of multiplicative type $L(A)$ over $\mathbb{Q}$ whose fixed tensors in any Weil cohomology of a power A^s of A are exactly the *Lefschetz classes*, i.e., those in the algebra generated by divisor classes. We call $L(A)$ the *Lefschetz group* of A.

Now take k to be the algebraic closure $\mathbb{Q}^{\mathrm{al}}$ of $\mathbb{Q}$ in $\mathbb{C}$, and let w_0 be a prime of $\mathbb{Q}^{\mathrm{al}}$ dividing p. It follows from the theory of Néron models, that A has good reduction at w_0 (Serre and Tate 1968, Theorem 6) and so defines an abelian variety A_0 over the residue field $\mathbb{F}$ at w. There is a canonical inclusion $L(A_0) \hookrightarrow L(A)$ (see A.3, Remark).

Let $H^r(A, \mathbb{Q})$ denote the usual cohomology group of the complex manifold $A(\mathbb{C})$, and let $H^r(A, \mathbb{Q}(m)) = H^r(A, (2\pi i)^m \mathbb{Q})$—it is a rational Hodge structure of weight $r - 2m$. The action of $L(A)$ on $H^{2r}(A^s, \mathbb{Q}(r))$ defines a decomposition

$$H^{2r}(A^s, \mathbb{Q}(r)) \otimes \mathbb{Q}^{\mathrm{al}} = \bigoplus_{\chi \in X^*(L(A))} H^{2r}(A^s, \mathbb{Q}(r))_\chi$$

where $(-)_\chi$ is the subspace on which $L(A)$ acts through its character χ. We say that χ is *algebraic* if $H^{2r}(A^s, \mathbb{Q}(r))_\chi$ contains a nonzero algebraic class for some r and s. The set of algebraic characters of $L(A)$ is stable under the action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$, and if χ is algebraic then $H^{2r}(A^s, \mathbb{Q}(r))_\chi$ consists entirely of algebraic classes [1]. By composition, an algebraic character of $L(A)$ defines a character of $L(A_0)$.

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[1] The algebraic characters are precisely those that are trivial on the subgroup $M(A)$ of $L(A)$ (see A.3).

A model $A_1/\mathbb{F}_q$ of A_0 over a finite field defines a Frobenius endomorphism π of A_0. Some power of π lies in $L(A_0)(\mathbb{Q})$, and we define $P(A_0)$ to be the smallest algebraic subgroup of $L(A_0)$ containing a power of π (see A.3).

THEOREM 1.1. *If*

$$P(A_0) = \bigcap \mathrm{Ker}(\chi : L(A_0) \to \mathbb{G}_m)$$

(intersection over the algebraic characters of $L(A)$), then the Tate conjecture holds for all powers of A_0.

An element of $H^{2r}(A, \mathbb{Q}(r)) \cap H^{0,0}$ is called a *Hodge class of degree r* on A. We say that *the Hodge conjecture holds for A* if the $\mathbb{Q}$-vector space of Hodge classes on A of degree r is spanned by the classes of algebraic cycles for all r. The *Mumford–Tate group* $\mathrm{MT}(A)$ of A is the largest algebraic subgroup of $L(A)$ fixing the Hodge classes on all powers of A.

COROLLARY 1.2. *If the Hodge conjecture holds for all powers of A and*

$$P(A_0) = L(A_0) \cap \mathrm{MT}(A) \quad (\textit{intersection inside } L(A)),$$

then the Tate conjecture holds for all powers of A_0.

A Hodge class is said [2] to be *exotic* if it is not in the $\mathbb{Q}$-algebra generated by Hodge classes of degree 1. Lefschetz showed that all Hodge classes of degree 1 are divisor classes, and so the nonexotic Hodge classes are exactly the Lefschetz classes (in particular, they are algebraic).

Let E be a CM-field of degree $2n$, $n > 2$, over $\mathbb{Q}$ containing a quadratic imaginary field Q. Choose an embedding $\varrho_0 : Q \to \mathbb{Q}^{\mathrm{al}}$, and let $\{\sigma_0, \ldots, \sigma_{n-1}\}$ be the set of extensions of ϱ_0 to E. Then $\Phi_0 =_{\mathrm{df}} \{\sigma_0, \iota \circ \sigma_1, \ldots, \iota \circ \sigma_{n-1}\}$ (ι denotes complex conjugation on $\mathbb{C}$) is a CM-type on E and $\Phi =_{\mathrm{df}} \{\varrho_0\}$ is a CM-type on Q. Let (A, i) and (B, j) be abelian varieties over $\mathbb{Q}^{\mathrm{al}}$ of CM-types (E, Φ_0) and (Q, Φ) respectively. We let Q act diagonally on $A \times B^{n-2}$.

LEMMA 1.3. *The exotic Hodge classes on $A \times B^{n-2}$ are exactly the nonzero elements of the subspace*

$$W(A, B) \stackrel{\mathrm{df}}{=} \Big(\bigwedge\nolimits_Q^{2n-2} H^1(A \times B^{n-2}, \mathbb{Q}) \Big)(n-1)$$

of $H^{2n-2}(A \times B^{n-2}, \mathbb{Q}(n-1))$.

As $A \times B^{n-2}$ has dimension $2n - 2$, $H^1(A \times B^{n-2}, \mathbb{Q})$ has dimension $4n - 4$ over $\mathbb{Q}$, and so $\bigwedge_Q^{2n-2} H^1(A \times B^{n-2}, \mathbb{Q})$ has dimension 1 over Q. The action of an endomorphism of an abelian variety on its cohomology groups preserves algebraic classes, and so, if $W(A, B)$ contains one nonzero algebraic class, then it is spanned as a $\mathbb{Q}$-space by algebraic classes.

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[2] Following Tate 1994, p. 82.

THEOREM 1.4. *If some exotic Hodge class on $A \times B^{n-2}$ is algebraic, then the Hodge conjecture holds for all abelian varieties of the form $A^s \times B^t$, $s, t \in \mathbb{N}$.*

The abelian varieties A and B over $\mathbb{Q}^{\mathrm{al}}$ reduce modulo w_0 to abelian varieties A_0 and B_0 over $\mathbb{F}$. Let K be the Galois closure of $\sigma_0 E$ in $\mathbb{Q}^{\mathrm{al}}$, and let $D(w_0)$ be the decomposition group of w_0 in $\mathrm{Gal}(K/\mathbb{Q})$.

THEOREM 1.5. *Assume p splits in Q and that $\mathrm{Gal}(K/\sigma_0 E) \cdot D(w_0)$ is a subgroup of $\mathrm{Gal}(K/\mathbb{Q})$.*

(a) *For all $\ell \neq p$, the exotic ℓ-adic Tate classes on $A_0 \times B_0^{n-2}$ are exactly the nonzero elements of the subspace*

$$W(A_0, B_0) \stackrel{\mathrm{df}}{=} \Big(\bigwedge\nolimits_{Q \otimes_{\mathbb{Q}} \mathbb{Q}_\ell}^{2n-2} H^1(A_0 \times B_0^{n-2}, \mathbb{Q}_\ell) \Big)(n-1)$$

of $H^{2n-2}(A_0 \times B_0^{n-2}, \mathbb{Q}_\ell(n-1))$.

(b) *If some exotic Hodge class on $A \times B^{n-2}$ is algebraic, then the Tate conjecture holds for all abelian varieties over $\mathbb{F}$ of the form $A_0^s \times B_0^t$, $s, t \in \mathbb{N}$.*

REMARK 1.6. (a) Note that, under the hypotheses of the theorem, the $\mathbb{Q}$-algebra of Hodge classes on $A \times B^{n-2}$ is larger than the tensor product of the similar algebras for A and B^{n-2}, and the $\mathbb{Q}_\ell$-algebra of Tate classes on $A_0 \times B_0^{n-2}$ is larger than the tensor product of the similar algebras for A_0 and B_0^{n-2}. Moreover, the groups $L(A \times B)$ and $\mathrm{MT}(A \times B)$ (resp. $L(A_0 \times B_0)$ and $P(A_0 \times B_0)$) are not distinguished by their fixed tensors in the cohomology of $A \times B$ (resp. $A_0 \times B_0$).

(b) The condition that $\mathrm{Gal}(K/\sigma_0 E) \cdot D(w_0)$ be a subgroup of $\mathrm{Gal}(K/\mathbb{Q})$ holds, for example, if E is Galois over $\mathbb{Q}$. Without it, the analysis becomes very complicated, and the theorem fails.

Examples. Let C be an abelian variety over $\mathbb{C}$, and let $i : Q \to \mathrm{End}^0(C)$ be a homomorphism of $\mathbb{Q}$-algebras, where, as above, Q is a quadratic imaginary extension of $\mathbb{Q}$. The pair (C, i) is said to be of *Weil type* if the tangent space to C at 0 is a free $Q \otimes_{\mathbb{Q}} \mathbb{C}$-module.

When (C, i) is of Weil type, its dimension is even, say, $\dim C = 2m$, and the subspace $(\bigwedge_Q^{2m} H^1(C, \mathbb{Q}))(m)$ of $H^{2m}(C, \mathbb{Q}(m))$ consists of Hodge classes (Weil 1977)—they are called the *Weil classes* on C.

Let λ be a polarization of C whose Rosati involution induces complex conjugation on Q, and let E^λ be the Riemann form defined by λ. There exists a skew-Hermitian form $\phi : H_1(A, \mathbb{Q}) \times H_1(A, \mathbb{Q}) \to Q$ such that $\mathrm{Tr}_{Q/\mathbb{Q}} \circ \phi = E$. The discriminant of ϕ is an element of $\mathbb{Q}^\times / \mathrm{Nm}(Q^\times)$ which is independent of the choice of the polarization, and so can be denoted by $\det(C, i)$. The quotient $\mathbb{Q}^\times / \mathrm{Nm}(Q^\times)$ is an infinite group killed by 2, and for any $a \in \mathbb{Q}^\times / \mathrm{Nm}(Q^\times)$ with $(-1)^m a > 0$, there exists an m^2-dimensional family of abelian varieties of Weil type with determinant a (Weil 1977, van

Geemen 1994). We say that ϕ is *split* when there is an m-dimensional Q-subspace of $H_1(A, \mathbb{Q})$ on which ϕ is totally isotropic.

The Weil classes on C are known to be algebraic in the following cases:

(a) $Q = \mathbb{Q}[\sqrt{-3}]$, $m = 3$, and ϕ is split (Schoen 1998; see also van Geemen 1994, 7.3, p. 250);

(b) $Q = \mathbb{Q}[\sqrt{-3}]$, $m = 2$ (Schoen 1988 when ϕ is split and Schoen 1998 in general);

(c) $Q = \mathbb{Q}[\sqrt{-1}]$, $m = 2$, and ϕ is split (van Geemen 1996).

COROLLARY 1.7. *Let A, B, E, Q be as in Theorem* 1.5, *and let Q act diagonally on $A \times B^{n-2}$. If the Weil classes on $A \times B^{n-2}$ are algebraic, then the Hodge conjecture holds for all abelian varieties of the form $A^s \times B^t$, $s, t \in \mathbb{N} \times \mathbb{N}$, and the Tate conjecture holds for all abelian varieties of the form $A_0^s \times B_0^t$, $s, t \in \mathbb{N} \times \mathbb{N}$.*

Proof. In this case, $W(A, B)$ is the space of Weil classes on $A \times B^{n-2}$. ∎

EXAMPLE 1.8. Let $Q = \mathbb{Q}[\sqrt{-3}]$ and let p be a prime such that $p \equiv 1$ (mod 3); let F be a totally real cubic extension of $\mathbb{Q}$ that is Galois over $\mathbb{Q}$ or such that p splits in it, and let $E = F \cdot Q$; let $\Phi = \{\varrho_0\}$ and $\Phi_0 = \{\sigma_0, \iota\sigma_1, \iota\sigma_2\}$ be the CM-types on Q and E respectively defined above. Then, for all abelian 3-folds A of CM-type (E, Φ_0) and all elliptic curves B of CM-type (Q, Φ),

(a) the Hodge conjecture holds for the abelian varieties $A^s \times B^t$, $s, t \in \mathbb{N}$; the subspace $W(A, B)$ of $H^4(A \times B, \mathbb{Q}(2))$ consists of exotic Hodge classes;

(b) the Tate conjecture holds for the abelian varieties $A_0^s \times B_0^t$, $s, t \in \mathbb{N}$; the subspace $W(A_0, B_0)$ of $H^4(A_0 \times B_0, \mathbb{Q}_\ell(2))$ consists of exotic Tate classes.

2. Proofs. Notations concerning groups of multiplicative type are reviewed at the start of Appendix A.

Proof of 1.1. After the Theorem in A.3, in order to prove Theorem 1.1, it suffices to show that its hypotheses imply that $M(A_0) = P(A_0)$.

As numerical equivalence agrees with homological equivalence on abelian varieties in characteristic zero (see B.1), we may regard $M(A)$ as the subgroup of $L(A)$ fixing the algebraic classes in $H^{2r}(A^s, \mathbb{Q}(r))$ for all r, s, i.e., as the intersection of the kernels of the algebraic characters on $L(A)$. Hence

$$L(A_0) \cap M(A) = \bigcap_{\chi \text{ algebraic}} \mathrm{Ker}(\chi : L(A_0) \to \mathbb{G}_m).$$

Thus, the hypotheses of Theorem 1.1 imply that $L(A_0) \cap M(A) = P(A_0)$. Since

$$L(A_0) \cap M(A) \supset M(A_0) \supset P(A_0),$$

this implies that $M(A_0) = P(A_0)$.

