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# The size function $h^\circ$ for a pure cubic field

by

PAOLO FRANCINI (Roma)

**Introduction.** Let $F$ be a number field and $\mathcal{S}_\infty$ the set of its infinite primes. Assume that $\#(\mathcal{S}_\infty) = r_1 + r_2$, where $r_1$ is the number of real places of $F$ and $r_2$ is the number of complex places. An *Arakelov divisor* is a pair $D = (J_D, \mathbf{v})$, where $J_D$ is a fractional ideal of the ring of integers $\mathcal{O}_F$ and $\mathbf{v} \in \mathbb{R}^{\mathcal{S}_\infty}$. We say that $J_D$ is the *finite part* of $D$, while $\mathbf{v}$ is the *infinite part*. We denote by $\mathcal{N}(J_D)$ the ordinary norm of $J_D$ as a fractional ideal, and we define the *norm* of $D$ in the following way:

$$\mathcal{N}(D) = \frac{\prod_{\sigma\in\mathcal{S}_\infty} e^{\mathbf{v}_\sigma}}{\mathcal{N}(J_D)},$$

where $\mathbf{v} = (\mathbf{v}_\sigma)_{\sigma\in\mathcal{S}_\infty}$, and we define the *degree* of $D$ as $\deg(D) = \log \mathcal{N}(D)$.

The set of Arakelov divisors of a number field forms a group, which we denote by $\mathcal{D}iv(F)$. The group operation consists in multiplying the fractional ideals and summing the vectors in $\mathbb{R}^{\mathcal{S}_\infty}$. Thus we have the isomorphism $\mathcal{D}iv(F) \approx \mathcal{I}(F) \times \mathbb{R}^{\mathcal{S}_\infty}$, where $\mathcal{I}(F)$ is the group of fractional ideals of $\mathcal{O}_F$.

The set of zero degree Arakelov divisors is a subgroup $\mathcal{D}iv^\circ(F)$ of $\mathcal{D}iv(F)$, as is $\mathcal{PD}iv(F)$, the set of *principal* Arakelov divisors. These are the divisors of the type

$$(x) = (x^{-1}\mathcal{O}_F, ((-[F_\sigma : \mathbb{R}] \log|\sigma(x)|)_{\sigma\in\mathcal{S}_\infty})), \qquad x \in F^*.$$

Here, $|z|$ stands for the standard absolute value $\sqrt{z\bar{z}}$ of the complex number $z$ and $F_\sigma$ is the completion of the field $F$ ($\mathbb{R}$ or $\mathbb{C}$) at the place $\sigma$. We define the *Picard group* of $F$ as

$$\mathcal{P}ic(F) = \frac{\mathcal{D}iv(F)}{\mathcal{PD}iv(F)}$$

and then we put

$$\mathcal{P}ic^\circ(F) = \frac{\mathcal{D}iv^\circ(F)}{\mathcal{PD}iv(F)}.$$

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The vector $\mathbf{v}$, the infinite part of $D$, assigns a metric to the fields $F_\sigma$, and hence to $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$, in the following manner:

$$\|1_\sigma\|_D^2 = d_\sigma e^{-2\mathbf{v}_\sigma/d_\sigma} = \begin{cases} e^{-2\mathbf{v}_\sigma}, & \sigma \text{ real}, \\ 2e^{-\mathbf{v}_\sigma}, & \sigma \text{ complex}; \end{cases}$$

here and in what follows, we set $d_\sigma = [F_\sigma : \mathbb{R}]$ and $1_\sigma$ is the image of $1 \in F$ in $F_\sigma$. In this way, we associate to an Arakelov divisor $D$ a Hermitian line bundle over $\mathcal{O}_F$, i.e., $J_D$ with the above metric on $J_D \otimes_{\mathbb{Z}} \mathbb{R}$. The group $\mathcal{P}ic(F)$, defined as above, is isomorphic to the group of the isomorphism classes of Hermitian line bundles over $\mathcal{O}_F$. We refer to [Gr] for a more detailed account of this equivalent approach. We also observe that we have the exact sequence (see [GS])

$$0 \to T \to \mathcal{P}ic^\circ(F) \to \mathcal{C}l(F) \to 0,$$

where $T$ is a real torus of dimension $\#(\mathcal{S}_\infty) - 1$.

The fractional ideal $J_D$, viewed as a real lattice in the Euclidean space $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ (which is endowed with the metric specified by the infinite part of $D$), may be considered as an analogue of the space $H^\circ(D)$, the global sections of the divisor $D$ on a curve. When $S$ is a subset of a lattice in a Euclidean space, we attach to it the real number

$$k^\circ(S) = \sum_{x \in S} e^{-\pi\|x\|^2}.$$

In particular, we define $k^\circ(D)$ as the number $k^\circ(J_D)$. This definition is clearly equivalent to the following. For $x \in F$ and $D = (J_D, \mathbf{v}) \in \mathcal{D}iv(F)$, set

$$\begin{aligned} \Phi_D(x) &= \left( \left( \frac{d_\sigma}{e^{\mathbf{v}_\sigma}} \right)^{1/d_\sigma} \sigma(x) \right)_{\sigma \in \mathcal{S}_\infty} \\ &= ((e^{-\mathbf{v}_\sigma}\sigma(x))_{\sigma\,\text{real}}, (\sqrt{2}\, e^{-\mathbf{v}_\sigma/2}\sigma(x))_{\sigma\,\text{complex}}), \end{aligned}$$

which is a vector in $F \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$, and set $k^\circ(D) = k^\circ(\widetilde{D})$, where $\widetilde{D}$ is the lattice $\widetilde{D} = \{\Phi_D(x) \mid x \in J_D\} \subset \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$, taking the standard Euclidean inner product on $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$. We record the basic fact that $k^\circ(D)$ depends only on the class of $D$ in $\mathcal{P}ic(F)$.

