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Lower bound of real primitive L-function at $s = 1$

by

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1. Introduction and main results. Let χ be a real primitive Dirichlet character modulo k (> 1). It is well known that if $L(s,\chi)$ has no zero in the interval $(1 - c_1/\log k, 1)$, then $L(1,\chi) > c_2/\log k$, where c_1 and c_2 are positive constants and c_2 depends upon c_1. If, however, $L(s,\chi)$ has a real zero close to 1, the only non-trivial lower bounds that are known for $L(1,\chi)$ are ineffective. Siegel [9] proved that for any $\varepsilon > 0$,

$$L(1,\chi) > \frac{c(\varepsilon)}{k^{\varepsilon}},$$

where $c(\varepsilon)$ is an ineffective positive constant depending upon ε (see also Chowla [1], Estermann [2], Goldfeld [3] and Goldfeld and Schinzel [4]). Tatuzawa [10] proved that if $0 < \varepsilon < 1/11.2$ and $k > e^{1/\varepsilon}$, then with at most one exception

$$L(1,\chi) > \frac{0.655\varepsilon}{k^{\varepsilon}}.$$

Hoffstein [6] proved that if $0 < \varepsilon < 1/(6\log 10)$ and $k > 10^6$, then with at most one exception

$$L(1,\chi) > \min\left\{\frac{1}{7.735\log k}, \frac{\varepsilon}{0.349k^{\varepsilon}}\right\}.$$

In this paper we improve upon the result of Hoffstein. Using Lemma 4 of Hoffstein [5], the upper bound estimate of $L(1,\chi)$ of Louboutin [7], [8] and some arithmetic theory of biquadratic bicyclic number fields, we prove the following:

Theorem. *Let $0 < \varepsilon < 1/(6\log 10)$, and χ be a real primitive Dirichlet character modulo k which is greater than 10^6. Then with at most one*

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exception,

$$L(1,\chi) > \min\left\{\frac{1}{7.7388\log k}, \frac{32.260\varepsilon}{k^{\varepsilon}}\right\}.$$

2. Several lemmas

LEMMA 1 ([6, Lemma 2]). *Set $c = (3+2\sqrt{2})/2$ and let d_F denote the absolute value of the discriminant of a number field $F \neq \mathbb{Q}$. Then $\zeta_F(s)$ has at most one real zero β with $\beta > 1 - 1/(c\log d_F)$, and if it exists it is a simple zero.*

LEMMA 2. *Let K be an algebraic number field of degree $n > 1$ and assume that for each $m \geq 1$ there exists at least one integral ideal of K of norm m^2 (e.g. K is a quadratic or a biquadratic bicyclic number field). Assume also that $1/2 < \beta < 1$ and $\zeta_K(\beta) \leq 0$. Then the residue at $s = 1$ of the Dedekind zeta function $\zeta_K(s)$ of K satisfies*

$$\kappa_K \geq (1-\beta)\left(x^{\beta-1}\left(\frac{\pi^2}{6} - \frac{n+2}{[\sqrt{x}\,]}\right) - 2\,\frac{d_K}{x^{3/2}}\,\frac{\zeta^n(3/2)(n+1)!}{(4n-3)\pi^n}\right) \quad (x \geq 1).$$