Proof of 1.2. As we noted in the proof of 1.1,

$$M(A) = \bigcap_{\chi \text{ algebraic}} \text{Ker}(\chi : L(A) \to \mathbb{G}_m).$$

If the Hodge conjecture holds for the powers of A, then $\text{MT}(A) = M(A)$ (see A.3). If, in addition, $P(A_0) = L(A_0) \cap \text{MT}(A)$, then

$$P(A_0) = \bigcap_{\chi} \text{Ker}(\chi : L(A_0) \to \mathbb{G}_m)$$

(intersection over the algebraic characters of $L(A)$), and so (1.2) follows from (1.1).

Proofs of 1.3 and 1.4. Let $E, Q, \varrho_0, \{\sigma_0, \ldots, \sigma_{n-1}\}, \Phi$ and Φ_0 be as in the paragraph preceding the statement of Lemma 1.3. Let K be a CM-subfield of $\mathbb{Q}^{\text{al}}$, finite and Galois over $\mathbb{Q}$, containing the Galois closure of $\sigma_0 E$ in $\mathbb{Q}^{\text{al}}$, and let S^K be its Serre group (see A.4). For each i, $0 \leq i \leq n-1$, let

$$\Sigma_i = \{\tau \in \text{Gal}(K/\mathbb{Q}) \mid \tau \circ \sigma_0 = \sigma_i\}.$$

Then Σ_0 is the subgroup $\text{Gal}(K/\sigma_0 E)$ of $\text{Gal}(K/\mathbb{Q})$ and $\Sigma_0, \ldots, \Sigma_{n-1}, \iota\Sigma_0,$ $\ldots, \iota\Sigma_{n-1}$ are its left cosets. Let ψ_i be the characteristic function of $\Sigma_i \cup \bigcup_{j \neq i} \iota\Sigma_j$, and let ψ be the characteristic function of $\bigcup \Sigma_i = \{\tau \mid \tau \circ \varrho_0 = \varrho_0\}$. Note that Σ_K acts on the set $\{\Sigma_0, \ldots, \iota\Sigma_{n-1}\}$, and that if $\tau\Sigma_i = \Sigma_{i'}$, then $\tau\psi_i = \psi_{i'}$. The linear relations among $\psi_0, \ldots, \psi_{n-1}, \psi, \iota\psi$ regarded as elements of $X^*(S^K)$ are exactly the multiples of

$$(*) \qquad \psi_0 + \ldots + \psi_{n-1} + (n-2)\psi = (n-1)(\psi + \iota\psi).$$

Let (A, i) be an abelian variety of CM-type (E, Φ_0), and let $X^*(L(A))$ be identified with a quotient of $\mathbb{Z}^{\Sigma_E}$ (see A.5). The map

$$X^*(\varrho_{\Phi_0}) : X^*(L(A)) \to X^*(S^K)$$

(ibid.) sends $[\sigma_0]$ to ψ_0 and hence, by equivariance and linearity, it sends $[\sigma_i]$ to ψ_i and $[\sigma_0 + \iota\sigma_0]$ to $\psi_0 + \iota\psi_0 = \psi + \iota\psi$. Because $[\sigma_0], \ldots, [\sigma_{n-1}], [\sigma_0 + \iota\sigma_0]$ form a basis for $X^*(L(A))$ and $\psi_0, \ldots, \psi_{n-1}, \psi + \iota\psi$ are linearly independent in $X^*(S^K)$, we see that $X^*(\varrho_{\Phi_0}) : X^*(L(A)) \to X^*(S^K)$ is injective. Therefore, $\varrho_{\Phi_0} : S^K \to L(A)$ is surjective, and $\text{MT}(A) = L(A)$ (ibid.). Hence all Hodge classes on all powers of A are Lefschetz (A.3, Theorem). In particular, the Hodge conjecture holds for A and its powers.

Let (B, i) be an elliptic curve of CM-type (Q, Φ). In this case, $X^*(L(B)) = \mathbb{Z}^{\Sigma_Q}$ and $L(B) = (\mathbb{G}_m)_{Q/\mathbb{Q}}$. The map $X^*(\varrho_\Phi)$ sends ϱ_0 to ψ and $\iota\varrho_0$ to $\iota\psi$. As ψ and $\iota\psi$ are linearly independent in $X(S^K)$, this shows that $\text{MT}(B) = L(B)$, and so all Hodge classes on all powers of B are Lefschetz.

The abelian variety A is simple because its CM-type is primitive (this uses that $n > 2$). The product $A \times B$ is of CM-type $(E \times Q, \Phi')$ where

$\Phi' = \Phi_0 \sqcup \Phi$. The group $X^*(L(A \times B))$, regarded as a quotient of $\mathbb{Z}^{\Sigma_E \sqcup \Sigma_Q}$, has basis $\{[\sigma_0], \ldots, [\sigma_{n-1}], [\varrho_0], [\varrho_0 + \iota\varrho_0]\}$, and $X^*(\varrho_{\Phi'})$ sends

$$[\sigma_i] \mapsto \psi_i, \quad [\varrho_0] \mapsto \psi, \quad [\varrho_0 + \iota\varrho_0] \mapsto \psi + \iota\psi.$$

Since $(*)$ is the only relation among $\psi_0, \ldots, \psi_{n-1}, \psi, \iota\psi$, the kernel of $X^*(L(A \times B)) \to X^*(S^K)$ is free of rank 1 with generator

$$\chi = [\sigma_0 + \ldots + \sigma_{n-1} + (n-2)\varrho_0 - (n-1)(\varrho_0 + \iota\varrho_0)].$$

As $\mathrm{MT}(A \times B)$ is the image of S^K in $L(A \times B)$ (see A.5), this shows that there is an exact sequence

$$0 \to \mathrm{MT}(A \times B) \to L(A \times B) \xrightarrow{\chi} T \to 0$$

where T is the 1-dimensional torus over $\mathbb{Q}$ whose character group $\langle\chi\rangle$ is isomorphic to $\mathbb{Z}$ with $\mathrm{Gal}(K/\mathbb{Q})$ acting nontrivially through $\mathrm{Gal}(\varrho_0 Q/\mathbb{Q})$.

The exotic Hodge classes on $A \times B$ and its powers are those that lie in a rational subspace on which $L(A \times B)$ acts through the characters $m\chi$, $m \neq 0$.

We now prove 1.3. The Lefschetz group of $A \times B^{n-2}$ equals that of $A \times B$. It acts on

$$W(A, B) = \bigwedge_Q^n H^1(A) \otimes \bigwedge_Q^{n-2}((n-2)H^1(B)) \otimes \mathbb{Q}(n-1)$$

through the characters χ and $\iota\chi = -\chi$. Because χ is trivial on $\mathrm{MT}(A \times B)$, this space consists of Hodge classes, and because χ is not trivial on $L(A \times B)$, the Hodge classes are exotic. The group $L(A \times B)$ acts on no other subspace of a space $H^{2r}(A \times B^{n-2}, \mathbb{Q}(r)) \otimes \mathbb{Q}^{\mathrm{al}}$ through the characters $\pm\chi$, and so the elements of $W(A, B)$ are the only exotic Hodge classes on $A \times B^{n-2}$.

We now prove 1.4. If some exotic Hodge class in $A \times B^{n-2}$ is algebraic, then the character χ is trivial on $M(A \times B)$. Hence $M(A \times B) = \mathrm{MT}(A \times B)$. But $M(A^s \times B^t) = M(A \times B)$ and $\mathrm{MT}(A^s \times B^t) = \mathrm{MT}(A \times B)$ for any $s, t \geq 1$ (see A.5), and so the Hodge conjecture holds for $A^s \times B^t$ (see A.3).

Proof of 1.5. We shall compute the terms in the diagram

$$
\begin{array}{ccc}
S^K & \longrightarrow & L(A \times B) \\
\uparrow & & \uparrow \\
P^K & \longrightarrow & L(A_0 \times B_0)
\end{array}
$$

or, equivalently, in the corresponding diagram of character groups. In fact,

we shall prove that there is an exact commutative diagram

$(**)$

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \langle \chi \rangle & \longrightarrow & X^*(L(A \times B)) & \longrightarrow & X^*(S^K) \\
& & \Big\downarrow{\cong} & & \Big\downarrow & & \Big\downarrow \\
0 & \longrightarrow & \langle \chi_0 \rangle & \longrightarrow & X^*(L(A_0 \times B_0)) & \longrightarrow & X^*(P^K)
\end{array}
$$

The horizontal maps in the right-hand square are those defined in A.5 and A.7, the map $X^*(S^K) \to X^*(P^K)$ is that in the fundamental diagram (A.8), and the map $L(A_0 \times B_0) \to L(A \times B)$ comes from the inclusion $C(A_0 \times B_0) \subset C(A \times B)$ induced by the reduction map $\operatorname{End}^0(A \times B) \hookrightarrow \operatorname{End}^0(A_0 \times B_0)$. The character χ of $L(A \times B)$ is that defined above, and χ_0 is the composite of χ with $L(A_0 \times B_0) \to L(A \times B)$.

The "subgroup" condition in the statement of the theorem implies that $\Sigma_0 \cdot D(w_0)$ is a subgroup of $\operatorname{Gal}(K/\mathbb{Q})$, even though we are no longer assuming K to be the Galois closure of $\sigma_0 E$. In fact, we now assume that K is large enough to split $\operatorname{End}^0(A_0 \times B_0)$ (in the sense of A.6).

Let X be the set of primes of K dividing p. Suppose that the subsets $\Sigma_i \cdot w_0$ and $\Sigma_j \cdot w_0$ of X have nonempty intersection. Then $\tau_i w_0 = \tau_j w_0$ for some $\tau_i \in \Sigma_i$ and $\tau_j \in \Sigma_j$. Hence $\tau_i \in \tau_j D(w_0)$, and so

$$
\Sigma_i \cdot w_0 = \tau_i \Sigma_0 \cdot w_0 \subset \tau_j D(w_0) \Sigma_0 \cdot w_0 = \tau_j \Sigma_0 D(w_0) \cdot w_0 = \Sigma_j \cdot w_0.
$$

By symmetry, $\Sigma_i \cdot w_0 \supset \Sigma_j \cdot w_0$, and so the two sets are equal: we have shown that the sets $\Sigma_i \cdot w_0$ and their complex conjugates form a partition of X. Let $X_0, \ldots, X_{m-1}$ be the distinct elements of $\{\Sigma_i \cdot w_0 \mid 0 \leq i \leq n-1\}$ with X_0 chosen to be $\Sigma_0 w_0$, and let

$$
Y = \{X_0, \ldots, X_{m-1}, \iota X_0, \ldots, \iota X_{m-1}\}.
$$

The group Σ_K acts transitively on X and Y, and the stabilizers of w_0 and X_0 are $D(w_0)$ and $\Sigma_0 \cdot D(w_0)$ respectively. By using w_0 and X_0 as base points, we can identify the map of Σ_K-sets $X \to Y$ with $\Sigma_K/D(w_0) \to \Sigma_K/\Sigma_0 \cdot D(w_0)$. Each X_j then corresponds to the quotient of a left coset of $\Sigma_0 \cdot D(w_0)$ by the right action of $D(w_0)$. From these remarks, we see that

$$
\begin{aligned}
|X| &= (\Sigma_K : D(w_0)), \\
|Y| &= (\Sigma_K : \Sigma_0 \cdot D(w_0)) \quad (= 2m), \\
|X_j| &= (\Sigma_0 \cdot D(w_0) : D(w_0)).
\end{aligned}
$$

For $i \in \{0, \ldots, n-1\}$, define $j(i)$ to be the element of $\{0, \ldots, m-1\}$ such that $\Sigma_i \cdot w_0 = X_{j(i)}$. For each j, there are $(\Sigma_0 \cdot D(w_0) : \Sigma_0) = n/m$ sets Σ_i such that $\Sigma_i \cdot w_0 = X_j$.

We next compute the terms in the diagram

$$(**A)$$

$$
\begin{array}{ccc}
X^*(L(A)) & \longrightarrow & X^*(S^K) \\
\downarrow & & \downarrow \\
X^*(L(A_0)) & \longrightarrow & X^*(P^K)
\end{array}
$$

Recall that we have already shown that $X^*(L(A)) \to X^*(S^K)$ sends the element $[\sigma_i]$ of $X^*(L(A))$ to ψ_i.