We recall from [GS] that the function $h^\circ = \log k^\circ$ can be seen as an analogue of the function $h^\circ$ which associates to each divisor $D$ the dimension of the space $H^\circ(D)$. In this framework, Poisson's summation formula provides a Riemann–Roch theorem: this neatly leads to the functional equation for the Dedekind zeta function and the classical results about the finiteness of the class groups and to Dirichlet's unit theorem as well. All these facts are essentially equivalent to the compactness of the group $\mathcal{P}ic^\circ(F)$. We also mention [B], where an analogue of $H^1(D)$ is defined, together with an analogue of Serre's duality. Before stating the main result, we give another definition, following [K]:

DEFINITION. Let $d$ be a positive cube-free integer. Let $d = ab^2$ such that $ab$ is square-free. If $a^2 - b^2$ is not divisible by 9, the field $\mathbb{Q}(\sqrt[3]{d})$ is said to be a *pure cubic field of the first kind*, otherwise $\mathbb{Q}(\sqrt[3]{d})$ is called a *pure cubic field of the second kind.*

In this paper, we shall prove the following fact:

THEOREM. *Let $K$ be a pure cubic field of the first kind. Then the function $k^\circ$ on $\mathcal{P}ic^\circ(K)$ has its unique global maximum at the trivial bundle class.*

In [GS], the same property is conjectured for all number fields which are Galois over $\mathbb{Q}$ or over a complex quadratic field, corresponding to a standard fact about algebraic curves, i.e., that $h^\circ(D) = 0$ for any non-principal zero degree divisor $D$, and $h^\circ(D) = 1$ when $D$ is principal. In [F], the corresponding statement was proved for quadratic number fields. These facts support further the fruitful analogy between the geometric and the arithmetic situation.

We outline the steps of the proof. If $D$ is a zero degree Arakelov divisor, the main contributions to $k^\circ(\widetilde{D})$ come from the shortest lattice vectors. We notice that, for $x \in F$, the arithmetic-geometric mean inequality implies that

$$\langle \Phi_D(x), \Phi_D(x) \rangle \geq [F:\mathbb{Q}] \left( \frac{\mathcal{N}(x)}{\mathcal{N}(J_D)} \right)^{2/[F:\mathbb{Q}]}.$$

In particular, when $F$ is a cubic field, if we put $\mathcal{N}(x)/\mathcal{N}(J_D) = m$, then $\langle \Phi_D(x), \Phi_D(x) \rangle \geq 3\sqrt[3]{m^2}$. This permits us to reduce to a local problem and, thus, the question is to check that the trivial bundle is a stationary point for $k^\circ$. In general, this is not true if $F$ is not Galois over $\mathbb{Q}$. For instance, it is false if $F$ is the complex cubic field $\mathbb{Q}(\alpha)$ of discriminant $-23$. In the case of a pure cubic field of the first kind, the fact that the function $h^\circ$ has a local maximum at the trivial class comes from the existence of an orthogonal $\mathbb{Z}$-basis for the ring of integers such that each basis element has the same length under all the field embeddings $\sigma \in \mathcal{S}_\infty$. This property is not enjoyed by cubic fields of the second kind. For $\mathbb{Q}(\sqrt[3]{10})$ the statement of the conjecture does not hold.

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## 1. Some estimates

Lemma 1.1. *Let $B_n(x,\varepsilon)$ denote the open ball of $\mathbb{R}^n$ centered at $x$ of radius $\varepsilon$, and if $S$ is a subset of a lattice, set*

$$\varrho_m(S) = \sum_{x\in S} \|x\|^m e^{-\pi\|x\|^2},$$

*where $m \in \mathbb{N}$ (we remark that $\varrho_\circ(S) = k^\circ(S)$). Let $L$ be a lattice in $\mathbb{R}^3$ whose non-zero vectors have length at least $\sqrt{3}$. Then*

$$k^\circ(L - B_3(0,\sqrt{3}\sqrt[3]{2})) \le \frac{1.05}{10^5}, \qquad \varrho_4(L - B_3(0,\sqrt{3}\sqrt[3]{4})) \le \frac{2}{10^7}.$$

*Proof.* If $v,w \in L$ and $v \neq w$, then $B_3(v,\sqrt{3}/2)$ and $B_3(w,\sqrt{3}/2)$ are disjoint. Take $r \ge \sqrt{3}\sqrt[3]{2}$ and $\delta \le \sqrt{3}/2$. Set

$$A_{r,\delta} = \{(x,y,z) \in \mathbb{R}^3 \mid r^2 \le x^2+y^2+z^2 \le (r+\delta)^2\}$$

and, for $k \ge 0$, let $\mathcal{N}_k(r,\delta) = \#(A_{r+k\delta,\delta} \cap L)$. Consider the map

$$f : A_{r,\delta} \to \mathbb{R}^4,$$
$$u = (x,y,z) \mapsto \left(\frac{(r+\delta)x}{\|u\|}, \frac{(r+\delta)y}{\|u\|}, \frac{(r+\delta)z}{\|u\|}, r+\delta-\|u\|\right).$$