Proof. According to the proof of Lemma 4 of Hoffstein [5],

$$\begin{aligned}
&\frac{1}{2^{n+1}(n+1)!}\left(\frac{\pi^2}{6} - \frac{n+2}{[\sqrt{x}\,]}\right)\\
&\qquad \leq \frac{1}{2^{n+1}(n+1)!}\sum_{m\leq\sqrt{x}}\left(\frac{1}{m^2} - (n+1)\frac{m^2}{x^2}\right)\\
&\qquad \leq \frac{1}{2^{n+1}(n+1)!}\sum_{m\leq\sqrt{x}}\frac{1}{m^2}\left(1 - \frac{m^4}{x^2}\right)^{n+1}\\
&\qquad \leq \frac{1}{2^{n+1}(n+1)!}\sum_{m\leq\sqrt{x}}\frac{1}{m^{2\beta}}\left(1 - \frac{m^4}{x^2}\right)^{n+1}\\
&\qquad \leq \frac{1}{2^{n+1}(n+1)!}\sum_{N(\mathfrak{a})\leq x}\frac{1}{(N(\mathfrak{a}))^{\beta}}\left(1 - \frac{(N(\mathfrak{a}))^2}{x^2}\right)^{n+1}\\
&\qquad = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{\zeta_K(s+\beta)x^s}{s(s+2)\ldots(s+2n+2)}\,ds\\
&\qquad = J + \frac{\zeta_K(\beta)}{2^{n+1}(n+1)!} + \frac{\kappa_K x^{1-\beta}}{(1-\beta)(3-\beta)\ldots(2n+3-\beta)}\\
&\qquad \leq J + \frac{\kappa_K x^{1-\beta}}{(1-\beta)2^{n+1}(n+1)!}
\end{aligned}$$

where

$$J := \frac{1}{2\pi i} \int_{-1/2-\beta-i\infty}^{-1/2-\beta+i\infty} \frac{\zeta_K(s+\beta)x^s}{s(s+2)\dots(s+2n+2)}\,ds$$

satisfies

$$|J| \le \frac{d_K}{x^{3/2}} \frac{\zeta^n(3/2)}{(2\pi)^n(4n-3)}\,x^{1-\beta};$$

here we use the functional equation satisfied by $\zeta_K(s)$ and the bound $|\zeta_K(3/2 + it)| \le \zeta^n(3/2)$. This completes the proof of Lemma 2.

COROLLARY 3. *Let $c = (3+2\sqrt{2})/2$ and K be a quadratic number field with the absolute value of the discriminant $d_K > 10^6$. If $\zeta_K(\beta) \le 0$ for some β satisfying $1 - 1/(4c\log d_K) \le \beta < 1$, then $\kappa_K \ge 1.5063(1-\beta)$.*

Proof. Apply the previous result with $n = 2$ and $x = d_K^{0.95}$ for which $x^{\beta-1} \ge \exp(-0.95/(6+4\sqrt{2}))$.

LEMMA 4 (see [7] and [8]). (1) *If χ is a primitive even Dirichlet character modulo $k > 1$, then*

$$|L(1,\chi)| \le \frac{1}{2}\log k + \frac{2+\gamma-\log(4\pi)}{2} \le \frac{1}{2}(\log k + 0.05).$$

(2) *If χ is a primitive odd Dirichlet character modulo k, then*

$$|L(1,\chi)| \le \frac{1}{2}\log k + \frac{2+\gamma-\log\pi}{2} \le \frac{1}{2}(\log k + 1.44).$$

Here $\gamma = 0.577215\ldots$ denotes Euler's constant.

3. Proof of Theorem. Let $0 < \varepsilon < 1/(6\log 10)$ and let χ_1 be a real primitive Dirichlet character of least conductor $k_1 > 10^6$ such that

$$L(1,\chi_1) \le \frac{1}{7.7388\log k_1} \tag{1}$$

if it exists. By Corollary 3, $L(1,\chi_1)$ has a real zero β_1 such that

$$1-\beta_1 < \frac{1}{4c\log k_1}. \tag{2}$$

Let χ be another real primitive Dirichlet character modulo $k > 10^6$ different from χ_1. By our choice of k_1 we can assume $k \ge k_1$. Let the Kronecker symbols of quadratic number fields $\mathbb{Q}(\sqrt{d})$, $\mathbb{Q}(\sqrt{d_1})$ be χ, χ_1 respectively, where d and d_1 are fundamental discriminants. Then $|d| = k$, $|d_1| = k_1$. Let $F = \mathbb{Q}(\sqrt{d}, \sqrt{d_1})$. Then it has another quadratic number field $\mathbb{Q}(\sqrt{d_2})$ with the fundamental discriminant d_2 and $d_2 \mid dd_1$. Let the Kronecker symbol of $\mathbb{Q}(\sqrt{d_2})$ be χ_2 with conductor $k_2 = |d_2|$. Let