We use the map $\pi \mapsto f_\pi^K$ (see A.6, A.7) to identify $X^*(P^K)$ with

$$\{f \in \mathbb{Z}^X \mid \text{there exists an } m \in \mathbb{Z} \text{ such that } f + \iota f = mn_0\}.$$

Here $n_0 = [K_{w_0} : \mathbb{Q}_p] = |D(w_0)|$. The map $X^*(S^K) \to X^*(P^K)$ is

$$f = \sum_{\tau \in \Sigma_K} f(\tau)\tau \mapsto \sum_{\tau \in \Sigma_K} f(\tau)\tau w_0 = \sum_{w \in X} \Big(\sum_{\tau, \tau w_0 = w} f(\tau) \Big) w$$

(see A.8). When $f = \psi_i$, $w \in X_j$ occurs in the right-hand side with nonzero coefficient if and only if $j = j(i)$, in which case its coefficient is $|\Sigma_0 \cap D(w_0)| = n_0 m/n$. Thus the map sends ψ_i to $f_{j(i)}$ where f_j is the function determined by the conditions

$$
f_j(w) = \begin{cases} |\Sigma_0 \cap D(w_0)|, & w \in X_j, \\ 0, & w \in X_{j'}, \ j' \neq j, \end{cases}
$$
$$f_j(w) + f_j(\iota w) = n_0 \quad \text{for all } w.$$

We identify $X^*(L(A_0))$ with

$$\frac{\mathbb{Z}^{\Pi_{A_0}}}{\{g \mid g = \iota g, \ \sum g(\pi) = 0\}}$$

where Π_{A_0} is the set of conjugates of π_{A_0} in K (see A.7). Let $u = \varrho_0^{-1} w_0$, and let $v_0 = \sigma_0^{-1} w_0$. Note that $\sigma_i^{-1} w_0$ lies over u_0 and $(\iota \sigma_i)^{-1} w_0$ lies over ιu_0, $0 \leq i \leq n-1$. Using this, we find that the slope function of the Frobenius germ π_{A_0} of A_0 satisfies

$$
s_{\pi_{A_0}}(v) = \begin{cases} 1/|\Sigma_E(v_0)|, & v = v_0, \\ 0, & v \text{ lies over } u_0, \ v \neq v_0, \end{cases}
$$

where $\Sigma_E(v_0) = \{\sigma \in \Sigma_E \mid \sigma^{-1} w_0 = v_0\}$ (see A.8). As $s + \iota s = 1$, this determines s. Note that

$$|\Sigma_E(v_0)| = (\Sigma_0 \cdot D(w_0) : \Sigma_0) = (D(w_0) : \Sigma_0 \cap D(w_0)).$$

Note also that X_0 is the set of $w \in X$ lying over the prime $\sigma_0 v_0$ in $\sigma_0 E$. For

any $\tau \in \Sigma_i$ (i.e., such that $\tau \circ \sigma_0 = \sigma_i$), the diagram

$$
\begin{array}{ccc}
K & \xrightarrow[\approx]{\tau} & K \\
\downarrow & & \downarrow \\
\sigma_0 E & \xrightarrow{\approx} & \sigma_i E
\end{array}
\qquad
\begin{array}{l}
X_0 \mapsto \tau X_0 = X_{j(i)} \\[2ex]
\sigma_0 v_0 \mapsto \sigma_i v_0
\end{array}
$$

shows that $X_{j(i)}$ is the set of $w \in X$ lying over $\sigma_i v_0$ in $\sigma_i E_0$. In other words, $X_{j(i)}$ is the set of $w \in X$ such that $\sigma_i^{-1} w = v_0$. For $\sigma \in \Sigma_E$, $\sigma \pi_{A_0}$ is the Weil germ in K with

$$
f^K_{\sigma \pi_{A_0}}(w) = s_{\sigma \pi_{A_0}}(w) \cdot n_0 = s_{\pi_{A_0}}(\sigma^{-1} w) \cdot n_0.
$$

When $\sigma = \sigma_i$ and $w \in X_j$, this becomes

$$
f^K_{\sigma_i \pi_{A_0}}(w) = \begin{cases} |\Sigma_0 \cap D(w_0)|, & j = j(i), \\ 0, & j \neq j(i). \end{cases}
$$

Thus, $f_{\sigma_i \pi_{A_0}} = f_{j(i)}$. In particular, $\sigma_i \pi_{A_0}$ depends only on $j(i)$. As the functions f_j are distinct, we see that

$$
\Pi_{A_0} = \{\pi_0, \ldots, \pi_{m-1}, \iota \pi_0, \ldots, \iota \pi_{m-1}\}
$$

where $\pi_{j(i)} = \sigma_i \pi_{A_0}$. The map $X^*(L(A)) \to X^*(L(A_0))$ sends $[\sigma_i]$ to $[\pi_{j(i)}]$, and the map $X^*(L(A_0)) \to X^*(P^K)$ sends $[\pi_j]$ to f_j.

We have now computed all the terms in the diagram $(**\text{A})$. It is clear that it commutes.

We next compute the terms in the diagram

$$(**\text{B})
\qquad
\begin{array}{ccc}
X^*(L(B)) & \longrightarrow & X^*(S^K) \\
\downarrow & & \downarrow \\
X^*(L(B_0)) & \longrightarrow & X^*(P^K)
\end{array}
$$

Recall that $X^*(L(B))$ has basis $[\varrho_0]$, $[\iota \varrho_0]$, and that the map $X^*(L(B)) \to X^*(S^K)$ sends $[\varrho_0]$ to ψ and $[\iota \varrho_0]$ to $\iota \psi$. Here ψ is the characteristic function of $\bigcup \Sigma_i$. Clearly, $\mathbb{Q}[\pi_{B_0}] = Q$, and $X^*(L(B_0)) = \mathbb{Z}^{\Sigma_Q}$. The left-hand vertical map in the diagram is therefore the identity map. Let $f \in \mathbb{Z}^X$ be the function

$$
f(w) = \begin{cases} n_0 & \text{if } \varrho_0^{-1} w = u_0, \\ 0 & \text{otherwise.} \end{cases}
$$

Then $f \in X(P^K)$ and the bottom map sends $[\varrho_0] \mapsto f$. The right-hand vertical map sends ψ to f.

On combining the diagrams $(**\text{A})$ and $(**\text{B})$, we get the right-hand square in $(**)$. It remains to compute the kernel of $X^*(L(A_0 \times B_0)) \to$

$X^*(P^K)$. Note that

$$X^*(L(A_0 \times B_0)) = \frac{\mathbb{Z}^{\Sigma_{\Pi_{A_0}} \sqcup \Sigma_Q}}{\{g \mid g = \iota g \text{ and } \sum g(y) = 0\}}.$$

The elements $[\pi_0], \ldots, [\pi_{m-1}], [\varrho_0], [\varrho_0 + \iota \varrho_0]$ form a basis for $X^*(L(A_0 \times B_0))$. They are mapped respectively to $f_0, \ldots, f_{m-1}, f, f + \iota f$ in $X^*(P^K)$. Clearly,

$$\frac{n}{m}(f_0 + \ldots + f_{m-1}) + (n-2)f = (n-1)(f + \iota f),$$

and any relation among $f_0, \ldots, f_{m-1}, f, f + \iota f$ is a multiple of this one. Therefore, the kernel of $X^*(L(A_0 \times B_0)) \to X^*(P^K)$ is the free $\mathbb{Z}$-module of rank one generated by

$$\chi_0 = \left[\frac{n}{m}(\pi_0 + \ldots + \pi_{m-1}) + (n-2)\varrho_0 - (n-1)(\varrho_0 + \iota \varrho_0)\right].$$

The map $X^*(L(A \times B)) \to X^*(L(A_0 \times B_0))$ sends χ to χ_0, and so we have obtained the diagram $(**)$.

We now prove Theorem 1.5. The group $L(A_0 \times B_0)$ acts on the space $W(A_0, B_0)$ through the characters χ_0 and $\iota \chi_0 = -\chi_0$. Because χ_0 is trivial on $P(A_0 \times B_0)$, $W(A_0 \times B_0)$ consists of Tate classes, and because χ_0 is nontrivial on $L(A_0 \times B_0)$, the classes are exotic. The group $L(A_0 \times B_0)$ acts on no other subspace of a space $H^{2r}(A_0 \times B_0^{n-2}, \mathbb{Q}_\ell(r))$ through the character χ_0, and so $W(A_0 \times B_0)$ contains all the exotic Tate classes on $A_0 \times B_0^{n-2}$.

From $(**)$, we obtain an exact commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{MT}(A \times B) & \longrightarrow & L(A \times B) & \overset{\chi}{\longrightarrow} & T & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow{\scriptstyle \cong} & & \\
0 & \longrightarrow & P(A_0 \times B_0) & \longrightarrow & L(A_0 \times B_0) & \overset{\chi_0}{\longrightarrow} & T_0 & \longrightarrow & 0
\end{array}
$$

It follows that

$$P(A_0 \times B_0) = L(A_0 \times B_0) \cap \mathrm{MT}(A \times B).$$

If some exotic Hodge class on $A \times B^{n-2}$ is algebraic, then the Hodge conjecture holds for all powers of $A \times B^{n-2}$ (see Theorem 1.4), and so Theorem 1.5(b) follows from Corollary 1.2.

REMARK 2.1. It follows from the above calculations that $P(A_0) = L(A_0)$ and $P(B_0) = L(B_0)$, and so all Tate classes on A_0 and B_0 are Lefschetz.

REMARK 2.2. Choose E to be Galois over $\mathbb{Q}$, and identify it with K. In this case, the maps $X^*(L(A \times B)) \to X^*(S^K)$ and $X^*(L(A_0 \times B_0)) \to$

$X^*(P^K)$ are surjective, and so we obtain an exact commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \langle \chi \rangle & \longrightarrow & X^*(L(A \times B)) & \longrightarrow & X^*(S^K) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \cong} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \langle \chi_0 \rangle & \longrightarrow & X^*(L(A_0 \times B_0)) & \longrightarrow & X^*(P^K) & \longrightarrow & 0
\end{array}
$$

The vertical arrows are surjective, and so

$$ 0 \to \mathrm{Ker}(X^*(L(A \times B)) \to X^*(L(A_0 \times B_0))) \to X^*(S^K) \to X^*(P^K) \to 0 $$

is exact. Hence

$$ 0 \to P^K \to S^K \to L(A \times B)/L(A_0 \times B_0) $$

is exact, which implies that

$$ 0 \to P^K \to S^K \to L^K/T^K $$

is exact (notations as in Milne 1999b) because $L(A \times B)/L(A_0 \times B_0)$ $\to L^K/T^K$ is injective. Therefore

$$ P^K = S^K \cap T^K \quad \text{(intersection inside } L^K \text{)}, $$

and we recover ibid., Theorem 6.1.

Appendix A. Abelian varieties with many endomorphisms

A.1. Notations. Throughout, $\mathbb{Q}^{\mathrm{al}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, and $\Gamma = \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$. Complex conjugation on $\mathbb{C}$, or a subfield of $\mathbb{C}$, is denoted by ι or $x \mapsto \bar{x}$. In A.8, we fix a prime w_0 of $\mathbb{Q}^{\mathrm{al}}$ dividing p, and denote the residue field at w_0 by $\mathbb{F}$. We denote the restriction of w_0 to a subfield of $\mathbb{Q}^{\mathrm{al}}$ by the same symbol. For a finite étale $\mathbb{Q}$-algebra E, $\Sigma_E = \mathrm{Hom}(E, \mathbb{Q}^{\mathrm{al}})$. For a subfield K of $\mathbb{Q}^{\mathrm{al}}$ Galois over $\mathbb{Q}$, Σ_K can be identified with $\mathrm{Gal}(K/\mathbb{Q})$.

A *CM-algebra* E is a finite product of finite field extensions of $\mathbb{Q}$ admitting an involution ι_E that is nontrivial on each factor and such that $\sigma(\iota_E x) = \overline{\sigma(x)}$ for all $\sigma : E \to \mathbb{C}$; equivalently, E is a finite product of CM-fields.

For a finite set Y, $\mathbb{Z}^Y$ denotes the set of functions $f : Y \to \mathbb{Z}$. We sometimes denote such a function by $\sum f(y)y$; for example, the function $f = y_1$ takes the value 1 on y_1 and 0 on all $y \neq y_1$.

For a group of multiplicative type T over $\mathbb{Q}$, $X^*(T) =_{\mathrm{df}} \mathrm{Hom}(T_{\mathbb{Q}^{\mathrm{al}}}, \mathbb{G}_m)$ is the character group. We often use the pairing

$$ \chi, \mu \mapsto \langle \chi, \mu \rangle \overset{\mathrm{df}}{=} \chi \circ \mu : X^*(T) \times X_*(T) \to \mathrm{End}(\mathbb{G}_m) \cong \mathbb{Z} $$

to identify the cocharacter group $X_*(T) =_{\mathrm{df}} \mathrm{Hom}_{\mathbb{Q}^{\mathrm{al}}}(\mathbb{G}_m, T_{\mathbb{Q}^{\mathrm{al}}})$ of T with the $\mathbb{Z}$-linear dual of $X^*(T)$. The group Γ acts on $X_*(T)$ and $X^*(T)$, and $\langle \gamma\chi, \gamma\mu \rangle = \langle \chi, \mu \rangle$ for all $\gamma \in \Gamma$, $\chi \in X^*(T)$, and $\mu \in X_*(T)$.

Let $\varrho : T \to \mathrm{GL}(V)$ be a representation of a group T of multiplicative type on a finite-dimensional $\mathbb{Q}$-vector space V. For any subfield Ω of $\mathbb{C}$ that splits T, there is a decomposition

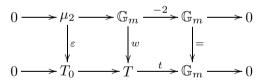$$V \otimes_{\mathbb{Q}} \Omega \cong \bigoplus_{\chi \in X^*(T)} V_\chi$$

where V_χ is the subspace of $V \otimes_{\mathbb{Q}} \Omega$ on which T acts through χ. If V_χ is nonzero, then we say that χ *occurs* in V. When Ω is Galois over $\mathbb{Q}$, a subspace $\bigoplus_{\chi \in \Xi} V_\chi$, $\Xi \subset X^*(T)$, is defined over $\mathbb{Q}$ (i.e., of the form $W \otimes_{\mathbb{Q}} \Omega$ for some subspace $W \subset V$) if and only if Ξ is stable under Γ. The subspace of vectors in V fixed by T (in the sense of Milne 1999a, §3) is denoted V^T.

For a finite étale $\mathbb{Q}$-algebra E, $(\mathbb{G}_m)_{E/\mathbb{Q}} =_{\mathrm{df}} \mathrm{Res}_{E/\mathbb{Q}}(\mathbb{G}_m)$ (Weil restriction of scalars), so that $X^*((\mathbb{G}_m)_{E/\mathbb{Q}}) = \mathbb{Z}^{\Sigma_E}$. Under this identification, an element $f = \sum f(\sigma)\sigma$ of $\mathbb{Z}^{\Sigma_E}$ maps an element a of $E^\times = (\mathbb{G}_m)_{E/\mathbb{Q}}(\mathbb{Q})$ to $a^f = \prod(\sigma a)^{f(\sigma)}$. We sometimes identify a subset Δ of Σ_E with the character $\sum_{\sigma \in \Delta} \sigma$; for example, if V is an E-vector space, then $(V^{\otimes r} \otimes \Omega)_\Delta$ is the subspace on which $a \in E$ acts as $\prod_{\sigma \in \Delta} \sigma a$.