Clearly,

$$f(A_{r,\delta}) = \{(x,y,z,t) \in \mathbb{R}^4 \mid x^2+y^2+z^2 = r+\delta,\ 0 \le t \le \delta\};$$

moreover, if $a,b \in A_{r,\delta}$, then $d(a,b) \le d(f(a),f(b))$. So, if $a$ and $b$ are two distinct points in $A_{r,\delta} \cap L$, then the two balls $B_4(f(a),\sqrt{3}/2)$ and $B_4(f(b),\sqrt{3}/2)$ are disjoint. For $h \in [0,\delta]$, let $C_h$ be the hyperplane

$$C_h = \{(x,y,z,t) \in \mathbb{R}^4 \mid t = h\}.$$

The intersection $f(A_{r,\delta}) \cap C_h$ is a 2-dimensional sphere of radius $r+\delta$. The intersection $B_4(f(a),\sqrt{3}/2) \cap C_h$ is an open ball whose center lies in $f(A_{r,\delta})$ and whose radius $\widetilde{r}$ satisfies $\widetilde{r}^2 + h^2 = 3/4$. For the area $\alpha$ of the cone with vertex $f(a)$ and bounded by the circle $f(A_{r,\delta}) \cap \partial(B_4(f(a),\sqrt{3}/2) \cap C_h)$, we have

$$\alpha = \frac{\pi\widetilde{r}^2}{2(r+\delta)}\sqrt{4(r+\delta)^2 - \widetilde{r}^2} \ge \frac{\pi(3/4-\delta^2)}{2(r+\delta)}\sqrt{4(r+\delta)^2 - 3/4 + \delta^2}.$$

The surface $f(A_{r,\delta}) \cap B_4(f(a),\sqrt{3}/2) \cap C_h$ has a larger area than this cone, so the volume of $B_4(f(a),\sqrt{3}/2) \cap f(A_{r,\delta})$ satisfies

$$\mathrm{vol}(f(A_{r,\delta}) \cap B_4(f(a),\sqrt{3}/2)) > \frac{\pi\delta(3-4\delta^2)}{16(r+\delta)}\sqrt{16(r+\delta)^2 - 3 + 4\delta^2}.$$

Since the volume of $f(A_{r+\delta,\delta})$ is $4\pi\delta(r+\delta)^2$ and the lattice $L$ is symmetric with respect to the origin, for $k \geq 0$ we have

$$(1)\qquad \mathcal{N}_k(r,\delta) \leq 2\left\lfloor \frac{\frac{1}{2}\operatorname{vol}(f(A_{r+k\delta,\delta}))}{\operatorname{vol}(B_4(f(a),\sqrt{3}/2)\cap f(A_{r+k\delta,\delta}))}\right\rfloor$$
$$\leq 2\left\lfloor \frac{32(r+(k+1)\delta)^3}{(3-4\delta^2)+\sqrt{16(r+(k+1)\delta)^2+4\delta^2-3}}\right\rfloor = \mathcal{M}_k(\delta).$$

Therefore,

$$k^\circ(L - B(0,\sqrt{3}\,\sqrt[3]{2})) < \sum_{k\in\mathbb{N}} \mathcal{M}_k(\delta)e^{-\pi(\sqrt{3}\,\sqrt[3]{2}+k\delta)^2},$$
$$\varrho_4(L - B(0,\sqrt{3}\,\sqrt[3]{4})) < \sum_{k\in\mathbb{N}} \mathcal{M}_k(\delta)(\sqrt{3}\,\sqrt[3]{4}+k\delta)^4 e^{-\pi(\sqrt{3}\,\sqrt[3]{4}+k\delta)^2},$$

for every $\delta < \sqrt{3}/2$. From these inequalities, one finds the numerical results. ∎

Now, let $K = \mathbb{Q}(\sqrt[3]{d})$ be a pure cubic field. It has one real place and one complex place. We consider $K$ as a subset of $\mathbb{R}$, via its real embedding, and we denote by $\sigma$ its complex place, represented by the field homomorphism $\sigma : K \hookrightarrow \mathbb{C}$ given by

$$\sigma : \sqrt[3]{d} \mapsto \sqrt[3]{d}\,\zeta,$$

where $\zeta = e^{2\pi i/3}$. With this notation, if $D = (J_D,(\alpha,\beta)) \in K$, then

$$k^\circ(D) = \sum_{x\in J_D} \exp(-\pi(e^{-2\alpha}|x|^2 + 2e^{-\beta}|\sigma(x)|^2)).$$

A zero degree Arakelov divisor with finite part $J$ has the form

$$D_s = \left(J, \left(\log s, \log\frac{\mathcal{N}(J)}{s}\right)\right)$$

for some $s > 0$. As explained in the introduction, we associate to $D_s$ the real lattice $\widetilde{D}_s = \Phi_{D_s}(J) \subset \mathbb{R}\times\mathbb{C} \approx \mathbb{R}^3$, where

$$\Phi_{D_s} : a \mapsto (x_a, y_a, z_a) = \left(\frac{a}{s}, \sqrt{\frac{2s}{\mathcal{N}(J)}}\,\mathcal{R}e(\sigma(a)), \sqrt{\frac{2s}{\mathcal{N}(J)}}\,\mathcal{I}m(\sigma(a))\right).$$

We have the equality

$$\widetilde{D}_s = \begin{pmatrix} s^{-1} & 0 & 0\\ 0 & \sqrt{s} & 0\\ 0 & 0 & \sqrt{s}\end{pmatrix}\widetilde{D}_1$$

and, moreover,

$$x_a(y_a^2 + z_a^2) = 2\,\frac{\mathcal{N}(a)}{\mathcal{N}(J)}$$

for every $s > 0$. Therefore $\Phi_{D_s}(a)$ lies on the surface in $\mathbb{R}^3$ described by the equation

$$x(y^2+z^2) = 2\,\frac{\mathcal{N}(a)}{\mathcal{N}(J)}.$$

On this surface, the points that are nearest to the origin are at distance $\sqrt{3}(\mathcal{N}(a)/\mathcal{N}(J))^{1/3}$. The upper bound of Lemma 1.1 permits us to rule out some cases for the maximum of $k^\circ$ and to reduce the problem to a connected neighborhood of the trivial class.