$$\alpha = -3/2 - \beta_1, \qquad x = d_F^A, \qquad A > 0.8, \tag{3}$$

and notice that $d_F \le (kk_1)^2 \le k^4$. Applying Lemma 2 with $n = 4$ and $x = d_F^{0.8} > 10^{9.6}$ and $1-\beta_1 < 1/(4c\log k_1) \le 1/((6+4\sqrt{2})\log 10^6) < 0.00621$, we obtain

$$1.60452(1-\beta_1) < \kappa_F x^{1-\beta_1} = L(1,\chi)L(1,\chi_1)L(1,\chi_2)x^{1-\beta_1}. \tag{4}$$

If $L(s,\chi) \ne 0$ in the range $1 - 1/(4c\log k) < s < 1$, then by Corollary 3 we have

$$L(1,\chi) > \frac{1.5063}{4c\log k} > \frac{1}{7.7388\log k}$$

and the result follows. If $L(\beta,\chi) = 0$ for some β such that $1 - 1/(4c\log k) < \beta < 1$ then both β and β_1 are zeros of $\zeta_F(s)$. Since $\beta > 1 - 1/(4c\log k) \ge 1 - 1/(c\log d_F)$, we must have

$$1-\beta_1 > \frac{1}{c\log d_F} \ge \frac{1}{2c\log(kk_1)} \tag{5}$$

by Lemma 1. Let $A = 2/(1.5+\beta_1)$. Then

$$2A(1-\beta_1) = \frac{4(1-\beta_1)}{2.5-(1-\beta_1)} < \frac{4}{10c - \frac{1}{\log k_1}} \cdot \frac{1}{\log k_1} < \frac{0.1376}{\log k_1},$$

by (3) and $k_1 \ge 10^6$, and we have

$$x^{1-\beta_1} = d_F^{A(1-\beta_1)} \le (kk_1)^{2A(1-\beta_1)} < (kk_1)^{0.1376/\log k_1}. \tag{6}$$

From Lemma 4 we have

$$L(1,\chi_2) < \frac{1}{2}(\log k_2 + 1.44) \le \frac{1}{2}(\log(kk_1) + 1.44) < 0.5261\log(kk_1) \tag{7}$$

for $kk_1 \ge 10^{12}$. Combining (1) and (4)–(7) we get

$$L(1,\chi) > \frac{4.04947}{\log k_1 \big(1 + \frac{\log k}{\log k_1}\big)^2 (kk_1)^{\frac{0.1376}{\log k_1}}}. \tag{8}$$

Let

$$\eta = \frac{\log k}{\log k_1}, \qquad \eta \ge 1.$$

If $\eta \le 7.54$, then $L(1,\chi) > 1/(7.7388\log k)$ by

$$\frac{4.04947\eta}{(\eta+1)^2 e^{0.1376(1+\eta)}} \cdot 7.7388 > 1, \qquad 1 \le \eta \le 7.54.$$

Hence we may assume $\eta > 7.54$; in this case $2\log(\eta+1) < 0.5023(\eta+1)$ and

$$L(1,\chi) > \frac{2.13546}{(\log k_1)k^{0.6399/\log k_1}}.$$

For fixed $\delta > 0$, $xe^{\delta/x}$ decreases until it reaches a minimum at $x = \delta$. Let $x = \log k_1$, $\delta = 0.6399\log k$. Then $\log k > 7.54\log k_1 > (0.6425)^{-1}\log k_1$,

$\log k_1 > 1/\varepsilon$. By (8), we have

$$L(1,\chi) > \frac{2.13546\varepsilon}{k^{0.6399\varepsilon}} = \frac{\varepsilon}{k^{\varepsilon}} \cdot 2.13546 k^{0.3601\varepsilon}.$$

Since

$$2.13546 k^{0.3601\varepsilon} > 2.13546 e^{0.3601\varepsilon(7.54\log k_1)} > 2.13546 e^{0.3601\cdot 7.54} > 32.260,$$

we have

$$L(1,\chi) > \frac{32.260\varepsilon}{k^{\varepsilon}}.$$

This completes the proof of Theorem.

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