There is a natural correspondence (^{3}) between

- triples (T, w, t) comprising a group of multiplicative type T, a cocharacter w of T, and a character t, all defined over $\mathbb{Q}$, such that $t \circ w = -2$;

and

- pairs (T_0, ε) comprising a group of multiplicative type T_0 and an element ε of order 1 or 2 in $T_0(\mathbb{Q})$.

Given (T, w, t), define T_0 to be the kernel of t and ε to be $w(-1)$. Conversely, given (T_0, ε), define T by the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mu_2 & \longrightarrow & \mathbb{G}_m & \overset{-2}{\longrightarrow} & \mathbb{G}_m & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle =} & & \\
0 & \longrightarrow & T_0 & \longrightarrow & T & \overset{t}{\longrightarrow} & \mathbb{G}_m & \longrightarrow & 0
\end{array}
$$

in which $T = (T_0 \times \mathbb{G}_m)/\mu_2$. If $(T_1, w_1, t_1) \subset (T_2, w_2, t_2)$, then $T_1 = T_2$ if and only if $(T_1)_0 = (T_2)_0$.

Let $\varrho_0 : T_0 \to \mathrm{GL}(V)$ be a representation of T_0 such that $\varrho_0(\varepsilon)$ acts on V as multiplication by the scalar -1, and let W be a one-dimensional vector space with basis e. Then

$$(x, y) \mapsto (\varrho_0(x) \cdot y, y^{-2}) : T_0 \times \mathbb{G}_m \to \mathrm{GL}(V) \times \mathrm{GL}(W)$$

sends $(\varepsilon, \varepsilon)$ to 1, and therefore defines a homomorphism $\varrho : T \to \mathrm{GL}(V) \times \mathrm{GL}(W)$. Note that $(\varrho \circ w)(y)$ acts on V as y, and that the composite of ϱ

(^{3}) Experts will recognize the Tannakian significance of this correspondence (Saavedra 1972, V 3.1.4; Deligne and Milne 1982, p. 190).

with the projection to $\mathrm{GL}(W)$ is t. Let $s \in V^{\otimes i}$. If s is fixed by T_0, then i is even. There is a one-to-one correspondence

$$s \leftrightarrow s \otimes e^{\otimes j}$$

between the elements s of $V^{\otimes 2j}$ fixed by T_0 and the elements of $V^{\otimes 2j} \otimes W^{\otimes j}$ fixed by T.

For a smooth projective variety X, $\mathcal{Z}^r(X)$ is the space of algebraic cycles on X of codimension r with coefficients in $\mathbb{Q}$, and $\mathcal{Z}^r_{\mathrm{num}}(X)$ is the quotient of $\mathcal{Z}^r(X)$ by numerical equivalence. The space $\mathcal{Z}^*_{\mathrm{num}}(X) = \bigoplus_r \mathcal{Z}^r_{\mathrm{num}}(X)$ becomes a $\mathbb{Q}$-algebra under intersection product. An *algebraic class* in a cohomology group with coefficients in a field Ω is an element of the Ω-subspace spanned by the classes of algebraic cycles.

For an abelian variety A over an algebraically closed field k of characteristic zero, we often implicitly assume that there is given an embedding $\sigma : k \to \mathbb{C}$ so that we can define $H^r(A, \mathbb{Q})$ to be rth cohomology group of the complex manifold $(\sigma A)(\mathbb{C})$. We let $\mathrm{Hom}^0(A, B) = \mathrm{Hom}(A, B) \otimes_{\mathbb{Z}} \mathbb{Q}$.

For Hodge structures and class field theory, we follow the usual conventions of those areas rather than the conventions of Deligne used in my previous papers. For example, $z \in \mathbb{C}^{\times}$ acts on a Hodge structure of type (r, s) as $z^r \bar{z}^s$, and the Artin reciprocity maps send prime elements to the Frobenius element $x \mapsto x^q$.

We sometimes use $[x]$ to denote an equivalence class containing x, and $|X|$ to denote the order of a finite set X.

For an explanation of the various cohomology groups of varieties, and their Tate twists, see Deligne 1982, §1.

This section summarizes results due to many mathematicians. Omitted proofs can be found in Milne 1999a, 1999b, Tate 1968/69, or in the references for those articles.

A.2. Generalities. Let A be an abelian variety over an algebraically closed field k. The reduced degree ([4]) of the $\mathbb{Q}$-algebra $\mathrm{End}^0(A)$ is $\leq 2 \dim A$, and when equality holds the abelian variety is said ([5]) to have *many endomorphisms*. An isotypic ([6]) abelian variety has many endomorphisms if and only if $\mathrm{End}^0(A)$ contains a field of degree $2 \dim A$ over $\mathbb{Q}$, and an arbitrary

---

([4]) Let R be a semisimple algebra of finite degree over $\mathbb{Q}$. Then R is a product of simple algebras, say, $R = R_1 \times \ldots \times R_m$, and the centre E_i of each R_i is a field. The *reduced degree* $[R : \mathbb{Q}]_{\mathrm{red}}$ of R over $\mathbb{Q}$ is defined to be $\sum_{i=1}^m [R_i : E_i]^{1/2}[E_i : \mathbb{Q}]$.

([5]) Often such an abelian variety is said to admit "complex multiplication", but this conflicts with classical terminology—see Lange and Birkenhake 1992, p. 268. Also "multiplication" for "endomorphism" seems archaic.

([6]) An abelian variety is said to be *isotypic* if it is isogenous to a power of a simple abelian variety.

abelian variety has many endomorphisms if and only if each isotypic isogeny factor of it does. Equivalent conditions:

(a) the $\mathbb{Q}$-algebra $\mathrm{End}^0(A)$ contains an étale subalgebra of degree $2 \dim A$ over $\mathbb{Q}$;

(b) for a Weil cohomology $X \mapsto H^*(X)$ with coefficient field Ω, the centralizer of $\mathrm{End}^0(A)$ in $\mathrm{End}_\Omega(H^1(A))$ is commutative (in which case it equals $C(A) \otimes_\mathbb{Q} \Omega$ where $C(A)$ is the centre of $\mathrm{End}^0(A)$);

(c) (characteristic zero) A has CM-type, i.e., its Mumford–Tate group (see A.3 below) is commutative (hence a torus);

(d) (characteristic $p \neq 0$) A is isogenous to an abelian variety defined over $\mathbb{F}$ (theorems of Tate and Grothendieck).

Let $k \subset k'$ be algebraically closed fields. The functor $A \mapsto A_{k'}$ from the category of abelian varieties over k to the similar category over k' is fully faithful, because the map on torsion points $A(k)_{\mathrm{tors}} \to A(k')_{\mathrm{tors}}$ is bijective and $A(k)_{\mathrm{tors}}$ is Zariski dense in A. That the functor becomes essentially surjective on the categories of abelian varieties with many endomorphisms up to isogeny is a result of Grothendieck (Oort 1973). Thus, in large part, the theory of abelian varieties with many endomorphisms up to isogeny over an algebraically closed field depends only on the characteristic of the field.

A.3. The groups attached to an abelian variety with many endomorphisms. Let A be an abelian variety with many endomorphisms over an algebraically closed field k, and let $C(A)$ be the centre of $\mathrm{End}^0(A)$. Every Rosati involution on $\mathrm{End}^0(A)$ stabilizes $C(A)$, and the different Rosati involutions restrict to the same involution ([7]) on $C(A)$, which we denote $\dagger$. Each factor of $C(A)$ is either a CM-field, on which $\dagger$ acts as complex conjugation, or is $\mathbb{Q}$.

The Lefschetz group. We define $L(A)_0$ to be the group of multiplicative type over $\mathbb{Q}$ such that, for all commutative $\mathbb{Q}$-algebras R,

$$L(A)_0(R) = \{\alpha \in C(A) \otimes R \mid \alpha\alpha^\dagger = 1\}.$$

Let $\varepsilon = -1 \in L(A)_0(\mathbb{Q})$, and let $(L(A), w, t)$ be the triple associated (as in A.1) with $(L(A)_0, \varepsilon)$.

Then

$$L(A)(\mathbb{Q}) \cong \{\alpha \in C(A)^\times \mid \alpha\alpha^\dagger \in \mathbb{Q}^\times\},$$

and, on $\mathbb{Q}$-points, w is $x \mapsto x$ and t is $x \mapsto (xx^\dagger)^{-1}$.

The motivic group. Because $L(A)_0$ is a subgroup of $\mathrm{End}^0(A)^\times$, it acts on $\mathcal{Z}^*_{\mathrm{num}}(A^s)$ for all s, and we define $M(A)_0$ to be the largest algebraic subgroup

---

([7]) The Rosati involution defined by a polarization $\lambda : A \to A^\vee$ is $\alpha \mapsto \alpha^\dagger = \lambda^{-1} \circ \alpha^\vee \circ \lambda$. Let μ be a second polarization, and let $\beta = \lambda^{-1} \circ \mu$. If $\alpha \in C(A)$, then $\alpha^\dagger \in C(A)$ and $\mu^{-1} \circ \alpha^\vee \circ \mu = \beta^{-1} \circ \alpha^\dagger \circ \beta = \alpha^\dagger$.

of $L(A)_0$ acting trivially on these $\mathbb{Q}$-algebras. Then $-1 \in M(A)_0(\mathbb{Q})$, and we let $(M(A), w, t)$ be the triple associated with $(M(A)_0, -1)$.

The Mumford–Tate group. When k has characteristic zero, $L(A)_0$ acts on the $\mathbb{Q}$-algebra of Hodge classes on A^s for all s, and we define $\mathrm{MT}(A)_0$ to be the subgroup of $L(A)_0$ fixing the elements of these $\mathbb{Q}$-algebras. Again $-1 \in \mathrm{MT}(A)_0$, and we let $(\mathrm{MT}(A), w, t)$ be the triple associated with $(\mathrm{MT}(A)_0, -1)$.

The group P. Let $k = \mathbb{F}$. A model A_1 of A over a finite field $\mathbb{F}_q$ defines a Weil q-number π_1, whose class π_A in $W(p^\infty)$ (see A.6 below) is independent of the choice of A_1. The group $P(A)$ is defined to be the smallest algebraic subgroup of $L(A)$ containing some power of π_1 — again, it is independent of the choice of A_1.

Let π_1 be a Weil p^{2n}-number representing π_A. Then $\pi_1/p^n \in L(A)_0$, and $P(A)_0$ is the smallest algebraic subgroup of $L(A)_0$ containing some power of π_1/p^n.

Let H^* be a Weil cohomology with coefficients in a field Ω. Since $L(A)_0 \subset (\mathbb{G}_m)_{C(A)/\mathbb{Q}}$, there is a natural action of $L(A)_0$ on $H^1(A, \Omega)$, and ε acts as -1. Hence (see A.1) there is a natural action of $L(A)$ on

$$H^r(A^s, \Omega)(m) \cong \left(\bigwedge^r \Big(\bigoplus_{s \text{ copies}} H^1(A, \Omega) \Big) \right) \otimes (\Omega(1))^{\otimes m}.$$

LEMMA. *Let A be an abelian variety with many endomorphisms over an algebraically closed field k, and let H^* be a Weil cohomology with coefficients in a field Ω. Let $H^{2*}(A^s)(*) = \bigoplus_r H^{2r}(A^s)(r)$. Then, for all s,*

(a) $H^{2*}(A^s)(*)^{L(A)}$ *is the Ω-subalgebra of $H^{2*}(A^s)(*)$ generated by the classes of divisors on A^s (i.e., it is the space of Lefschetz classes);*

(b) $H^{2*}(A^s)(*)^{M(A)}$ *is the space of algebraic classes in $H^{2*}(A^s)(*)$, provided numerical equivalence coincides with homological equivalence for H^*;*

(c) $H^{2*}(A^s)(*)^{\mathrm{MT}(A)}$ *is the space of Hodge classes on A^s when k has characteristic zero and H^* is the cohomology defined by an embedding $k \to \mathbb{C}$;*

(d) $H^{2*}(A^s, \mathbb{Q}_\ell(*))^{P(A)}$ *is the space of ℓ-adic Tate classes on A^s when $k = \mathbb{F}$ and H^* is ℓ-adic étale cohomology.*

Proof. Statement (a) is proved in Milne 1999a (Theorem 4.4).

For (b), recall that theorems of Jannsen and Deligne show that the category of motives over k, based on abelian varieties and defined using the numerical equivalence classes of algebraic cycles as correspondences, is Tannakian (Jannsen 1992). Almost by definition, $M(A)$ is the fundamental group of the Tannakian subcategory $\langle A \rangle^\otimes$ of this category generated by A and the Tate object. When numerical equivalence coincides with homological equivalence, the Weil cohomology defines a fibre functor, and there is

a natural map

$$\text{Hom}(\mathbb{1}, h^{2r}(A^s)(r)) \otimes_{\mathbb{Q}} \Omega \to \text{Hom}_{\Omega}(\Omega, H^{2r}(A^s)(r))^{M(A)},$$

which the theory of Tannakian categories shows to be bijective. But

$$\text{Hom}(\mathbb{1}, h^{2r}(A^s)(r)) = \mathcal{Z}^r_{\text{num}}(A^s).$$

Statement (c) is proved in Deligne 1982 (see the proof of 3.4).

Almost by definition of $P(A)$, $H^{2*}(A^s, \mathbb{Q}_\ell(*))^{P(A)}$ consists of the classes fixed by the Frobenius germ π_A, and these are exactly the Tate classes. ∎

Thus (Deligne 1982, 3.1), under the hypotheses in each part of the lemma, knowing the group $?(A)_\Omega$ is equivalent to knowing the corresponding spaces of fixed classes: $?(A)_\Omega$ is the largest algebraic subgroup of $\text{GL}(H^1(A))$ $\times \mathbb{G}_m$ fixing the particular classes on all A^s, and the particular classes are exactly those fixed by $?(A)_\Omega$.