COROLLARY 1.2. *Let $K$ be a cubic field and $D \in \mathcal{D}iv^\circ(K)$ such that $J_D$ is not a principal fractional ideal. Then $k^\circ(D) < k^\circ(\mathcal{O}_K)$.*

*Proof.* Let $x \in J_D$. The distance of the lattice point $\Phi_D(x)$ from the origin is at least $\sqrt{3}(\mathcal{N}(x)/\mathcal{N}(J_D))^{1/3}$. If $J_D$ is not principal, then $\mathcal{N}(x) \geq 2\,\mathcal{N}(J_D)$, hence a non-zero vector in the lattice $\widetilde{D}$ has length at least $\sqrt{3}\,\sqrt[3]{2}$. Applying Lemma 1.1 we get

$$k^\circ(D) = k^\circ(\widetilde{D}) < 1 + \frac{1.05}{10^5} < 1 + 2e^{-3\pi} < k^\circ(\mathcal{O}_K). \ \blacksquare$$

Therefore, in order to determine the maximum of $k^\circ$, we may consider only the Arakelov divisors whose finite part is a principal fractional ideal, i.e., the connected component of $\mathcal{P}ic^\circ(K)$. A zero degree Arakelov divisor $D$ with principal finite part is equivalent to a divisor of the form

$$D_s = (\mathcal{O}_K, (\log s, -\log s)).$$

Notice that $\|\Phi_{D_s}(x)\|^2 = x^2/s^2 + 2s\,|\sigma(x)|^2$. Setting, for $s > 0$,

$$g(s) = k^\circ(D_s) = \sum_{x \in \mathcal{O}_K} e^{-\pi\|\Phi_{D_s}(x)\|^2},$$

we have to show that the function $g$ attains its maximum at $s = 1$.

A basic fact about pure cubic fields is the following (see [K, 2.5.1]):

PROPOSITION 1.3. *Let $d = ab^2$, with $ab$ a square-free integer, and let $K = \mathbb{Q}(\sqrt[3]{d})$. Then, if $K$ is of the first kind, a $\mathbb{Z}$-basis for $\mathcal{O}_K$ is given by*

$$\{1, \sqrt[3]{ab^2}, \sqrt[3]{a^2b}\}.$$

*If $K$ is of the second kind, a $\mathbb{Z}$-basis for $\mathcal{O}_K$ is*

$$\{(1 + a\sqrt[3]{ab^2} + b\sqrt[3]{a^2b})/3, \sqrt[3]{ab^2}, \sqrt[3]{a^2b}\}.$$

By this proposition, if $K = \mathbb{Q}(\sqrt[3]{ab^2})$ with $ab$ a square-free integer, then

$$\Delta_K = \begin{cases} -27a^2b^2 & \text{if } a^2 - b^2 \neq 0 \pmod 9, \\ -3a^2b^2 & \text{if } a^2 - b^2 = 0 \pmod 9. \end{cases}$$

COROLLARY 1.4. *Let $K$ be a pure cubic field and let $\varepsilon > 1$ be a fundamental unit of $\mathcal{O}_F$. Let $D_s = (\mathcal{O}_K, (\log s, -\log s)) \in \mathcal{D}iv^\circ(K)$. If $s \in [1/\sqrt{\varepsilon}, 0.92] \cup [1.09, \sqrt{\varepsilon}]$, then $k^\circ(D_s) < k^\circ(\mathcal{O}_K)$.*

*Proof.* Given $s \in [1/\sqrt{\varepsilon}, \sqrt{\varepsilon}\,]$, consider the set

$$B_s = \{\Phi_{D_s}(x) \mid 0 \neq x \in \mathcal{O}_K,\ \|\Phi_{D_s}(x)\|^2 < 3\sqrt[3]{4}\}.$$

If $\Phi_{D_s}(x) \in B_s$, then $x$ is a unit of $\mathcal{O}_F$, so $x = \pm\varepsilon^n$ for some $n \in \mathbb{Z}$. When $s \notin [0.92, 1.09]$, we have

$$2e^{-\pi(2s+1/s^2)} + \frac{1.05}{10^5} < 2e^{-3\pi}.$$

Moreover, from [PZ, 5.6], we have $\varepsilon \geq \sqrt[3]{|\Delta_F|}/3$, so that

$$B_s = \{\Phi_{D_s}(1), \Phi_{D_s}(-1)\}$$

for every $s \notin [0.92, 1.09]$. Therefore, for $s \in [1/\sqrt{\varepsilon},\, 0.92] \cup [1.09, \sqrt{\varepsilon}\,]$, we have

$$k^\circ(D_s) < 1 + 2e^{-\pi(2s+1/s^2)} + \frac{1.05}{10^5} < 1 + 2e^{-3\pi} < k^\circ(\mathcal{O}_F). \ \blacksquare$$

To complete the proof of the Theorem, we show that the function $g(s) = k^\circ(D_s)$ on the interval $[0.92, 1.09]$ has its maximum at $s = 1$. To prove this, it is enough to check that $g'(1) = 0$ and that $g''(s) < 0$ for every $s \in [0.92, 1.09]$. We first prove the latter statement.

LEMMA 1.5. *Let $K$ be a complex cubic field and let the function $g$ be defined as above. Then $g''(s) < 0$ for all $s \in [0.92, 1.09]$.*

*Proof.* For every $s > 0$, we have

$$\begin{aligned} g''(s) &= \frac{2\pi}{s^2} \sum_{x\in\mathcal{O}_K} \left(2\pi\left(s|\sigma(x)|^2 - \frac{x^2}{s^2}\right)^2 - 3\,\frac{x^2}{s^2}\right) e^{-\pi\|\Phi_{D_s}(x)\|^2} \\ &= \frac{2\pi}{s^2} \sum_{x\in\mathcal{O}_K} \left(2\pi\left(\frac{x^2}{s^2} + 2s|\sigma(x)|^2\right)^2 - 3\,\frac{x^2}{s^2} - 6\pi s^2|\sigma(x)|^4 - 12\pi\,\frac{|x\sigma(x)|^2}{s}\right) \\ &\quad \times e^{-\pi\|\Phi_{D_s}(x)\|^2}. \end{aligned}$$

Hence, $g''(s) < 0$ if and only if

$$2s^2 \sum_{x\in\mathcal{O}_K} \|\Phi_{D_s}(x)\|^4 e^{-\pi\|\Phi_{D_s}(x)\|^2} < 3 \sum_{x\in\mathcal{O}_K} \left(\frac{x^2}{\pi} + 4sx\mathcal{N}(x) + 2\,\frac{s^4\mathcal{N}(x)^2}{x^2}\right) e^{-\pi\|\Phi_{D_s}(x)\|^2}.$$

We now arrange the terms of this inequality in a different manner. For $m \in \mathbb{N}$ and $s > 0$, set

$$B_{(s,m)} = \{x \in \mathcal{O}_K \mid 3\sqrt[3]{m^2} \leq \|\Phi_{D_s}(x)\|^2 < 3\sqrt[3]{(m+1)^2}\,\}.$$

Notice that if $x \in B_{(s,m)}$, then $|\mathcal{N}(x)| = |x| \cdot |\sigma(x)|^2 \le m$. So the above inequality may be rewritten as

$$2s^2 \sum_{x\in B_{(s,2)}} \left( \|\Phi_{D_s}(x)\|^4 - 3\left(\frac{x^2}{2\pi s^2} + \frac{2x\mathcal{N}(x)}{s} + \frac{s^2\mathcal{N}(x)^2}{x^2}\right)\right) e^{-\pi\|\Phi_{D_s}(x)\|^2}$$

$$+ 2s^2 \sum_{x\in B_{(s,3)}} \left( \|\Phi_{D_s}(x)\|^4 - 3\left(\frac{x^2}{2\pi s^2} + \frac{2x\mathcal{N}(x)}{s} + \frac{s^2\mathcal{N}(x)^2}{x^2}\right)\right)$$

$$\times e^{-\pi\|\Phi_{D_s}(x)\|^2} + 2s^2 \sum_{x\notin B_{(s,2)}\cup B_{(s,3)}} \|\Phi_{D_s}(x)\|^4 e^{-\pi\|\Phi_{D_s}(x)\|^2}$$

$$< 3 \sum_{x\notin B_{(s,2)}\cup B_{(s,3)}} \left( \frac{x^2}{\pi} + 4sx\mathcal{N}(x) + 2\,\frac{s^4\mathcal{N}(x^2)}{x^2}\right) e^{-\pi\|\Phi_{D_s}(x)\|^2}.$$

We give some estimates of the terms involved in these sums. For the right-hand side, we have

$$\sum_{x\notin B_{(s,2)}\cup B_{(s,3)}} \left( \frac{x^2}{\pi} + 4sx\mathcal{N}(x) + 2\,\frac{s^4\mathcal{N}(x^2)}{x^2}\right) e^{-\pi\|\Phi_{D_s}(x)\|^2}$$

$$> 2(1/\pi + 4s + 2s^4) e^{-\pi(2s+1/s^2)},$$

and (in the left-hand side)

$$\sum_{x\notin B_{(s,2)}\cup B_{(s,3)}} \|\Phi_{D_s}(x)\|^4 e^{-\pi\|\Phi_{D_s}(x)\|^2}$$

$$= 2(2s + 1/s^2)^2 e^{-\pi(2s+1/s^2)} + \sum_{\|\Phi_{D_s}(x)\| \ge \sqrt{3}\,\sqrt[3]{4}} \|\Phi_{D_s}(x)\|^4 e^{-\pi\|\Phi_{D_s}(x)\|^2}$$

$$< 2(2s + 1/s^2)^2 e^{-\pi(2s+1/s^2)} + 2/10^7,$$

from Lemma 1.1. By direct calculations, one also finds the inequalities that follow. If $|\mathcal{N}(x)| = 1$ and $x \in B_{(s,2)}$, then

$$\left( \|\Phi_{D_s}(x)\|^4 - 3\left(\frac{x^2}{2\pi s^2} + \frac{2x\mathcal{N}(x)}{s} + \frac{s^2\mathcal{N}(x)^2}{x^2}\right)\right) e^{-\pi\|\Phi_{D_s}(x)\|^2}$$

$$= \left( \frac{x^4}{s^4} - 2\,\frac{|x|}{s} + \frac{s^2}{x^2} - \frac{3x^2}{2\pi s^2}\right) e^{-\pi(\frac{x^2}{s^2} + 2\frac{s}{|x|})} < \frac{2.72}{10^6},$$

while for $|\mathcal{N}(x)| = 1$ and $x \in B_{(s,3)}$, we have

$$\left( \|\Phi_{D_s}(x)\|^4 - 3\left(\frac{x^2}{2\pi s^2} + \frac{2x\mathcal{N}(x)}{s} + \frac{s^2\mathcal{N}(x)^2}{x^2}\right)\right) e^{-\pi\|\Phi_{D_s}(x)\|^2} < \frac{6.7}{10^8}.$$