THEOREM. (a) *For any abelian variety A with many endomorphisms over an algebraically closed field k of characteristic zero,* $\text{MT}(A) \subset M(A) \subset L(A)$, *and*

> (i) *the Hodge conjecture holds for all powers of A if and only if* $\text{MT}(A) = M(A)$;
> (ii) *all Hodge classes on all powers of A are Lefschetz if and only if* $\text{MT}(A) = L(A)$.

When $k = \mathbb{Q}^{\text{al}}$, "Hodge" can be replaced by "Tate" in the above statements.
> (b) *For any abelian variety A_0 over $\mathbb{F}$, $P(A_0) \subset M(A_0) \subset L(A_0)$, and*

> (i) *all ℓ-adic Tate classes on all powers of A_0 are algebraic for one (or all) ℓ if and only if $P(A_0) = M(A_0)$;*
> (ii) *all ℓ-adic Tate classes on all powers of A_0 are Lefschetz for one (or all) ℓ if and only if $P(A_0) = L(A_0)$.*

Proof. Since every character of $L(A)$ occurs in a space of the form $H^r(A^s)(m)$, we see that the subgroups of $L(A)$ are determined by their invariants in these spaces. Thus (a) of the Theorem is an immediate consequence of the Lemma. That "Hodge" can be replaced by "Tate" follows from Pohlmann 1968.

If $P(A_0) = M(A_0)$, then the Lemma shows that the ℓ-adic Tate conjecture holds for all powers of A_0 and all ℓ in the set in Proposition B.2, but if the ℓ-adic Tate conjecture holds for one ℓ then it holds for all (Tate 1994, 2.9). Conversely, if the ℓ-adic Tate conjecture holds for all powers of A_0 and a single ℓ, then numerical equivalence coincides with ℓ-homological equivalence for that ℓ (Tate 1994, 2.9, (c)⇒(b)), and the preceding Lemma then shows that $P(A_0)_{\mathbb{Q}_\ell} = M(A_0)_{\mathbb{Q}_\ell}$. As $P(A_0) \subset M(A_0)$, this implies that $P(A_0) = M(A_0)$.

The proof of the remaining statement is similar. ■

EXAMPLE. If A has dimension 1, then either $\text{End}^0(A)$ is a quadratic imaginary field E or a quaternion algebra D with centre $\mathbb{Q}$. In the first case, all the groups attached to A equal $(\mathbb{G}_m)_{E/\mathbb{Q}}$ and in the second, all the groups attached to A equal $\mathbb{G}_m$. Hence, there are no exotic Hodge or Tate classes on any power of an elliptic curve, and the Hodge and Tate conjectures hold.

REMARK. Let A be an abelian variety with many endomorphisms over $\mathbb{Q}^{\text{al}}$, and let A_0 be its reduction to an abelian variety over A_0. The reduction map $\text{End}^0(A) \to \text{End}^0(A_0)$ is injective, and the image of the centre of $\text{End}^0(A)$ contains the centre of $\text{End}^0(A_0)$ (because the latter is generated as a $\mathbb{Q}$-algebra by a Frobenius element which lifts to an element of the centre of $\text{End}^0(A)$). Therefore, $L(A) \supset L(A_0)$.

A.4. Classification over $\mathbb{C}$ of abelian varieties with many endomorphisms. Let E be a CM-algebra. A *CM-type* on E is the choice of one out of every pair of complex conjugate homomorphisms $E \to \mathbb{C}$. It can variously be considered as:

(a) a partition $\Sigma_E = \Phi \cup \iota\Phi$;

(b) a function $\varphi : \Sigma_E \to \mathbb{Z}$ such that, for all σ, $\varphi(\sigma) \geq 0$ and $\varphi(\sigma) + \varphi(\iota\sigma) = 1$;

(c) the choice of an isomorphism $E \otimes_{\mathbb{Q}} \mathbb{R} \to \mathbb{C}^{\Sigma_F}$ where F is the product of the maximal real subfields of the factors of E.

Here Φ is the support of φ and φ is the characteristic function of Φ.

Let A be a simple abelian variety over $\mathbb{C}$ with many endomorphisms. Then $\text{End}^0(A)$ is a CM-field E, and the action of E on $\Gamma(A, \Omega^1)$ defines a CM-type Φ on E, which is primitive, i.e., not the extension of a CM-type on a proper CM-subfield of E. The map $A \mapsto (E, \Phi)$ defines a bijection from the set of isogeny classes of simple abelian varieties over $\mathbb{C}$ with many endomorphisms to the set of isomorphism classes of pairs (E, Φ). It remains to classify the pairs (E, Φ).

Fix a (large) CM-field $K \subset \mathbb{Q}^{\text{al}}$, finite and Galois over $\mathbb{Q}$. The *Serre group* S^K of K is the quotient of $(\mathbb{G}_m)_{K/\mathbb{Q}}$ whose character group consists of the $f \in \mathbb{Z}^{\Sigma_K}$ for which there is an integer $\text{wt}(f)$ (the *weight* of f) such that $f(\tau) + f(\iota\tau) = \text{wt}(f)$ for all $\tau \in \Sigma_K$, that is,

$$X^*(S^K) = \{f \in \mathbb{Z}^{\Sigma_K} \mid f + \iota f \text{ is constant}\}.$$

The *reflex field* of (E, Φ) is the fixed field of the subgroup $\{\tau \in \Gamma \mid \tau\Phi = \Phi\}$ of Γ. We classify the pairs (E, Φ) whose reflex field is contained in K. Let φ be the characteristic function of Φ. For each $\sigma : E \to \mathbb{Q}^{\text{al}}$ and $\tau \in \text{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q})$, define

$$\psi_\sigma(\tau) = \varphi(\tau^{-1} \circ \sigma).$$

Then $\psi_\sigma(\tau)$ depends only on $\tau|K$, and for any $\varrho \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$, $\psi_{\varrho\circ\sigma} = \varrho\psi_\sigma$. Thus, $\{\psi_\sigma\}$ is a Γ-orbit in $\mathbb{Z}^{\Sigma_K}$. The map $(E, \Phi) \mapsto \{\psi_\sigma\}$ is a bijection from the set of isomorphism classes of pairs (E, Φ) comprising a CM-field and a primitive CM-type whose reflex field is contained in K to the set of Γ-orbits of elements f of $X^*(S^K)$ such that $f(\tau) \geq 0$ for all τ and $\mathrm{wt}(f) = 1$.

A.5. Calculation of the groups over $\mathbb{C}$. Let A be an abelian variety with many endomorphisms over $\mathbb{C}$. Then A is isogenous to a product $A_1^{s_1} \times \ldots \times A_t^{s_t}$ with the A_i simple and pairwise nonisogenous, and

$$L(A) \cong L(A_1 \times \ldots \times A_t) \quad (\text{in fact } L(A)_0 \cong L(A_1)_0 \times \ldots \times L(A_t)_0),$$
$$M(A) \cong M(A_1 \times \ldots \times A_t),$$
$$\mathrm{MT}(A) \cong \mathrm{MT}(A_1 \times \ldots \times A_t).$$

Thus, in the following, we assume that A is a product of pairwise nonisogenous simple abelian varieties. Then $E =_{\mathrm{df}} \mathrm{End}^0(A)$ is a CM-algebra. The action of E on $H^{1,0}(A)$ defines a CM-type Φ on E, and the Rosati involution is ι_E.

The Lefschetz group. The group $L(A)$ is the subgroup of $(\mathbb{G}_m)_{E/\mathbb{Q}}$ whose character group is

$$\frac{\mathbb{Z}^{\Sigma_E}}{\{g \mid g = \iota g \text{ and } \sum g(\sigma) = 0\}}.$$

The weight map $w : \mathbb{G}_m \to L(A)$ corresponds to the map

$$[g] \mapsto \mathrm{wt}(g) \overset{\mathrm{df}}{=} \sum_{\sigma \in \Sigma_E} g(\sigma)$$

on characters, and the homomorphism $t : L(A) \to \mathbb{G}_m$ giving the action of $L(A)$ on the Tate object $\mathbb{Q}(1)$ sends $1 \in X^*(\mathbb{G}_m)$ to the element of $X^*(L(A))$ represented by $-\sigma - \iota\sigma$ for any $\sigma \in \Sigma_E$.

The group $L(A)_0$ is the subgroup of $(\mathbb{G}_m)_{E/\mathbb{Q}}$ whose character group is

$$\frac{\mathbb{Z}^{\Sigma_E}}{\{g \mid g = \iota g\}}.$$

The map $\mu_2 \to L(A)_0$ corresponds to the map on characters $[g] \mapsto \sum g(\sigma)$ mod 2.

When A is simple, the map $\sigma \mapsto \psi_\sigma$ is bijective and commutes with the action of Γ, and so it identifies $L(A)$ with the torus whose character group is

$$\frac{\mathbb{Z}^\Psi}{\{g \mid g = \iota g \text{ and } \sum g(\psi) = 0\}}, \quad \Psi = \{\psi_\sigma \mid \sigma \in \Sigma_E\}.$$

The Mumford–Tate group. The Hodge decomposition on $H^r(A^s, \mathbb{Q})(m)$ is defined over $\mathbb{Q}^{\mathrm{al}}$, i.e., there is a decomposition

$$H^r(A^s, \mathbb{Q})(m) \otimes \mathbb{Q}^{\mathrm{al}} \cong \bigoplus_{i+j=r-2m} H^r(A^s)(m)^{i,j}$$

that becomes the Hodge decomposition when tensored with $\mathbb{C}$. Since $L(A)_0 \subset (\mathbb{G}_m)_{E/\mathbb{Q}}$, there is a natural action of $L(A)_0$ on $H^1(A, \mathbb{Q})$, and ε acts as -1. Hence (see A.1) there is a natural action of $L(A)$ on

$$H^r(A^s, \mathbb{Q})(m) \cong \left(\bigwedge^r \left(\bigoplus_{s \text{ copies}} H^1(A, \mathbb{Q}) \right) \right) \otimes (\mathbb{Q}(1))^{\otimes m}.$$

For $\chi = [g] \in X^*(L(A))$, $(H^r(A^s)(m))_\chi$ is of Hodge type

$$\left(\sum_{\sigma \in \Phi} g(\sigma), \sum_{\sigma \in \iota \Phi} g(\sigma) \right).$$

Every character of $L(A)$ occurs in $H^r(A^s, \mathbb{Q})(m)$ for some r, s, m, and if $[g]$ occurs in $H^r(A^s)(m)$, then $\mathrm{wt}(g) = r - 2m$. A character χ of $L(A)$ is trivial on $\mathrm{MT}(A)$ if and only if $\bigoplus_{\tau \in \Gamma} H^{2r}(A^s)(r)_{\tau\chi}$ is purely of type $(0, 0)$ for some r, s for which the space is nonzero. Hence, a character $\chi = [g]$ of $L(A)$ is trivial on $\mathrm{MT}(A)$ if and only if

$$\sum_{\sigma \in \Phi} g(\tau \circ \sigma) = 0 \quad \text{for all } \tau \in \Gamma.$$

The motivic group. Let $\chi \in X^*(L(A))$. Then χ is trivial on $M(A)$ if and only if $H^{2r}(A^s)(r)_\chi$ contains a nonzero algebraic class for some r and s, in which case the spaces $H^{2r}(A^s)(r)_\chi$ consist entirely of algebraic classes for all r and s (see (b) of the lemma in A.3).

Second description of $\mathrm{MT}(A)$. There is another description of $\mathrm{MT}(A)$ that is useful. Let K be a CM-subfield of $\mathbb{Q}^{\mathrm{al}}$, finite and Galois over $\mathbb{Q}$, and let S^K be its Serre group. Let $\tau_0 \in \Sigma_K$ be the given embedding of K into $\mathbb{Q}^{\mathrm{al}}$. Then $f \mapsto f(\tau_0)$ is a cocharacter μ^K of S^K with the property that $\mu^K + \iota\mu^K$ is fixed by Γ and so is defined over $\mathbb{Q}$. The pair (S^K, μ^K) is universal: if T is a second torus over $\mathbb{Q}$ and $\mu \in X_*(T)$ is defined over K and $\mu + \iota\mu$ is defined over $\mathbb{Q}$, then there is a unique homomorphism $\varrho_\mu : S^K \to T$ such that $(\varrho_\mu)_{\mathbb{Q}^{\mathrm{al}}} \circ \mu^K = \mu$. On characters, ϱ_μ sends $\chi \in X^*(T)$ to the element f of $X^*(S^K)$ with $f(\tau) = \langle \chi, \tau\mu \rangle$ for all τ.

Let A be an abelian variety of CM-type (E, Φ), and let μ_Φ be the cocharacter of $L(A)$ sending a character $[g]$ of $L(A)$ to $\sum_{\sigma \in \Phi} g(\sigma)$. If K contains the reflex field of Φ, then μ_Φ is defined over K. Moreover $\mu_\Phi + \iota\mu_\Phi$ is $[g] \mapsto \mathrm{wt}(g)$, which is defined over $\mathbb{Q}$, and so there is a unique homomorphism $\varrho_\Phi : S^K \to L(A)$ such that $\varrho_\Phi \circ \mu^K = \mu_\Phi$. It sends a character g of

$L(A)$ to the character f of S^K such that

$$f(\tau) = \langle [g], \tau\mu_\Phi \rangle = \langle \tau^{-1}[g], \mu_\Phi \rangle = \sum_{\sigma \in \Phi} g(\tau \circ \sigma).$$

The image of this homomorphism is $\mathrm{MT}(A)$. It is obvious that this description agrees with the previous one.

REMARK. The roles of K and E should be carefully distinguished. The first is a "large" CM-subfield of $\mathbb{Q}^{\mathrm{al}}$ Galois over $\mathbb{Q}$; the second is a finite product of abstract CM-fields acting on A. The field K contains the reflex field of (E, Φ). There are homomorphisms

$$(\mathbb{G}_m)_{K/\mathbb{Q}} \twoheadrightarrow S^K \to L(A) \hookrightarrow (\mathbb{G}_m)_{E/\mathbb{Q}}.$$

The image of the middle homomorphism is $\mathrm{MT}(A)$.

A.6. Classification over $\mathbb{F}$ of abelian varieties. A *Weil q-number* [8] *of weight m* is an element π of a field of characteristic zero such that $q^N\pi$ is an algebraic integer for some N and $\sigma(\pi) \cdot \iota(\sigma(\pi)) = q^m$ for all homomorphisms $\sigma : \mathbb{Q}[\pi] \hookrightarrow \mathbb{C}$. The conditions imply that $q^N\pi$ is a unit at all finite primes v of $\mathbb{Q}[\pi]$ not dividing p, and hence that the same is true for π. For any prime v dividing p of a field containing π, we let

$$s_\pi(v) = \frac{\mathrm{ord}_v(\pi)}{\mathrm{ord}_v(q)};$$

thus $s_\pi(v) + s_\pi(\iota v) = \mathrm{wt}(\pi)$. A Weil q-number is determined up to a root of 1 (as an element of an algebraic number field) by the numbers $s_\pi(v)$ because they determine all of its valuations. We call s_π the *slope function* of π. A Weil q-number that is itself an integer is called a *Weil q-integer.*

Weil germs. Let π be a Weil p^n-number and π' a Weil $p^{n'}$-number in some field. We say π and π' are *equivalent* if $\pi^{n'}$ and π'^n differ by a root of 1. A *Weil germ* is an equivalence class of Weil numbers. The weight and slope function of a Weil germ π are the weight and slope function of any representative of it, and $\mathbb{Q}[\pi]$ is defined to be the smallest field containing a representative of π. A Weil germ is determined by its slope function.

Let $W(p^\infty)$ denote the set of Weil germs represented by elements of $\mathbb{Q}^{\mathrm{al}}$. It is an abelian group endowed with an action of Γ. Let $W(p^\infty)_{m,+}$ denote the subset of $W(p^\infty)$ consisting of Weil germs of weight m represented by algebraic integers; thus,

$$W(p^\infty)_{m,+} = \{\pi \in W(p^\infty) \mid s_\pi(v) \geq 0, \ s_\pi(v) + s_\pi(\iota v) = m \ \forall v\}.$$

Classification of abelian varieties. Let A_0 be a simple abelian variety over $\mathbb{F}$, and let A_1 be a model of A_0 over $\mathbb{F}_q \subset \mathbb{F}$ with the property that

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[8] This conflicts with an earlier terminology (e.g., Tate 1968/69) which calls a "Weil q-number" what we call a "Weil q-integer of weight 1".

$\text{End}(A_1) = \text{End}(A_0)$. The Frobenius endomorphism π_{A_1} of A_1 is a Weil q-integer of weight 1 in $C(A_0)$, and we let π_{A_0} denote the germ represented by π_{A_1}—it is independent of the choice of $A_1/\mathbb{F}_q$. The conjugates of π_{A_0} in $\mathbb{Q}^{\text{al}}$ form a Γ-orbit Π_{A_0} in $W(p^\infty)$, and the map $A_0 \mapsto \Pi_{A_0}$ is a bijection from the set of isomorphism classes of simple abelian varieties over $\mathbb{F}$ onto the set of Γ-orbits in $W(p^\infty)_{1,+}$.

The various invariants of A_0 can be read off from Π_{A_0} as follows. The images of $\mathbb{Q}[\pi_{A_0}]$ in $\mathbb{Q}^{\text{al}}$ are the fixed fields of the stabilizers of the different elements of Π_{A_0}, and so $[\mathbb{Q}[\pi_{A_0}] : \mathbb{Q}] = |\Pi_{A_0}|$. The division algebra $D =_{\text{df}} \text{End}^0(A_0)$ has centre $\mathbb{Q}[\pi_{A_0}]$, and D splits at no real prime of $\mathbb{Q}[\pi_{A_0}]$, splits at each finite prime not dividing p, and has invariant

$$\text{inv}_v(D) = s_\pi(v)[\mathbb{Q}[\pi_{A_0}]_v : \mathbb{Q}_p] \bmod \mathbb{Z},$$

at each prime v dividing p. By class field theory, the order of D in the Brauer group of $\mathbb{Q}[\pi_{A_0}]$ is the smallest positive integer e such that $e \cdot \text{inv}_v(D) \in \mathbb{Z}$ for all v, and $[D : \mathbb{Q}[\pi_{A_0}]]^{1/2} = e$. Moreover,

$$2 \dim A_0 = [D : \mathbb{Q}[\pi_{A_0}]]^{1/2} \cdot [\mathbb{Q}[\pi_{A_0}] : \mathbb{Q}],$$

and so A_0 has many endomorphisms. The set of slopes of the Dieudonné module of A_0 is $\{s_{\pi_{A_0}}(v) \mid v|p\}$, and an s in this set has multiplicity

$$\sum_{v,\, s_{\pi_{A_0}}(v)=s} \frac{2 \dim A_0 \cdot [\mathbb{Q}[\pi_{A_0}]_v : \mathbb{Q}_p]}{[\mathbb{Q}[\pi_{A_0}] : \mathbb{Q}]}.$$

It remains to classify the Weil germs.

Classification of Weil germs. Fix a CM-subfield K of $\mathbb{Q}^{\text{al}}$, finite and Galois over $\mathbb{Q}$. For a Weil germ π in $\mathbb{Q}^{\text{al}}$ and a prime w of $\mathbb{Q}^{\text{al}}$ dividing p, let

$$f_\pi^K(w) = s_\pi(w)[K_w : \mathbb{Q}_p].$$

Define $W^K(p^\infty)$ to be the set of Weil germs in $\mathbb{Q}^{\text{al}}$ represented by an element of K and such that $f_\pi^K(w) \in \mathbb{Z}$. Since $W(p^\infty) = \bigcup_K W^K(p^\infty)$, it suffices to describe $W^K(p^\infty)$ for each K.

Let F be the maximal real subfield of K, and let X and Y be the sets of primes in K and F respectively dividing p. Then there is an exact sequence

$$0 \to W^K(p^\infty) \to \mathbb{Z}^X \times \mathbb{Z} \to \mathbb{Z}^Y \to 0.$$

The first map is $\pi \mapsto (f_\pi^K, \text{wt}(\pi))$ and the second is

$$(f, m) \mapsto f|Y - n_0 \cdot m \cdot \sum_{v \in Y} v$$

where $n_0 = [K_w : \mathbb{Q}_p]$ for any prime w of K dividing p (it is independent of w). Thus $\pi \mapsto f_\pi^K$ identifies $W^K(p^\infty)$ with the set of $f \in \mathbb{Z}^X$ such that $f(w) + f(\iota w) = n_0 \cdot m$ for some integer m (independent of w).

Under $A_0 \leftrightarrow \Pi_{A_0}$, the abelian varieties corresponding to orbits of Γ in $W^K(p^\infty) \cap W(p^\infty)_{1,+}$ are those with the property that, for every $\sigma :$ $\mathbb{Q}[\pi_{A_0}] \hookrightarrow \mathbb{Q}^{al}$, $\sigma\mathbb{Q}[\pi_{A_0}] \subset K$ and $\text{End}^0(A_0) \otimes_{\mathbb{Q}[\pi_{A_0}],\sigma} K$ is a matrix algebra. Thus, there is a one-to-one correspondence between the isogeny classes of abelian varieties over $\mathbb{F}$ whose endomorphism algebra is split by K in this sense and the $\text{Gal}(K/\mathbb{Q})$-orbits of $f \in \mathbb{Z}^X$ such that $f(w) + f(\iota w) = n_0$ and $f(w) \geq 0$ for all w.

REMARK. Given a possible slope function for a Weil germ π, the Dieudonné module of the corresponding abelian variety imposes restrictions on the possible factorizations of p in $\mathbb{Q}[\pi]$. For example, suppose that $A(\pi)$ has slopes 0, $1/2$, and 1, that the multiplicity of $1/2$ is 2, and that 0 and 1 do occur. Then the Dieudonné module of $A(\pi)$ has a simple isogeny factor of rank 2, which implies that a prime w for which $s_\pi(w) = 1/2$ must be of degree 2 (if it had degree 1, the action of $\mathbb{Q}[\pi]_w$ on the Dieudonné module would split off an isogeny factor of rank 1). Thus, the endomorphism algebra of such an abelian variety is commutative.

A.7. Calculation of the groups over $\mathbb{F}$. Let A_0 be an abelian variety over $\mathbb{F}$. Then A_0 is isogenous to a product $A_1^{s_1} \times \ldots \times A_t^{s_t}$ with the A_i simple and pairwise nonisogenous, and $G(A_0) \cong G(A_1 \times \ldots \times A_t)$ for $G = L$, M, or P; moreover, $L(A_0)_0 \cong L(A_1)_0 \times \ldots \times L(A_t)_0$. Thus, in the following, we assume that A_0 is a product of pairwise nonisogenous simple abelian varieties. Then $E =_{df} C(A_0)$ is either a CM-algebra or the product of a CM-algebra with $\mathbb{Q}$—the second case occurs when one of the isogeny factors of A_0 is a supersingular elliptic curve. The Rosati involution is complex conjugation on each CM-factor of E and the identity map $\mathbb{Q}$.

The Lefschetz group. The description of $L(A_0)$ as a subgroup of $(\mathbb{G}_m)_{E/\mathbb{Q}}$ in terms of characters is the same as in the complex case.

Thus, the group $L(A_0)$ is the subgroup of $(\mathbb{G}_m)_{E/\mathbb{Q}}$ whose character group is

$$\frac{\mathbb{Z}^{\Sigma_E}}{\{g \mid g = \iota g \text{ and } \sum g(\sigma) = 0\}}.$$

The weight map $w : \mathbb{G}_m \to L(A_0)$ corresponds to the map

$$[g] \mapsto \text{wt}(g) \overset{df}{=} \sum_{\sigma \in \Sigma_E} g(\sigma)$$

on characters, and the homomorphism $t : L(A_0) \to \mathbb{G}_m$ giving the action of $L(A_0)$ on the Tate object sends 1 to the element of $X^*(L(A_0))$ represented by $-\sigma - \iota\sigma$, any $\sigma \in \Sigma_E$.

It suffices to describe $L(A_0)_0$ in the case that A_0 is simple. When A_0 is supersingular elliptic curve, $L(A_0)_0 = \mu_2$; otherwise $L(A_0)_0$ is the subgroup

of $(\mathbb{G}_m)_{E/\mathbb{Q}}$ whose character group is

$$\frac{\mathbb{Z}^{\Sigma_E}}{\{g \mid g = \iota g\}}.$$

The map $\mu_2 \to L(A)_0$ corresponds to the map on characters $[g] \mapsto \sum g(\sigma)$ mod 2.

When A_0 is simple, the map $\sigma \mapsto \sigma(\pi_{A_0}) : \Sigma_E \to \Pi_{A_0}$ is bijective and commutes with the action of Γ, and so identifies $L(A_0)$ with the torus whose character group is

$$\frac{\mathbb{Z}^{\Pi_{A_0}}}{\{g \mid g = \iota g \text{ and } \sum g(\pi) = 0\}}.$$

The group $P(A_0)$. By definition, $P(A_0) \subset L(A_0)$, and a character $[g]$ of $L(A_0)$ is trivial on $P(A_0)$ if and only if $g(\pi_{A_0}) = 1$, where $g(\pi_{A_0})$ is the Weil germ $\prod_{\sigma \in \Sigma_E} (\sigma\pi_{A_0})^{g(\sigma)}$. A Weil germ is 1 if and only if its slopes are all zero, and so $[g]$ is trivial on $P(A_0)$ if and only if

$$\sum_{\sigma \in \Sigma_E} g(\sigma)s_{\sigma\pi_{A_0}}(w) = 0 \quad \text{for all } w.$$

Note that $s_{\sigma\pi_{A_0}}(w) = s_{\pi_{A_0}}(\sigma^{-1}w)$. Similarly, a character $[g]$ of $L(A_0)_0$ is trivial on $L(A_0)_0$ if and only if $g(\pi_{A_0}/p^{1/2}) = g$ where $p^{1/2}$ also denotes the Weil germ represented by the Weil p-number $p^{1/2}$.

The motivic group. Fix a prime $\ell \in S(A_0)$ (see Appendix B). Let Ω_λ be a finite Galois extension of $\mathbb{Q}_\ell$ splitting $L(A_0)$, and let $\chi \in X^*(L(A_0))$. Then χ is trivial on $M(A_0)$ if and only if $H^{2r}(A_0^s, \Omega_\lambda(r))_\chi$ contains a nonzero algebraic class for some r and s, in which case all the spaces $H^{2r}(A_0^s, \Omega_\lambda(r))_\chi$ consist entirely of algebraic classes.

Second description of $P(A_0)$. Let K be a CM-subfield of $\mathbb{Q}^{\text{al}}$, finite and Galois over $\mathbb{Q}$, and let P^K be the torus over $\mathbb{Q}$ such that $X^*(P^K) = W^K(p^\infty)$ (as a Γ-module). Assume that K is large enough to contain the conjugates of $\mathbb{Q}[\pi_{A_0}]$ and to split $\text{End}^0(A_0)$. For any character χ of $L(A_0)$, $\chi(\pi) \in W^K(p^\infty)$. Thus we have a homomorphism $[g] \mapsto [g(\pi)] : X^*(L(A_0)) \to W^K(p^\infty)$, which clearly commutes with the action of Γ. It corresponds to a homomorphism $\varrho_{A_0} : P^K(p^\infty) \to L(A_0)$, whose image is $P(A_0)$.