If $|\mathcal{N}(x)|=2$ and $x\in B_{(s,2)}$, then

$$\Big(\|\Phi_{D_s}(x)\|^4-3\Big(\frac{x^2}{2\pi s^2}+\frac{2x\mathcal{N}(x)}{s}+\frac{s^2\mathcal{N}(x)^2}{x^2}\Big)\Big)e^{-\pi\|\Phi_{D_s}(x)\|^2}$$
$$=\Big(\frac{x^4}{s^4}-4\,\frac{|x|}{s}+4\,\frac{s^2}{x^2}-\frac{3x^2}{2\pi s^2}\Big)e^{-\pi(\frac{x^2}{s^2}+4\frac{s}{|x|})}<\frac{1.2}{10^7},$$

while for $|\mathcal{N}(x)|=2$ and $x\in B_{(s,3)}$, we have

$$\Big(\|\Phi_{D_s}(x)\|^4-3\Big(\frac{x^2}{2\pi s^2}+\frac{2x\mathcal{N}(x)}{s}+\frac{s^2\mathcal{N}(x)^2}{x^2}\Big)\Big)e^{-\pi\|\Phi_{D_s}(x)\|^2}<\frac{3}{10^8}.$$

If $|\mathcal{N}(x)|=3$ and $x\in B_{(s,3)}$, i.e., $(x^2/s^2+6s/|x|)\in[3\sqrt[3]{9},6\sqrt[3]{2})$, then

$$\Big(\|\Phi_{D_s}(x)\|^4-3\Big(\frac{x^2}{2\pi s^2}+\frac{2x\mathcal{N}(x)}{s}+\frac{s^2\mathcal{N}(x)^2}{x^2}\Big)\Big)e^{-\pi\|\Phi_{D_s}(x)\|^2}$$
$$=\Big(\frac{x^4}{s^4}-6\,\frac{|x|}{s}+9\,\frac{s^2}{x^2}-\frac{3x^2}{2\pi s^2}\Big)e^{-\pi(\frac{x^2}{s^2}+6\frac{s}{|x|})}<\frac{2}{10^9}.$$

Moreover, from (1), used in the proof of Lemma 1.1, we deduce that $\#(B_{(s,2)})\le 38$ and $\#(B_{(s,3)})\le 44$. We also point out that, for $s\in[0.92,1.09]$, there cannot be more than 4 units in $B_{(s,2)}$, because of the inequality $\varepsilon\ge\sqrt[3]{|\Delta_F|}/3$ from [PZ, 5.6]. Therefore, the conclusion follows if

$$4s^2(2s+1/s^2)e^{-\pi(2s+1/s^2)}+2s^2\,\frac{2}{10^5}<6\,(1/\pi+4s+2s^4)e^{-\pi(2s+1/s^2)}$$

for all $s\in[0.92,1.09]$, i.e., if

$$2\Big(\frac{3}{2\pi s^2}+\frac{2}{s}-s^2-\frac{1}{s^4}\Big)e^{-\pi(2s+1/s^2)}>\frac{2}{10^5}.$$

This fact is readily checked. ■

**2. Local calculations.** The last step of the proof is to establish the vanishing of $g'(1)$, where $g$ is the function introduced in the above section. We start with some general considerations.

For a number field $F$, set

$$V_\circ=\Big\{\mathbf{v}\in\mathbb{R}^{\mathcal{S}_\infty}\ \Big|\ \sum_{\sigma\in\mathcal{S}_\infty}\mathbf{v}_\sigma=0\Big\}.$$

We define the function $f:\mathbb{R}^{\mathcal{S}_\infty}\to\mathbb{R}$ as follows:

$$f(\mathbf{v})=\sum_{x\in\mathcal{O}_F}\exp\Big(-\pi\sum_{\sigma\in\mathcal{S}_\infty}d_\sigma e^{-2\mathbf{v}_\sigma/d_\sigma}|\sigma(x)|^2\Big),$$

with the notation $d_\sigma=[F_\sigma:\mathbb{R}]$.

PROPOSITION 2.1. *Let $F$ be a number field and let $\| \|_\circ$ denote the metric associated with the trivial Arakelov divisor. Then $\mathbf{v} = 0$ is a stationary point of the restriction of $f$ to $V_\circ$ if and only if*

$$\sum_{x\in\mathcal{O}_F} |\sigma(x)|^2 e^{-\pi\|x\|_\circ^2}$$

*is independent of $\sigma \in \mathcal{S}_\infty$.*

*Proof.* Put $\widetilde{f}(\mathbf{v},\lambda) = f(\mathbf{v}) - \lambda\sum_{\sigma\in\mathcal{S}_\infty}\mathbf{v}_\sigma$. We have

$$\begin{aligned}\frac{\partial\widetilde{f}}{\partial\mathbf{v}_\sigma}(\mathbf{v},\lambda) &= \frac{\partial f}{\partial\mathbf{v}_\sigma}(\mathbf{v}) - \lambda \\ &= -\lambda + 2\pi\sum_{x\in\mathcal{O}_F}|\sigma(x)|^2\exp\Big(-\pi\sum_{\sigma\in\mathcal{S}_\infty} d_\sigma e^{-2\mathbf{v}_\sigma/d_\sigma}|\sigma(x)|^2\Big).\end{aligned}$$

Hence, $\frac{\partial\widetilde{f}}{\partial\mathbf{v}_\sigma}(0,\lambda) = 0$ if and only if

$$\sum_{x\in\mathcal{O}_F}|\sigma(x)|^2 e^{-\pi\|x\|_\circ^2} = \frac{\lambda}{2\pi} \quad \text{for each } \sigma\in\mathcal{S}_\infty. \ \blacksquare$$