EXAMPLE. Let A_0 be isogenous to a product of elliptic curves, $A_0 \sim A_1 \times \ldots \times A_t$, no two of which are isogenous. The centre E of the endomorphism algebra of A_0 is the product $E = \prod E_i$ of the centres of the endomorphism algebras of the A_i. For each i, choose an embedding $\sigma_i : E_i \hookrightarrow \mathbb{Q}^{\text{al}}$. A character g of $(\mathbb{G}_m)_{E/\mathbb{Q}}$ is trivial on $L(A_0)_0$ if and only if, for each i for which A_i is ordinary $g(\sigma_i) = g(\iota\sigma_i)$, and for each i (there is at most one) for which A_i is supersingular $2 \mid g(\sigma_i)$.

Let $\pi \in E$ be a Weil q-number representing π_{A_0}; let $\pi = (\pi_1, \ldots, \pi_t)$. Then g is trivial on $P(A_0)_0$ if and only if $g(\pi^N) = q^{N \cdot \text{wt}(g)/2}$ for some N. The statement (Spiess 1999, Proposition)

Let $\alpha_1, \ldots, \alpha_{2m}$ be Weil q-numbers of elliptic curves over $\mathbb{F}_q$ such that $\alpha_1 \ldots \alpha_{2m} = q^m$; then, after possibly renumbering the α_i and replacing each α_i with α_i^N for some N, $\alpha_{2j-1}\alpha_{2j} = q$ for $j = 1, \ldots, m$.

implies that this holds only if g is trivial on $L(A_0)_0$. Thus $P(A_0) = L(A_0)$, and so no product of elliptic curves over $\mathbb{F}$ has an exotic Tate class.

EXAMPLE. Let A_0 be a simple abelian variety over $\mathbb{F}$ and let π be its Frobenius germ. Assume that there is a prime v_1 of degree 1 of $\mathbb{Q}[\pi]$ such that $s_\pi(v_1) = 0$, $s_\pi(\iota v_1) = 1$, and $s_\pi(v) = 1/2$ for $v \neq v_1$. Let π_1 be a Weil q-number representing π, and let g be a character of $(\mathbb{G}_m)_{\mathbb{Q}[\pi]/\mathbb{Q}}$. For any prime w of $\mathbb{Q}^{\text{al}}$ dividing p,

$$\text{ord}_w(g(\pi_1/q^{1/2})) = \frac{\text{ord}_{v_1} q}{2}(-g(\sigma) + g(\iota\sigma))$$

where σ is the unique embedding of $\mathbb{Q}[\pi]$ such that $\sigma^{-1}w = v_1$. Therefore, g is trivial on $P(A_0)_0$ if and only if $g = \iota g$, i.e., if and only if g is trivial on $L(A_0)_0$. Thus $P(A_0) = L(A_0)$, and no power of A_0 has an exotic Tate class. In particular, the Tate conjecture holds for the powers of A_0.

The abelian varieties of "K3-type" of Zarhin 1993 are covered by this example (they are the varieties for which, additionally, $[\mathbb{Q}[\pi] : \mathbb{Q}] = 2 \dim A_0$).

EXAMPLE. Let A_0 be a simple abelian variety of dimension > 1 over $\mathbb{F}$ and let π be its Frobenius germ. Assume that there is a prime v_1 of $\mathbb{Q}[\pi]$ whose decomposition group is $\{1, \iota\}$ for which $s_\pi(v_1) = 1/2 = s_\pi(\iota v_1)$; assume moreover that $s_\pi(v) = 0$ or 1 for all other primes. Let π_1 be a Weil q-number representing π, and let χ be a character of $X^*(L(A_0)_0)$ that is trivial on $P(A_0)_0$. If $\chi = m\chi_1$ for some $\chi_1 \in X^*(L(A_0)_0)$, then χ_1 is also trivial on $P(A_0)_0$. Thus, we may assume that χ is not divisible in $X^*(L(A_0)_0)$. Let $g = \sum g(\sigma)\sigma$ be an element of $\mathbb{Z}^{\Sigma_{\mathbb{Q}[\pi]}}$ representing χ and such that $g(\sigma) \neq 0 \Rightarrow g(\iota\sigma) = 0$. For any prime w of $\mathbb{Q}^{\text{al}}$ dividing p,

$$\text{ord}_w(g(\pi_1))/\text{ord}_w(q) \equiv \tfrac{1}{2}g(\sigma) \bmod \mathbb{Z}$$

where σ is such that $\sigma^{-1}w = v_1$. Hence $g(\sigma)$ is even. As w ranges over the primes dividing p, σ ranges over the elements of $\Sigma_{\mathbb{Q}[\pi_{A_0}]}$ for which $g(\sigma) \neq 0$. This contradicts the fact that χ is not divisible. Hence $\chi = 0$, and we see that $P(A_0) = L(A_0)$. Hence no power of A_0 has an exotic Tate class.

The "almost ordinary" abelian varieties of Lenstra and Zarhin 1993 are covered by this example.

A.8. Reduction of abelian varieties with many endomorphisms: the fundamental diagram. Fix a prime w_0 of $\mathbb{Q}^{\text{al}}$ dividing p, and let $\mathbb{F}$

be the residue field. As we noted in Section 1, it follows from the theory of Néron models that an abelian variety A over $\mathbb{Q}^{\mathrm{al}}$ with many endomorphisms has good reduction at w_0 to an abelian variety A_0 over $\mathbb{F}$. We shall explain the map $A \mapsto A_0$ in terms of the above classifications.

Assume A is isotypic, and let E be a CM-subfield of $\mathrm{End}^0(A)$ for which $H^1(A, \mathbb{Q})$ is free of rank 1, and let Φ be the CM-type on E defined by its action on $H^{1,0}$. Let π_{A_0} be the Weil germ of A_0 in E. We fix an embedding $\varrho_0 : E \hookrightarrow \mathbb{Q}^{\mathrm{al}}$, and explain how to construct $\varrho_0(\pi_{A_0})$. Let K be a CM-subfield of $\mathbb{Q}^{\mathrm{al}}$, finite and Galois over $\mathbb{Q}$, and large enough to contain all conjugates of E. As a subfield of $\mathbb{Q}^{\mathrm{al}}$, K acquires a prime w_0. For some h, $\mathfrak{P}_{w_0}^h$ will be principal, say $\mathfrak{P}_{w_0}^h = (a)$. Let $\alpha = a^{2n}$ where n is the index of the unit group of the maximal real subfield of K in the full unit group of K. Then $\psi_{\varrho_0}(\alpha)$, where ψ_{ϱ_0} is the CM-type on K defined in A.4, is a well-defined Weil $p^{2nhf(\mathfrak{P}_{w_0}/p)}$-integer of weight 1 lying in $\varrho_0 E$. Its inverse image in E represents π_{A_0}.

Assume now that E is a field. The value of the function $s_{\pi_{A_0}}$ on a prime v of E dividing p is given by

$$(\ast\ast\ast) \qquad s_{\pi_{A_0}}(v) = \frac{|\Phi(v)|}{|\Sigma_E(v)|}$$

where

$$\Sigma_E(v) = \{\sigma \in \Sigma_E \mid v = \sigma^{-1} w_0\}, \qquad \Phi(v) = \Phi \cap \Sigma_E(v).$$

Suppose A is simple, and that it corresponds to a $\mathrm{Gal}(K/\mathbb{Q})$-orbit Ψ in $X^*(S^K)$. An element $f \in X^*(S^K)$ can be regarded as a function $f : \Sigma_K \to \mathbb{Z}$. Define $\bar{f}$ to be the function $X \to \mathbb{Z}$ such that $\bar{f}(w) = \sum_{\tau w_0 = w} f(\tau)$, i.e., if f is $\sum f(\tau)\tau$, then $\bar{f}$ is $\sum f(\tau)\tau w_0$. Then A_0 is isogenous to a power of a simple abelian variety, which corresponds (as in A.6) to the $\mathrm{Gal}(K/\mathbb{Q})$-orbit $\{\bar{f} \mid f \in \Psi\} \subset W^K(p^\infty)$.

Let K be a CM-field, finite and Galois over $\mathbb{Q}$, and let F be the maximal totally real subfield of K. If no p-adic prime of F splits in K, then $S^K = \mathbb{G}_m$ and the only elements of $W^K(p^\infty)$ are those represented by the Weil p-numbers $p^{m/2}$. Otherwise, all the p-adic primes in F split in K, and there is an exact commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X^*(S^K) & \xrightarrow[\binom{g}{\mathrm{wt}(g)}]{g} & \mathbb{Z}^{\Sigma_K} \times \mathbb{Z} & \xrightarrow[g|F - m\sum_{\sigma \in \Sigma_F}\sigma]{\binom{g}{m}} & \mathbb{Z}^{\Sigma_F} & \longrightarrow & 0 \\
& & \Big\downarrow{\scriptstyle g}\ [g(\alpha)] & & \binom{\tau}{m}\Big\downarrow\binom{\tau w_0}{m} & & \sigma\Big\downarrow\sigma v_0 & & \\
0 & \longrightarrow & W^K(p^\infty) & \xrightarrow[\binom{f_\pi^K}{\mathrm{wt}(\pi)}]{\pi} & \mathbb{Z}^X \times \mathbb{Z} & \xrightarrow[f|Y - n_0 \cdot m\sum_{v \in Y} v]{\binom{f}{m}} & \mathbb{Z}^Y & \longrightarrow & 0
\end{array}
$$

The element above (or to the left of) an arrow is mapped to the element below (or to the right) by the arrow. The symbol $g(\alpha)$ denotes $\prod \sigma(\alpha)^{g(\sigma)}$, $n_0 = [K_{w_0} : \mathbb{Q}_p]$, and v_0 is the prime on F induced by w_0.

We saw above that an abelian variety (A, i) of CM-type (E, Φ) reduces modulo the prime w_0 of $\mathbb{Q}^{\mathrm{al}}$ to an isotypic abelian variety A_0 whose Weil germ is determined by $(***)$. Every simple abelian variety arises in this way: let A_0 be a simple abelian variety over $\mathbb{F}$, and let E be a CM-field that can be embedded as a maximal subfield of $\mathrm{End}^0(A)$ containing $\mathbb{Q}[\pi_{A_0}]$; algebraic number theory shows that E exists, and it is an elementary exercise to show that there exist CM-types Φ on E such that $s_{\pi_{A_0}}$ is given by the formula $(***)$; let A be an abelian variety over $\mathbb{Q}^{\mathrm{al}}$ of CM-type (E, Φ); it is uniquely determined up to isogeny, and A_0 is isogenous to the reduction of A at w_0.

Thus, to give a lifting (up to isogeny) of A_0 to characteristic zero is to give a CM maximal subfield E of $\mathrm{End}^0(A)$ and a CM-type on E satisfying $(***)$.

Appendix B. Numerical equivalence on abelian varieties with many endomorphisms

Let A be an abelian variety of dimension g over an algebraically closed field. In characteristic zero, two cycles in $\mathcal{Z}^r(A)$ are *homologically equivalent* if their classes in $H^{2r}(A, \mathbb{Q}(r))$ are equal, and in characteristic $p \neq 0$, they are *ℓ-homologically equivalent*, $\ell \neq p$, if their classes in the étale cohomology group $H^{2r}(A, \mathbb{Q}_\ell(r))$ are equal. Because of Poincaré duality and the compatibility of intersection products with cup products, homological equivalence implies numerical equivalence. It is generally conjectured that they coincide.

Part (a) of the following theorem is a special case of a theorem of Lieberman (1968, Theorem 4), and part (b) is a theorem of Clozel (1999). The proof is based on that of Clozel.

THEOREM B.1. (a) *For any abelian variety A with many endomorphisms over an algebraically closed field k of characteristic zero, homological equivalence coincides with numerical equivalence on $\mathcal{Z}^r(A)$, all r.*

(b) *For any abelian variety A_0 over $\mathbb{F}$, there exists a set S of primes ℓ of density > 0 (depending on A_0) for which ℓ-homological equivalence coincides with numerical equivalence on $\mathcal{Z}^r(A_0)$, all r.*

Proof. In the proof, we ignore Tate twists, i.e., we choose an identification of $\mathbb{Q} \approx \mathbb{Q}(1)$ (or $\mathbb{Q}_\ell \approx \mathbb{Q}_\ell(1)$).

First consider the characteristic zero case. Choose ([9]) an étale CM-algebra $E \subset \mathrm{End}^0(A)$ such that $H^1(A, \mathbb{Q})$ is free of rank 1 as an E-module and E is stable under the Rosati involution defined by some ample divi-

---

[9] For each isotypic isogeny factor A_i of A, choose a CM-field E_i in $\mathrm{End}^0(A_i)$ of degree $2 \dim A_i$, and let $E = \prod E_i$. Write $H_1(A, \mathbb{Q}) = E \cdot x_0$. For any $c \in E^\times$ such that $\iota_E c = -c$, $a x_0, b x_0 \mapsto \mathrm{Tr}_{E/\mathbb{Q}}(cab)$ is a Riemann form on A, and we can take D to be any divisor whose class it is. When A, E, and Φ are as in this paragraph, one says that (A, i), where i is the inclusion $E \hookrightarrow \mathrm{End}^0(A)$, is of *CM-type (E, Φ)*.

sor D. The action of $E \otimes_{\mathbb{Q}} \mathbb{R}$ on $H^{1,0}$ defines a CM-type Φ on E. We have $\mathrm{Hom}(E, \mathbb{Q}^{\mathrm{al}}) = \Phi \sqcup \overline{\Phi}$.

Let Ω be the smallest subfield of $\mathbb{Q}^{\mathrm{al}}$ containing σE for every homomorphism $\sigma : E \to \mathbb{Q}^{\mathrm{al}}$. It is a CM-field, finite and Galois over $\mathbb{Q}$. Let $H^r(A, \mathbb{Q})_\Omega = H^r(A, \mathbb{Q}) \otimes \Omega$, and let $H^1(A)_\sigma$ be the subspace of $H^1(A, \mathbb{Q})_\Omega$ on which E acts through σ. Then $H^1(A, \mathbb{Q})_\Omega = \bigoplus_{\sigma \in \Phi \sqcup \overline{\Phi}} H^1(A)_\sigma$ and $H^1(A)_\sigma$ is one-dimensional. As $H^r(A, \mathbb{Q})_\Omega = \bigwedge_\Omega^r H^1(A, \mathbb{Q})_\Omega$, it follows that

$$H^r(A, \mathbb{Q})_\Omega = \bigoplus_{I, J, |I| + |J| = r} H^r(A)_{I,J}$$

where I and J are subsets of Φ and $\iota \Phi$ respectively, and $H^r(A)_{I,J} =_{\mathrm{df}} H^r(A)_{I \sqcup J}$ is the subspace on which $e \in E$ acts as $\prod_{\sigma \in I \sqcup J} \sigma e$—it is of dimension 1 and of Hodge type $(|I|, |J|)$. For $x \in H^r(A, \mathbb{Q})_\Omega$, let $x_{I,J}$ denote the projection of x on $H^r(A)_{I,J}$. Because $x \mapsto x_{I,J}$ is multiplication by an idempotent $e_{I,J}$ of $E \otimes \Omega$, it sends algebraic classes to algebraic classes.

Let L be the class in $H^2(A, \mathbb{Q})$ of the divisor D. Because L is algebraic, its isotypic components in $H^2(A, \mathbb{Q})_\Omega$ are of type $(\sigma, \iota\sigma)$, $\sigma \in \Sigma_E$, and, because L defines a nondegenerate form on $H_1(A, \mathbb{Q})$, each such component is nonzero.

For each σ, choose a nonzero element ω_σ of $H^1(A)_\sigma$. Then $(\omega_\sigma)_{\Phi \sqcup \iota\Phi}$ is a basis for $H^1(A, \mathbb{Q})_\Omega$. We may suppose that the ω_σ have been chosen so that the $(\sigma, \iota\sigma)$ component of L is $\omega_\sigma \omega_{\iota\sigma}$. Denote $\prod_{\sigma \in I} \omega_\sigma \prod_{\sigma \in J} \omega_\sigma$ by $\omega_{I,J}$ — it is a basis for $H^r(A)_{I,J}$. For $i \leq g = \dim A$,

$$L^{g-i} = \Big(\sum_{\sigma \in \Sigma_E} \omega_\sigma \omega_{\iota\sigma} \Big)^{g-i} = \sum_M (g-i)! \omega_{M, \iota M}$$

where M runs over the subsets of Φ with $|M| = g - i$. In particular, $\omega_{M, \iota M}$ is algebraic. Moreover,

$$L^{g-i} \omega_{IJ} = \sum_{|M| = g-i} (g-i)! \omega_{I \cup M, J \cup \iota M}.$$

Only the subsets M disjoint from both I and J contribute to the sum.

We shall need the following theorem of Lieberman (Kleiman 1968, 2A11 and 2.2):

Let $\mathcal{A}^r$ be the space of algebraic classes in $H^{2r}(A, \mathbb{Q})$; then for $2r \leq g$, the map $L^{g-2r} : \mathcal{A}^r \to \mathcal{A}^{g-r}$ is an isomorphism.

Suppose ω_{IJ} is algebraic with $|I| + |J| = 2r \leq g$. Let $M = I \cap \iota J$, so that there exist I_0 and J_0 for which $I = I_0 \sqcup M$, $J = J_0 \sqcup \iota M$, $I_0 \cap \iota J_0 = \emptyset$. We shall prove by induction on $|I \cap \iota J|$ that ω_{I_0, J_0} is also algebraic. If $|I \cap \iota J| = 0$, there is nothing to prove. If not, $|I \cup \iota J| \leq 2r - 1$, and there exists a subset M of Φ with $g - 2r + 1$ elements disjoint from $I \cup \iota J$. Then $\omega_{I \sqcup M, J \sqcup \iota M}$ is nonzero

and algebraic. By Lieberman's theorem, there exists an $x \in \mathcal{A}^{r-1}$ such that $L^{g-2r+2}x = \omega_{I \sqcup M, J \sqcup \iota M}$. If $\omega_{I', J'}$ occurs with nonzero coefficient in x, then it is algebraic. But if $\omega_{I', J'}$ is chosen so that $\omega_{I \sqcup M, J \sqcup \iota M}$ occurs with nonzero coefficient in $L^{g-2r+2}\omega_{I', J'}$, then $I_0' = I_0$, $J_0' = J_0$. Since $|I' \cap \iota J'| = |I \cap J| - 2$, the induction hypothesis shows that ω_{I_0, J_0} is algebraic.

We now prove the theorem in the case of characteristic zero. We have to show that, for each $r \leq g$, the cup-product pairing

$$\mathcal{A}^r \times \mathcal{A}^{g-r} \to \mathbb{Q}$$

is nondegenerate. Lieberman's theorem shows that the two spaces have the same dimension, and so it suffices to show that the left kernel is zero. Thus, let x be a nonzero element of $\mathcal{A}^r$, $r \leq g$, and suppose $\omega_{I, J}$ occurs with nonzero coefficient in x. It suffices to show that $\omega_{I', J'}$ is algebraic, where I' and J' are the complements of I and J in Φ and $\iota\Phi$ respectively. From the last paragraph, we know that $\omega_{I, J} = \omega_{I_0 \sqcup M, J_0 \sqcup \iota M}$ with ω_{I_0, J_0} algebraic and I_0, ιJ_0, and M disjoint. Because $\mathcal{A}^j \otimes_{\mathbb{Q}} \Omega$ is stable under $\mathrm{Gal}(\Omega/\mathbb{Q})$, $\iota\omega_{I_0, J_0} = \omega_{\iota J_0, \iota I_0}$ is algebraic. But $\omega_{I', J'} = \omega_{\iota J_0, \iota I_0} \cdot \omega_{N, \iota N}$ where N is the complement of $I_0 \sqcup \iota J_0 \sqcup M$ in Φ, which is obviously algebraic.

We now prove the theorem in the case $k = \mathbb{F}$. After possibly replacing A_0 with an isogenous variety, we may assume that it lifts to an abelian variety A with many endomorphisms in characteristic zero (see A.8). Let E be a CM-algebra for A as in the first paragraph of the proof. If ℓ is such that ι is in the decomposition group of some prime λ of Ω dividing ℓ, then the same argument as in characteristic zero case applies once one replaces $\mathbb{Q}$ with $\mathbb{Q}_\ell$ and Ω with Ω_λ (Lieberman's theorem holds for every Weil cohomology; in particular, it holds for the étale cohomology). The Frobenius density theorem shows that the set of primes ℓ such that ι is the Frobenius element at a prime λ dividing ℓ has density $1/[\Omega : \mathbb{Q}]$. For such a prime ℓ, ι is in the decomposition group of λ. ∎

Let A_0 be an abelian variety over $\mathbb{F}$, and let E_0 be the centre $C(A_0)$ of $\mathrm{End}^0(A_0)$. When A_0 does not have a supersingular elliptic curve as an isogeny factor, we define Ω_0 to be the composite of the fields σE_0 for $\sigma \in \Sigma_{E_0}$; otherwise we define it to be the composite of these fields with $\mathbb{Q}[\sqrt{-p}]$. Define $S(A_0)$ to be the set of finite primes $\ell \neq p$ such that ι is contained in the decomposition group of λ for one (hence every) prime λ of Ω_0 dividing ℓ. Note that $S(A_0)$ depends only on the finite set of simple isogeny factors of A_0; in particular, $S(A_0) = S(A_0^s)$.

PROPOSITION B.2. *Statement* (b) *of Theorem* B.1 *holds with* $S = S(A_0)$.

Proof. Suppose A_0 is isogenous to $A_1^{s_1} \times \ldots \times A_t^{s_t}$ with the A_i simple and nonisogenous in pairs. Assume initially that none of the A_i is a supersingular elliptic curve. Then each $C(A_i)$ is a CM-field.

For each i, let $D_i = \text{End}^0(A_i)$, let $m_i = [D_i : C(A_i)]^{1/2}$, and let $C(A_i)_+$ be the maximal real subfield of $C(A_i)$. Fix an $\ell \in S(A_0)$. For each i, there exists a field F_i cyclic of degree m_i over $C(A_i)_+$ and such that each real and ℓ-adic prime of $C(A_i)_+$ splits in F_i and the local degree at each p-adic prime is m_i (Artin and Tate 1961, p. 105, Theorem 5). Let $E_i = F_i \cdot C(A_i)$. Then E_i is a CM-field that splits D_i (Tate 1968/69, p. 7) and can be realized as a subfield of D_i. Therefore (Tate 1968/69, Théorème 2), A_i is isogenous to the reduction of an abelian variety $\widetilde{A}_i$ with $\text{End}^0(\widetilde{A}_i) = E_i$.

After replacing A_0 with an isogenous variety, we may suppose that it lifts to the abelian variety $A =_{\text{df}} \widetilde{A}_1^{s_1} \times \ldots \times \widetilde{A}_t^{s_t}$. The étale algebra $E =_{\text{df}} E_1^{s_1} \times \ldots \times E_t^{s_t}$ acts on A diagonally, and satisfies the conditions in the first paragraph of the proof of Theorem B.1. The field Ω generated by the images of E in $\mathbb{Q}^{\text{al}}$ is $\Omega_0 \cdot F_1 \ldots F_t$. Because of our choice of the F_i, every ℓ-adic prime in this field is fixed ([10]) by ι. This completes the proof of the proposition in this case.

When we add a factor $A_{t+1}^{s_{t+1}}$ to A_0 with A_{t+1} a supersingular elliptic curve, Ω is replaced with $\Omega \cdot E_{t+1}$ where E_{t+1} can be taken to be any quadratic field in which p does not split. If we choose $E_{t+1} = \mathbb{Q}[\sqrt{-p}]$, then $\Omega = \Omega_0 \cdot F_1 \ldots F_t$ still holds, and the same argument applies. ∎

Let A be an abelian variety with many endomorphisms over $\mathbb{Q}^{\text{al}}$, and let A_0 be its reduction at the prime w_0. Fix an $\ell \neq p$. Then there are canonical isomorphisms $H^i(A, \mathbb{Q}_\ell(j)) \to H^i(A_0, \mathbb{Q}_\ell(j))$ for all i and j (proper and smooth base change theorems in étale cohomology). We say that a cohomology class $\gamma \in H^{2r}(A, \mathbb{Q}(r))$ is w_0-*algebraic* if its image γ_ℓ in $H^{2r}(A_0, \mathbb{Q}_\ell(r))$ is in the $\mathbb{Q}$-span of the algebraic classes on A_0. Every algebraic class is w_0-algebraic, but not every w_0-algebraic class is algebraic.

THEOREM B.3. *For any nonzero w_0-algebraic class α on A, there exists a w_0-algebraic class α' such that $\alpha \cup \alpha' \neq 0$.*

Proof. Let $\mathcal{A}^r(w_0)$ be the $\mathbb{Q}$-space of w_0-algebraic classes in $H^{2r}(A, \mathbb{Q}(r))$. The proof of the characteristic zero case of the Theorem in A.3 will apply with "algebraic" replaced by "w_0-algebraic" once we have shown that Lieberman's theorem holds for $\mathcal{A}^r(w_0)$: for $2r \leq g$, $L^{g-2r} : \mathcal{A}^r(w_0) \to \mathcal{A}^{g-r}(w_0)$ is an isomorphism.

This map is automatically injective, and so we only have to prove surjectivity.

Let γ be a w_0-algebraic class in $H^{2g-2r}(A, \mathbb{Q}(g-r))$; by assumption, the image γ_ℓ of γ in $H^{2g-2r}(A_0, \mathbb{Q}_\ell(g-r))$ equals the class α_ℓ of some

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([10]) Let F_0 be the maximal totally real subfield of Ω_0. The condition that ι fixes all ℓ-adic primes in Ω_0 means that, for each ℓ-adic prime v of F_0, $\Omega_0 \otimes_{F_0} (F_0)_v$ is a field. Because ℓ splits in $F_1 \ldots F_t$, this property is retained by Ω.

$\alpha \in \mathcal{Z}^{g-r}(A_0)$. There exists a $\gamma' \in H^{2r}(A, \mathbb{Q}(r))$ such that $L^{g-2r}\gamma' = \gamma$ (hard Lefschetz theorem), and Lieberman's theorem says that there is an $\alpha' \in \mathcal{Z}^r(A_0)$ such that $L^{g-2r}\alpha' = \alpha$. The images of α' and γ' in $H^{2r}(A_0, \mathbb{Q}_\ell(r))$ map to α_ℓ and γ_ℓ respectively under the isomorphism $L^{g-r} : H^{2r}(A_0, \mathbb{Q}_\ell(r)) \to H^{2g-2r}(A_0, \mathbb{Q}_\ell(g - r))$. As $\alpha_\ell = \gamma_\ell$, this proves that γ' is w_0-algebraic. ∎

COROLLARY B.4. *Suppose that the ℓ-adic cohomology class c_ℓ of $c \in \mathcal{Z}^r(A_0)$ is nonzero. If c_ℓ is the image of a rational cohomology class on A (i.e., of an element of $H^{2r}(A, \mathbb{Q}(r))$), then c is not numerically equivalent to zero.*

Proof. Immediate consequence of the theorem and the compatibility of the cup-product pairings. ∎

The corollary implies that, if every algebraic class on A_0 "lifts" to a rational cohomology class in characteristic zero, then ℓ-adic homological equivalence on A_0 coincides with numerical equivalence.

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