An example of a field $K$ for which the sum $\sum_x |\sigma(x)|^2 e^{-\pi\|x\|_\circ^2}$ is not independent of $\sigma \in \mathcal{S}_\infty$ is the complex cubic field of discriminant $-23$, namely $K = \mathbb{Q}(\alpha)$, where $\alpha$ is a root of the polynomial $p(t) = t^3 - t + 1$. A $\mathbb{Z}$-basis of $\mathcal{O}_K$ is $\{1, \alpha, \alpha^2 - 1\}$. Choosing as $\alpha$ the real root of $p$ and denoting by $\sigma$ the complex place of $K$, we have

$$\begin{aligned}&\sum_{x\in\mathcal{O}_K}(x^2 - |\sigma(x)|^2)e^{-\pi(\|x\|_\circ^2)} \\ &\quad = \sum_{(l,m,n)\in\mathbb{Z}^3}(m^2 + (1/\alpha)n^2 + 3\alpha lm + (1-3/\alpha)ln - (\alpha+3)mn) \\ &\qquad\quad \times e^{-\pi(3l^2+(1-3/\alpha)m^2+(3/\alpha^2-2/\alpha)n^2-2ln+2\alpha mn)} > \frac{4}{10^8}.\end{aligned}$$

This implies that, in the case of this field $\mathbb{Q}(\alpha)$, the function $k^\circ$ does not attain its maximum at the trivial class.

Given a number field $F$, let $\widetilde{F}$ be the set of all the field embeddings of $F$ into $\mathbb{C}$. We consider the pairing $\langle -,-\rangle_\circ$ on $F$, defined as follows:

$$\langle x,y\rangle_\circ = \sum_{\varphi\in\widetilde{F}}\varphi(x)\overline{\varphi(x)} = \sum_{\sigma\in\mathcal{S}_\infty} d_\sigma\sigma(x)\overline{\sigma(x)}.$$

This pairing is the restriction to $F$ of the trace form on the finite étale $\mathbb{R}$-algebra $F\otimes_{\mathbb{Q}}\mathbb{R}$, namely the form which gives the metric of the trivial Arakelov divisor.

LEMMA 2.2. *Let $F$ be a number field of degree $n$ over $\mathbb{Q}$ and let $\{\omega_j \mid 1 \le j \le n\}$ be a $\mathbb{Z}$-basis of $\mathcal{O}_F$ which is an orthogonal system with respect to the pairing $\langle -,-\rangle_\circ$. If, for every $j$, the value of $|\sigma(\omega_j)|$ is independent of $\sigma \in \widetilde{F}$, then the number*

$$\sum_{x\in\mathcal{O}_F} |\sigma(x)|^2 e^{-\pi\|x\|_\circ^2} = \sum_{x\in\mathcal{O}_F} |\sigma(x)|^2 e^{-\pi\sum_{\varphi\in\widetilde{F}}|\varphi(x)|^2}$$

*is also independent of $\sigma\widetilde{F}$.*

*Proof.* Let $x \in \mathcal{O}_F$. One can write $x = \sum_{1\le j\le n} z_j(x)\,\omega_j$ for some $z_j(x) \in \mathbb{Z}$. Thus, for $\varphi \in \widetilde{F}$, we also have

$$|\varphi(x)|^2 = \sum_{i,j} z_i(x)\, z_j(x)\, \varphi(\omega_i)\overline{\varphi(\omega_j)},$$

and then, by orthogonality,

$$\begin{aligned}\|x\|_\circ^2 &= \sum_{\sigma\in\mathcal{S}_\infty}[F_\sigma:\mathbb{R}]|\sigma(x)|^2 = \sum_{\varphi\in\widetilde{F}}|\varphi(x)|^2 = \sum_{\varphi\in\widetilde{F}}\sum_{i,j} z_i(x)z_j(x)\varphi(\omega_i)\overline{\varphi(\omega_j)}\\ &= \sum_{i,j} z_i(x)z_j(x)\sum_{\varphi\in\widetilde{F}}\varphi(\omega_i)\overline{\varphi(\omega_j)} = \sum_{1\le j\le n}(z_j(x))^2\sum_{\varphi\in\widetilde{F}}|\varphi(\omega_i)|^2.\end{aligned}$$

Therefore, for each $\sigma \in \mathcal{S}_\infty$, we obtain

$$\begin{aligned}(2)\quad \sum_{x\in\mathcal{O}_F}|\sigma(x)|^2 e^{-\pi\|x\|_\circ^2} &= \sum_{x=\sum_j z_j\omega_j}|\sigma(x)|^2 e^{-\pi\|x\|_\circ^2}\\ &= \sum_{\mathbf{z}\in\mathbb{Z}^n}\Big(\sum_{i,j}\mathbf{z}_i\mathbf{z}_j\sigma(\omega_i)\overline{\sigma(\omega_j)}\Big)e^{-\pi\sum_i \mathbf{z}_i^2(\sum_{\varphi\in\widetilde{F}}|\varphi(\omega_i)|^2)}\\ &= \sum_{\mathbf{z}\in\mathbb{Z}^n}\Big(\sum_j \mathbf{z}_j^2\,|\sigma(\omega_j)|^2\Big)e^{-n\pi\sum_i \mathbf{z}_i^2|\sigma(\omega_i)|^2}.\end{aligned}$$

Indeed, by hypothesis, $|\varphi(\omega_j)| = |\sigma(\omega_j)|$ for every $\sigma \in \widetilde{F}$ and the number $\sum_{i,j}\mathbf{z}_i\mathbf{z}_j\sigma(\omega_i)\overline{\sigma(\omega_j)}$ changes its sign upon the substitution $(\mathbf{z}_1,\dots,\mathbf{z}_n) \leftrightarrow (-\mathbf{z}_1,\dots,\mathbf{z}_n)$ while the term $e^{-n\pi\sum_j \mathbf{z}_j^2|\sigma(\omega_j)|^2}$ does not. The expression that we have now recovered for $\sum_{x\in\mathcal{O}_F}|\sigma(x)|^2 e^{-\pi\|x\|_\circ^2}$ is visibly independent of $\sigma \in \widetilde{F}$. ∎

COROLLARY 2.3. *Let $d \in \mathbb{Z}$ be such that the polynomial $p(x) = x^n - d$ is irreducible over $\mathbb{Z}$ and $\mathcal{O}_{\mathbb{Q}(\sqrt[n]{d})} = \mathbb{Z}[\sqrt[n]{d}]$. Define $V_\circ$ and $f$ as above. Then $\mathbf{v} = 0$ is a stationary point of the restriction of $f$ to $V_\circ$.*

*Proof.* Set $F = \mathbb{Q}(\sqrt[n]{d})$ and $\omega_k = \sqrt[n]{d}^{\,k}$. The set $\{\omega_k \mid 0 \le k < n\}$ is an integral basis of $\mathcal{O}_F$. For $0 \le k < n$, let $\varphi_k : F \hookrightarrow \mathbb{C}$ be the field embedding

$$\varphi_k : \sqrt[n]{d} \mapsto \sqrt[n]{d}\, e^{2\pi k i/n},$$

with $i = \sqrt{-1}$. Clearly,

$$\widetilde{F} = \{\varphi_k \mid 0 \le k < n\},$$

and $|\varphi_k(\omega_h)| = \sqrt[n]{d}^{\,h}$, independently of $\varphi_k \in \widetilde{F}$. Moreover, if $g \neq h$, then

$$\langle \omega_g, \omega_h \rangle_\circ = \sum_{0 \le k < n} \varphi_k(\omega_g)\overline{\varphi_k(\omega_g)} = \sqrt[n]{d}^{\,g+h} \sum_{0 \le k < n} e^{2\pi(g-h)ki/n} = 0.$$

Therefore, applying Proposition 2.1 and Lemma 2.2, we obtain the required conclusion. ∎

We remark that the hypothesis that $F$ be of the first kind appears essential. Indeed, let $F = \mathbb{Q}(\sqrt[3]{ab^2})$ be a pure cubic field of the second kind. In this case,

$$\mathcal{O}_F = \left\{ \frac{l}{3} + (al + 3m)\,\frac{\sqrt[3]{ab^2}}{3} + (bl + 3n)\,\frac{\sqrt[3]{a^2 b}}{3} \;\middle|\; l, m, n \in \mathbb{Z} \right\}.$$

We can use the same argument that we applied in Lemma 2.2, taking the orthogonal system $(\omega_1, \omega_2, \omega_3) = (1/3, \sqrt[3]{ab^2}/3, \sqrt[3]{a^2b}/3)$ and letting $\mathbf{z} \in \mathbb{Z}^3$ run over all the strings of the form $(l,\, al + 3m,\, bl + 3n)$ with $l, m, n \in \mathbb{Z}$. But, in this situation, we cannot cancel the terms of the kind $(\sum_{i,j} \mathbf{z}_i \mathbf{z}_j\, \sigma(\omega_i)\overline{\sigma(\omega_j)})e^{-\pi \sum_j \mathbf{z}_j^2 (\sum_\varphi |\varphi(\omega_j)|^2)}$, with $i \neq j$, as we did in (2). Indeed,

$$\begin{aligned}
&\sum_{x \in \mathcal{O}_K} (x^2 - |\sigma(x)|^2) e^{-\pi(\|x\|_\circ^2)} \\
&\quad = \frac{1}{3} \sum_{(l,m,n) \in \mathbb{Z}^3} (l(al + 3m)\,\sqrt[3]{ab^2} + l(bl + 3n)\,\sqrt[3]{a^2b} + (al + 3m)(bl + 3n)ab) \\
&\qquad \times e^{-\frac{\pi}{3}(l^2 + (al+3m)^2 b\,\sqrt[3]{a^2b} + (bl+3n)^2 a\,\sqrt[3]{ab^2})}.
\end{aligned}$$

If $F = \mathbb{Q}(\sqrt[3]{10})$, this difference is

$$\begin{aligned}
&\frac{1}{3} \sum_{(l,m,n) \in \mathbb{Z}^3} (l(10l + 3m)\sqrt[3]{10} + l(l + 3n)\sqrt[3]{100} + 10(10l + 3m)(l + 3n)) \\
&\qquad\qquad \times e^{-\frac{\pi}{3}(l^2 + (10l+3m)^2\,\sqrt[3]{100} + (l+3n)^2 10\,\sqrt[3]{10})}.
\end{aligned}$$

With a numerical computation, one finds that this number is positive (greater than $4/10^{12}$). In view of Proposition 2.1, this means that the function $k^\circ$ on $\mathcal{P}ic^\circ(\mathbb{Q}(\sqrt[3]{10}))$ does not attain its maximum at the trivial class.

Finally, we state the following fact, which concludes the proof of the Theorem.

COROLLARY 2.4. *Let $K$ be a pure cubic field of the first kind and let $g$ be the function defined above. Then $g'(1) = 0$.*

*Proof.* Let $K = \mathbb{Q}(\sqrt[3]{ab^2})$ with $a$ and $b$ square-free integers. By Proposition 2.1 and Lemma 2.2, it is enough to check that if $\alpha, \beta \in \{1, \sqrt[3]{ab^2}, \sqrt[3]{a^2b}\}$ with $\alpha \neq \beta$, then

$$\langle \alpha, \beta \rangle_\circ = \alpha\overline{\beta} + \sigma(\alpha)\overline{\sigma(\beta)} + \overline{\sigma(\alpha)}\,\sigma(\beta) = 0$$

and $|\alpha| = |\sigma(\alpha)|$. This is immediate. ■

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Dipartimento di Metodi e Modelli Matematici
Università di Roma "La Sapienza"
Via Scarpa n°16
00161 Roma, Italy
E-mail: paolo.francini@uniroma1.it



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